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Local Solvability of Cauchy Problem for Kaup-Kupershmidt Equation

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Abstract This paper deals with the local solvability of initial value problem for Kaup-Kupershmidt equations. Indeed, using Bourgain method, we prove that the Cauchy problem of Kaup-Kupershmidt equation is local well-posed in H^s whenever $s > \frac{9}{8}$, which improves the former results in [5].

Keywords Kaup-Kupershmidt equation; initial value problem; local well-posed.

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1. Introduction

This paper studies the initial value problem (IVP) associated to Kaup-Kupershmidt equations:

$$\begin{cases} u_t + \alpha u_{xxx} + \beta u_{xxxxx} + \gamma_1 u u_{xx} = 0, & x \in R,\ t \in R, \\ u(0,x) = u_0(x), & x \in R \end{cases} \tag{1}$$

and

$$\begin{cases} u_t + \alpha u_{xxx} + \beta u_{xxxxx} + \gamma_2 u_x u_{xx} = 0, & x \in R,\ t \in R, \\ u(0,x) = u_0(x), & x \in R, \end{cases} \tag{2}$$

where γ_1, γ_2 are the nonlinear perturbed coefficients; α, $\beta \in R$ are the dispersive coefficients. These equations are important dispersive equations proposed first by Kaup in 1980 ([1]) and developed by Kupershmidt in 1994 ([2]).

There has been much interest in the traveling wave solutions of Kaup-Kupershmidt equation using either numerical methods, asymptotic methods or fundamental theoretical methods [3–4]. However, we should owe the first result of the well-posedness of Kaup-Kupershmidt equations to Tao and Cui [5]. They showed that IVP (1) and (2) are locally well-posed in $H^s(R)$ for $s > \frac{5}{4}$ and $s > \frac{301}{108}$, respectively. In this paper, using the Fourier transform restriction norm method, we shall lower the regularity of the initial value space to $s > \frac{9}{8}$, which improves the former result in [5].

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This paper will concentrate on IVP (1), since IVP (2) can be dealt with in the same way. To study IVP (1), we shall consider its equivalent formulation instead:

$$u = U(t)u_0 + \lambda \int_0^t U(t-s)[uu_{xx}](s)\mathrm{d}s,$$

where

$$U(t) = \mathcal{F}_x^{-1} e^{-it\phi(\xi)} \mathcal{F}_x$$

is the unitary operator associated to the linear equation, and the operator $\mathcal{F}_{(\cdot)}$ denotes the Fourier transform in the $(\cdot)$ variable. Here the phase function is

$$\phi(\xi) = \alpha\xi^3 - \beta\xi^5.$$

Before precisely stating the main results, we first introduce some definitions and notations.

Definition 1 *For $s, b \in R$, we define Bourgain space $X_{s,b}$ to be the completion of the Schwartz function space on R^2 with respect to the norm:*

$$\|u\|_{X_{s,b}} = \|(1+|\xi|)^s(1+|\tau-\phi(\xi)|)^b \mathcal{F}u(\xi,\tau)\|_{L^2_\xi L^2_\tau} = \|\langle\xi\rangle^s\langle\tau-\phi(\xi)\rangle^b \mathcal{F}u(\xi,\tau)\|_{L^2_\xi L^2_\tau},$$

where, $\langle\cdot\rangle = (1+|\cdot|)$ and $\mathcal{F}u = \widehat{u}(\xi,\tau)$ denotes the Fourier transform in t and x of u.

Obviously, $\|u\|_{X_{s_1,b_1}} \le \|u\|_{X_{s_2,b_2}}$ holds if $s_1 \le s_2$, $b_1 \le b_2$.

The main result of this work reads as follows:

Theorem 1 *Let $\alpha\beta < 0$. The initial value problem (1) and (2) are locally well-posed for given data in Sobolev spaces H^s, for $s > \frac{9}{8}$.*

The layout of this paper is as follows. In the next section, we establish some preliminary lemmas. Section 3 is devoted mainly to establishing a bilinear estimate in the Bourgain space, which is the core of this paper. The proof of the main theorem will be given in the last section.

2. Preliminary lemmas

Assume the symbol of partial differential operator $R(D)$, denoted by $R(\xi)$, satisfies:

(H_1) $R(\xi) = R_1(\xi) + R_2(\xi)$, where $R_i(\lambda\xi) = \lambda^{r_i} R_i(\xi)$, $\xi \in R$, $\lambda > 0$, $r_2 \ge r_1 \ge 2$, $i = 1, 2$.

(H_2) $R_1(\xi)R_2(\xi) > 0$, $\forall \xi \in R \setminus \{0\}$.

Then, we have

Lemma 1 ([6]) *The solution of Cauchy problem*

$$\begin{cases} u_t + iR(D)u = 0, & x \in R,\ t \in R, \\ u(0,x) = u_0(x), & x \in R, \end{cases}$$

$u(t) = S(t)u_0 = \mathcal{F}^{-1}[e^{-itR(\xi)}\hat{u}_0](x)$ *satisfies:*

$$\|S(t)u_0\|_{L^p} \le Ct^{-\frac{1}{r_1}(1-\frac{2}{p})}\|u_0\|_{L^{p'}},\ t \in R^+,$$

$$\|S(t)u_0\|_{L^p_x L^q_t} \le C\|u_0\|_{L^2},$$

where $2 \le p \le \infty$, $\frac{2}{q} = \frac{1}{r_1}(1-\frac{2}{p})$.

According to Lemma 1, we deduce directly

Lemma 2 *If $\alpha\beta < 0$, then the unitary group $U(t)$ $(t \in R)$ satisfies*

$$\|U(t)\varphi\|_{L_x^8 L_t^8} \leq \|\varphi\|_{L^2}. \tag{3}$$

To eliminate the singularity of phase function $\phi(\xi)$, we introduce the following Fourier restriction operators:

$$P^N f = \int_{|\xi|\geq N} e^{ix\xi}\widehat{f}(\xi)\mathrm{d}\xi,\ P_N f = \int_{|\xi|\leq N} e^{ix\xi}\widehat{f}(\xi)\mathrm{d}\xi,\ \forall N > 0.$$

Let $D_x^s = \mathcal{F}_x^{-1}|\xi|^s\mathcal{F}_x$, $a = \max\{1, \sqrt{\frac{3\alpha}{5\beta}}\}$. In the following, we always assume that $\alpha\beta < 0$.

Lemma 3 *The group $U(t)$ $(t \in R)$ satisfies*

$$\|D_x^2 U(t)P^{2a}\varphi\|_{L_x^\infty L_t^2} \leq \|\varphi\|_{L^2}, \tag{4}$$

$$\|U(t)P^{2a}\varphi\|_{L_x^4 L_t^\infty} \leq \|\varphi\|_{H^{\frac{1}{4}}}, \tag{5}$$

$$\|D_x^{\frac{1}{2}} U(t)P^{2a}\varphi\|_{L_x^6 L_t^6} \leq \|\varphi\|_{L^2}. \tag{6}$$

Proof Since $\phi'(\xi) = 3\alpha\xi^2 - 5\beta\xi^4$ and $\alpha\beta < 0$, it follows that if $|\xi| > 0$, $\phi(\xi)$ is invertible. Then

$$\begin{aligned}
P^{2a}U(t)\varphi &= \int_{|\xi|\geq 2a} e^{ix\xi}e^{-it\phi(\xi)}\widehat{\varphi}(\xi)\mathrm{d}\xi \\
&= \int_{|\phi^{-1}|\geq 2a} e^{ix\phi^{-1}}e^{-it\phi}\widehat{\varphi}(\phi^{-1})\frac{1}{\phi'}\mathrm{d}\phi \\
&= \mathcal{F}_t\Big(e^{ix\phi^{-1}}\widehat{\varphi}(\phi^{-1})\frac{1}{\phi'}\chi_{\{|\phi^{-1}|\geq 2a\}}\Big).
\end{aligned}$$

By Plancherel's theorem and the above estimate, we have

$$\begin{aligned}
\|P^{2a}U(t)\varphi\|_{L_t^2}^2 &= \Big\|\widehat{\varphi}(\phi^{-1})\frac{1}{\phi'}\chi_{\{|\phi^{-1}|\geq 2a\}}\Big\|_{L_\phi^2}^2 \\
&= \int_{|\phi^{-1}|\geq 2a}|\widehat{\varphi}(\phi^{-1})|^2\frac{1}{|\phi'|^2}\mathrm{d}\phi = \int_{|\xi|\geq 2a}|\widehat{\varphi}(\xi)|^2\frac{1}{|\phi'(\xi)|^2}\phi'(\xi)\mathrm{d}\xi \\
&\leq \int_{|\xi|\geq 2a}|\widehat{\varphi}(\xi)|^2\frac{1}{|\phi'(\xi)|}\mathrm{d}\xi = \int_{|\xi|\geq 2a}|\widehat{\varphi}(\xi)|^2\frac{1}{|5\beta\xi^4||1-\frac{3\alpha}{5\beta\xi^2}|}\mathrm{d}\xi \\
&\leq C\int_{|\xi|\geq 2a}|\widehat{\varphi}(\xi)|^2\frac{1}{|\xi|^4}\mathrm{d}\xi \leq C\|\varphi\|_{\dot{H}^{-2}}.
\end{aligned}$$

This implies the estimate (4).

With the help of Theorem 2.5 in [7], we have

$$\begin{aligned}
\|U(t)P^{2a}\varphi\|_{L_x^4 L_t^\infty}^2 &\leq \int_{|\xi|\geq 2a}|\mathcal{F}P^{2a}\varphi(\xi)|^2\Big|\frac{\phi'(\xi)}{\phi''(\xi)}\Big|^{\frac{1}{2}}\mathrm{d}\xi \\
&\leq \int_{|\xi|\geq 2a}|\mathcal{F}P^{2a}\varphi(\xi)|^2\Big|\frac{|5\beta\xi^4-3\alpha\xi^2|}{|20\beta\xi^3-6\alpha\xi|}\Big|^{\frac{1}{2}}\mathrm{d}\xi \\
&\leq \int_{|\xi|\geq 2a}|\mathcal{F}P^{2a}\varphi(\xi)|^2\Big|\frac{|5\beta\xi^4||1-\frac{3\alpha}{5\beta\xi^2}|}{|20\beta\xi^3||1-\frac{3\alpha}{10\beta\xi^2}|}\Big|^{\frac{1}{2}}\mathrm{d}\xi
\end{aligned}$$

$$\leq \int_{|\xi|\geq 2a} |\mathcal{F}P^{2a}\varphi(\xi)|^2|\xi|^{\frac{1}{2}}\mathrm{d}\xi \leq C\|P^{2a}\varphi\|^2_{H^{\frac{1}{4}}}.$$

This gives the estimate (5).

Finally, (6) follows by interpolation between (4) and (5). □

Let $\widehat{F}_\rho(\xi,\tau) = \frac{f(\xi,\tau)}{(1+|\tau-\phi(xi)|)^\rho}$. Then we have

Lemma 4 *If $\rho > \frac{1}{2}$, for any fixed N with $0 < N < \infty$, there holds that*

$$\|P_N F_\rho\|_{L^2_x L^\infty_t} \leq C\|f\|_{L^2_\xi L^2_\tau},$$

where the constant C depends on N.

Proof Consult the proof of Lemma 2.3 in [8].

Lemma 5 *(i) If $\rho > \frac{1}{2}\frac{4(q-2)}{3q}$, for $2 \leq q \leq 8$, then*

$$\|F_\rho\|_{L^q_x L^q_t} \leq C\|f\|_{L^2_\xi L^2_\tau}. \tag{7}$$

(ii) If $\rho > \frac{3}{8}$, then

$$\|D_x^{\frac{3}{8}} P^{2a} F_\rho\|_{L^4_x L^4_t} \leq C\|f\|_{L^2_\xi L^2_\tau}. \tag{8}$$

Proof Change variable $\tau = \lambda + \phi(\xi)$, then

$$\begin{aligned} F_\rho(x,t) &= \int\int e^{i(x\xi+t\tau)}\frac{f(\xi,\tau)}{(1+|\tau-\phi(\xi)|)^\rho}\mathrm{d}\xi\mathrm{d}\tau \\ &= \int e^{it\lambda}\Big(\int e^{i(x\xi+t\phi(\xi))}f(\xi,\lambda+\phi(\xi))\mathrm{d}\xi\Big)\frac{\mathrm{d}\lambda}{(1+|\lambda|)^\rho}. \end{aligned}$$

Therefore, using (3), Minkowshi's integral inequality and taking $\rho > \frac{1}{2}$, one easily shows that

$$\|F_\rho\|_{L^8_x L^8_t} \leq C\int \|f(\xi,\lambda+\phi(\xi))\|_{L^2_\xi}\frac{\mathrm{d}\lambda}{(1+|\lambda|)^\rho} \leq C\|f\|_{L^2_\xi L^2_\tau}. \tag{9}$$

By interpolation between (9) and the trivial inequality

$$\|F_0\|_{L^2_x L^2_t} \leq \|f\|_{L^2_\xi L^2_\tau}, \tag{10}$$

we obtain (7).

By (6) and Minkowshi's integral inequality and taking $\rho > \frac{1}{2}$, one easily shows that

$$\|D_x^{\frac{1}{2}} P^{2a} F_\rho\|_{L^6_x L^6_t} \leq C\int \|f(\xi,\lambda+\phi(\xi))\|_{L^2_\xi}\frac{\mathrm{d}\lambda}{(1+|\lambda|)^\rho} \leq C\|f\|_{L^2_\xi L^2_\tau}. \tag{11}$$

Then (8) follows by interpolation between (10) and (11). □

Lemma 6 *(i) Let $\rho > \frac{\theta}{2}$ with $\theta \in [0,1]$. Then*

$$\|D_x^{2\theta} P^{2a} F_\rho\|_{L_x^{\frac{2}{1-\theta}} L^2_t} \leq C\|f\|_{L^2_\xi L^2_\tau}.$$

(ii) Let $\rho > \frac{1}{2}$. Then

$$\|D_x^{-\frac{1}{4}} P^{2a} F_\rho\|_{L^4_x L^\infty_t} \leq C\|f\|_{L^2_\xi L^2_\tau}.$$

Proof The proof is similar to that of Lemma 5 with the help of (3) and (4).

Lemma 7 ([11]) *Assume that f, f_1, f_2 belong to Schwartz space on R^2. Then*

$$\int_* \overline{\widehat{f}}(\xi,\tau)\widehat{f_1}(\xi_1,\tau_1)\widehat{f_2}(\xi_2,\tau_2)\mathrm{d}\delta = \int \overline{f} f_1 f_2(x,t)\mathrm{d}x\mathrm{d}t,$$

where $\int_* \mathrm{d}\delta = \int_{\xi=\xi_1+\xi_2,\tau=\tau_1+\tau_2} \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\tau_1\mathrm{d}\tau_2$.

Let Z be any Abelian additive group with an invariant measure $\mathrm{d}\xi$. For any integer $k \geq 2$, we denote by $\Gamma_k(Z)$ the Hyperplane $\Gamma_k(Z) = \{(\xi_1,\xi_2,\ldots,\xi_k) \in Z^k,\ \xi_1+\xi_2+\cdots+\xi_k = 0\}$, and define a $[k,Z]$-multiplier to be any function $m : \Gamma_k(Z) \mapsto C$. If m is a $[k,Z]$-multiplier, we define $\|m\|_{[k,Z]}$ to be the best constant such that

$$\Big|\int_{\Gamma_k(Z)} m(\xi)\prod_{j=1}^k f_j\Big| \leq \|m\|_{[k,Z]}\prod_{j=1}^k \|f_j\|_{L^2(Z)}.$$

Lemma 8 ([10]) *If $m(\xi)$ and $M(\xi)$ are $[k,Z]$-multipliers satisfying $|m(\xi)| \leq |M(\xi)|$ for all $\xi \in \Gamma_k(Z)$, then*

$$\|m\|_{[k,Z]} \leq \|M\|_{[k,Z]}.$$

Lemma 9 *For $|\xi|$ large enough, e.g., $|\xi| > 2a$, there holds that*

$$\max\{|\sigma|,|\sigma_1|,|\sigma_2|\} \geq C|\xi_1||\xi_2||\xi|^3,$$

where

$$\xi=\xi_1+\xi_2,\ \tau=\tau_1+\tau_2;\ \sigma=\tau-\phi(\xi),\ \sigma_1=\tau_1-\phi(\xi_1),\ \sigma_2=\tau_2-\phi(\xi_2).$$

Proof Noticing the following equation first:

$$\sigma_1+\sigma_2-\sigma = -5\beta\xi_1\xi_2\xi(\xi^2-\xi\xi_1+\xi_1^2)+3\alpha\xi_1\xi_2\xi,$$

then since $\xi^2-\xi\xi_1+\xi_1^2 \geq \frac{3}{4}\xi^2$, $\alpha\beta<0$, we have

$$\begin{aligned}|\sigma_1+\sigma_2-\sigma| &= |-5\beta\xi_1\xi_2\xi(\xi^2-\xi\xi_1+\xi_1^2)+3\alpha\xi_1\xi_2\xi|\\ &= |\xi_1\xi_2\xi||-5\beta(\xi^2-\xi\xi_1+\xi_1^2)+3\alpha|\\ &\geq C|\xi_1||\xi_2||\xi|^3.\end{aligned}$$

□

3. Bilinear estimate

Theorem 2 *If $s \geq \frac{3}{4}$, $\frac{1}{2} < b < \frac{19}{32}$, then $\forall b' > \frac{13}{32}$, we have*

$$\|uv_{xx}\|_{X_{s,b-1}} \leq C\|u\|_{X_{s,b'}}\|v\|_{X_{s,b'}}. \tag{12}$$

Proof By duality and Plancherel theorem, it suffices to show that

$$\begin{aligned}I &= \int_* \frac{\langle\xi\rangle^s\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}\frac{f_1(\xi_1,\tau_1)}{\langle\xi_1\rangle^s\langle\sigma_1\rangle^{b'}}\frac{\xi_2^2 f_2(\xi_2,\tau_2)}{\langle\xi_2\rangle^s\langle\sigma_2\rangle^{b'}}\mathrm{d}\delta\\ &= \int_* \frac{\langle\xi\rangle^s\xi_2^2}{\langle\sigma\rangle^{1-b}\langle\xi_1\rangle^s\langle\sigma_1\rangle^{b'}\langle\xi_2\rangle^s\langle\sigma_2\rangle^{b'}}\overline{f}(\xi,\tau)f_1(\xi_1,\tau_1)f_2(\xi_2,\tau_2)\mathrm{d}\delta\\ &\leq C\Big\|\frac{\langle\xi\rangle^s\xi_2^2}{\langle\sigma\rangle^{1-b}\langle\xi_1\rangle^s\langle\sigma_1\rangle^{b'}\langle\xi_2\rangle^s\langle\sigma_2\rangle^{b'}}\Big\|_{[3,R\times R]}\|\bar{f}\|_{L^2_\xi L^2_\tau}\Pi_{j=1}^2\|f_j\|_{L^2_\xi L^2_\tau},\end{aligned}$$

for $\bar f \in L^2(R^2)$, $\bar f \geq 0$, where

$$f_1 = \langle\xi_1\rangle^s \langle\sigma_1\rangle^{b'} \widehat{u}(\xi_1,\tau_1),\ \ f_2 = \langle\xi_2\rangle^s \langle\sigma_2\rangle^{b'} \widehat{v}(\xi_2,\tau_2).$$

Thus, (11) holds only if

$$\Big\| \frac{\langle\xi\rangle^s \xi_2^2}{\langle\sigma\rangle^{1-b} \langle\xi_1\rangle^s \langle\sigma_1\rangle^{b'} \langle\xi_2\rangle^s \langle\sigma_2\rangle^{b'}} \Big\|_{[3,R\times R]} \leq C.$$

In order to bound I, we will split the domain of integration into several pieces as follows. By symmetry, it suffices to estimate the integral in the domain: $|\xi_1| \leq |\xi_2|$.

Let $\widehat{F}_\rho^j(\xi,\tau) = \frac{f_j(\xi,\tau)}{(1+|\tau-\phi(\xi)|)^\rho}$, $j=1,2$.

Case 1 $|\xi| \leq 4a$.

Case 1.1 $|\xi_1| \leq 2a$. It follows that $|\xi_2| \leq 6a$. Consequently,

$$\begin{aligned}
I &= \int_* \frac{\chi_{|\xi|\leq 4a} \langle\xi\rangle^s \overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}} \frac{\chi_{|\xi_1|\leq 2a} f_1(\xi_1,\tau_1)}{\langle\xi_1\rangle^s \langle\sigma_1\rangle^{b'}} \frac{\chi_{|\xi_2|\leq 6a} \xi_2^2 f_2(\xi_2,\tau_2)}{\langle\xi_2\rangle^r \langle\sigma_2\rangle^{b'}} \mathrm{d}\delta \\
&\leq C \int_* \frac{\chi_{|\xi|\leq 4a} \overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}} \frac{\chi_{|\xi_1|\leq 2a} f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}} \frac{\chi_{|\xi_2|\leq 6a} f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}} \mathrm{d}\delta \\
&\leq C \int \overline{F}_{1-b} \cdot F_{b'}^1 \cdot F_{b'}^2(x,t) \mathrm{d}x\mathrm{d}t \leq C\|F_{1-b}\|_{L_x^2 L_t^2} \|F_{b'}^1\|_{L_x^4 L_t^4} \|F_{b'}^2\|_{L_x^4 L_t^4} \\
&\leq C\|f\|_{L_\xi^2 L_\tau^2} \|f_1\|_{L_\xi^2 L_\tau^2} \|f_2\|_{L_\xi^2 L_\tau^2}
\end{aligned}$$

which follows by Lemmas 5 and 7.

Case 1.2 $|\xi_1| \geq 2a$. By the symmetry assumption, we obtain $|\xi_2| \geq 2a$ and $|\xi_1| \sim |\xi_2|$. Thus for $s \geq \frac{5}{8}$, we get

$$\begin{aligned}
I &= \int_* \frac{\chi_{|\xi|\leq 4a} \langle\xi\rangle^s \overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}} \frac{\chi_{|\xi_1|\geq 2a} f_1(\xi_1,\tau_1)}{\langle\xi_1\rangle^s \langle\sigma_1\rangle^{b'}} \frac{\chi_{|\xi_2|\geq 2a} \xi_2^2 f_2(\xi_2,\tau_2)}{\langle\xi_2\rangle^s \langle\sigma_2\rangle^{b'}} \mathrm{d}\delta \\
&\leq C \int_* \frac{\chi_{|\xi|\leq 4a} \overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}} \frac{\chi_{|\xi_1|\geq 2a} |\xi_1|^{1-s} f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}} \frac{\chi_{|\xi_2|\geq 2a} |\xi_2|^{1-s} f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}} \mathrm{d}\delta \\
&\leq C \int_* \frac{\chi_{|\xi|\leq 4a} \overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}} \frac{\chi_{|\xi_1|\geq 2a} |\xi_1|^{\frac{3}{8}} f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}} \frac{\chi_{|\xi_2|\geq 2a} |\xi_2|^{\frac{3}{8}} f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}} \mathrm{d}\delta \\
&= C \int P_{4a} \overline{F}_{1-b} \cdot D_x^{\frac{3}{8}} P^{2a} F_{b'}^1 \cdot D_x^{\frac{3}{8}} P^{2a} F_{b'}^2(x,t) \mathrm{d}x\mathrm{d}t \\
&\leq C\|F_{1-b}\|_{L_x^2 L_t^2} \|D_x^{\frac{3}{8}} P^{2a} F_{b'}^1\|_{L_x^4 L_t^4} \|D_x^{\frac{3}{8}} P^{2a} F_{b'}^2\|_{L_x^4 L_t^4} \\
&\leq C\|f\|_{L_\xi^2 L_\tau^2} \|f_1\|_{L_\xi^2 L_\tau^2} \|f_2\|_{L_\xi^2 L_\tau^2}
\end{aligned}$$

which follows also by Lemmas 5 and 7.

Case 2 $|\xi| \geq 4a$.

Case 2.1 $|\xi_1| \leq 2a$. We deduce that $|\xi_2| \geq 2a$ and $|\xi| < 2|\xi_2|$. Therefore,

$$I = \int_* \frac{\chi_{|\xi|\geq 4a} \langle\xi\rangle^s \overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}} \frac{\chi_{|\xi_1|\leq 2a} f_1(\xi_1,\tau_1)}{\langle\xi_1\rangle^s \langle\sigma_1\rangle^{b'}} \frac{\chi_{|\xi_2|\geq 2a} \xi_2^2 f_2(\xi_2,\tau_2)}{\langle\xi_2\rangle^s \langle\sigma_2\rangle^{b'}} \mathrm{d}\delta$$

$$
\begin{aligned}
&\le C\int_* \frac{\chi_{|\xi|\ge 4a}\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}\frac{\chi_{|\xi_1|\le 2a}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\ge 2a}|\xi_2|^2 f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}\mathrm{d}\delta\\
&\le C\int P^{4a}\overline{F}_{1-b}\cdot P_{2a}F^1_{b'}\cdot D_x^2P^{2a}F^2_{b'}(x,t)\mathrm{d}x\mathrm{d}t\\
&\le C\|F_{1-b}\|_{L^2_xL^2_t}\|P_{2a}F^1_{b'}\|_{L^2_xL^\infty_t}\|D^2_xP^{2a}F^2_{b'}\|_{L^\infty_xL^2_t}\\
&\le C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}
\end{aligned}
$$

where Lemmas 4–7 have been used.

Case 2.2 $|\xi_1|\ge 2a$. In this case, we have $2a\le|\xi_1|\le|\xi_2|$. According to Lemma 9, when $|\xi|>4a$, one of the following cases always occurs:

$$
(a)\ |\sigma|\ge|\xi|^3|\xi_1||\xi_2|,\quad (b)\ |\sigma_1|\ge|\xi|^3|\xi_1||\xi_2|,\quad (c)\ |\sigma_2|\ge|\xi|^3|\xi_1||\xi_2|.
$$

According to this, we subdivide case 2.2 into the following cases:

Case 2.2.1 (a) occurs. For $b-s+1\le\frac{3}{8}$ and $s\le 3(1-b)$, we obtain

$$
\begin{aligned}
I&=\int_*\frac{\chi_{|\xi|\ge 4a}\langle\xi\rangle^s\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}\frac{\chi_{|\xi_1|\ge 2a}f_1(\xi_1,\tau_1)}{\langle\xi_1\rangle^s\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\ge 2a}\xi_2^2f_2(\xi_2,\tau_2)}{\langle\xi_2\rangle^s\langle\sigma_2\rangle^{b'}}\mathrm{d}\delta\\
&\le\int_*\frac{\chi_{|\xi|\ge 4a}\langle\xi\rangle^s\overline{f}(\xi,\tau)}{\langle|\xi|^3|\xi_1||\xi_2|\rangle^{1-b}}\frac{\chi_{|\xi_1|\ge 2a}f_1(\xi_1,\tau_1)}{\langle\xi_1\rangle^s\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\ge 2a}\xi_2^2f_2(\xi_2,\tau_2)}{\langle\xi_2\rangle^s\langle\sigma_2\rangle^{b'}}\mathrm{d}\delta\\
&\le C\int_*\chi_{|\xi|\ge 4a}|\xi|^{s-3+3b}\overline{f}(\xi,\tau)\frac{\chi_{|\xi_1|\ge 2a}|\xi_1|^{b-s-1}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\ge 2a}|\xi_2|^{b-s+1}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}\mathrm{d}\delta\\
&\le C\int_*\chi_{|\xi|\ge 4a}\overline{f}(\xi,\tau)\frac{\chi_{|\xi_1|\ge 2a}|\xi_1|^{\frac{3}{8}}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{\chi_{|\xi_2|\ge 2a}|\xi_2|^{\frac{3}{8}}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}\mathrm{d}\delta\\
&\le C\int\overline{F}_0\cdot D_x^{\frac{3}{8}}P^{2a}F^1_{b'}\cdot D_x^{\frac{3}{8}}P^{2a}F^2_{b'}(x,t)\mathrm{d}x\mathrm{d}t\\
&\le C\|F_0\|_{L^2_xL^2_t}\|D_x^{\frac{3}{8}}P^{2a}F^1_{b'}\|_{L^4_xL^4_t}\|D_x^{\frac{3}{8}}P^{2a}F^2_{b'}\|_{L^4_xL^4_t}\\
&\le C\|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}.
\end{aligned}
$$

We have used Lemma 7 and Lemma 5 (2) in the above estimate. Here, $b\le\frac{19}{32}$ is required.

This means that if $b-s+1\le\frac{3}{8}$ and $s\le 3(1-b)$,

$$
\left\|\frac{\langle\xi\rangle^s\xi_2^2}{\langle\xi_1\rangle^s\langle\sigma\rangle^{1-b}\langle\xi_2\rangle^s\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b'}}\right\|_{[3,R\times R]}\le C. \tag{13}
$$

By Lemma 8, we deduce that (13) will also hold for $s>\frac{9}{8}$ (not $\frac{5}{8}$, to keep $b\le\frac{1}{2}$). As a matter of fact, since $\xi=\xi_1+\xi_2$, we first obtain $\langle\xi\rangle\le\langle\xi_1\rangle\langle\xi_2\rangle$. Then for any $\frac{19}{32}\ge b>\frac{1}{2}$, $b'\ge\frac{3}{8}$, if $s_1\ge s_2$, we have

$$
\begin{aligned}
m&=\frac{\langle\xi\rangle^{s_1}\xi_2^2}{\langle\xi_1\rangle^{s_1}\langle\sigma\rangle^{1-b}\langle\xi_2\rangle^{s_1}\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b'}}=\frac{\langle\xi\rangle^{s_1}}{\langle\xi_1\rangle^{s_1}\langle\xi_2\rangle^{s_1}}\frac{\xi_2^2}{\langle\sigma\rangle^{1-b}\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b'}}\\
&\le\frac{\langle\xi\rangle^{s_2}}{\langle\xi_1\rangle^{s_2}\langle\xi_2\rangle^{s_2}}\frac{\xi_2^2}{\langle\sigma\rangle^{1-b}\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b'}}=\frac{\langle\xi\rangle^{s_2}\xi_2^2}{\langle\xi_1\rangle^{s_2}\langle\sigma\rangle^{1-b}\langle\xi_2\rangle^{s_2}\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b'}}=M.
\end{aligned}
$$

Case 2.2.2 (b) occurs. For $s-3b'<\frac{3}{8}$, $2-s-b'<\frac{3}{8}$, we get

$$\begin{aligned}
I &= \int_* \frac{\chi_{|\xi|\geq 4a}\langle\xi\rangle^s\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}} \frac{\chi_{|\xi_1|\geq 2a}f_1(\xi_1,\tau_1)}{\langle\xi_1\rangle^s\langle\sigma_1\rangle^{b'}} \frac{\chi_{|\xi_2|\geq 2a}\xi_2^2 f_2(\xi_2,\tau_2)}{\langle\xi_2\rangle^s\langle\sigma_2\rangle^{b'}}\mathrm{d}\delta \\
&\leq \int_* \frac{\chi_{|\xi|\geq 4a}\langle\xi\rangle^s\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}} \frac{\chi_{|\xi_1|\geq 2a}f_1(\xi_1,\tau_1)}{\langle\xi_1\rangle^s(|\xi|^3|\xi_1||\xi_2|)^{b'}} \frac{\chi_{|\xi_2|\geq 2a}\xi_2^2 f_2(\xi_2,\tau_2)}{\langle\xi_2\rangle^s\langle\sigma_2\rangle^{b'}}\mathrm{d}\delta \\
&\leq C\int_* \frac{\chi_{|\xi|\geq 4a}|\xi|^{s-3b'}\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}} \cdot \frac{\chi_{|\xi_1|\geq 2a}f_1(\xi_1,\tau_1)}{|\xi_1|^{s+b'}} \cdot \frac{\chi_{|\xi_2|\geq 2a}|\xi_2|^{2-s-b'} f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}\mathrm{d}\delta \\
&\leq C\int_* \frac{\chi_{|\xi|\geq 4a}|\xi|^{\frac{3}{8}}\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}} \cdot \chi_{|\xi_1|\geq 2a}f_1(\xi_1,\tau_1) \cdot \frac{\chi_{|\xi_2|\geq 2a}|\xi_2|^{2-s-b'} f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}\mathrm{d}\delta \\
&\leq C\int_* \frac{\chi_{|\xi|\geq 4a}|\xi|^{\frac{3}{8}}\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}} \cdot \chi_{|\xi_1|\geq 2a}f_1(\xi_1,\tau_1) \cdot \frac{\chi_{|\xi_2|\geq 2a}|\xi_2|^{\frac{3}{8}} f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}\mathrm{d}\delta \\
&\leq C\|D_x^{\frac{3}{8}}P^{4a}F_{1-b}\|_{L_x^4L_t^4}\|P^{2a}F_0^1\|_{L_x^2L_t^2}\|D_x^{\frac{3}{8}}P^{2a}F_{b'}^2\|_{L_x^4L_t^4} \\
&\leq C\|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2},
\end{aligned}$$

which follows from Lemmas 5 and 7. Here, $b'\geq\frac{5}{16}$ and $b<\frac{5}{8}$ are required.

Similarly to Case 2.2.1, if $s>\frac{9}{8}$, we can also deduce the results from Lemma 8.

Case 2.2.3 (c) occurs. The argument is similar to Case 2.2.2. □

4. Proof of the main result

Let $\theta\in C_0^\infty(R)$ with $\theta\equiv 1$ on $[-\frac{1}{2},\frac{1}{2}]$ and $\operatorname{supp}\theta\subseteq(-1,1)$. Denoting $\theta_\delta(t)=\theta(\frac{t}{\delta})$, then we have

Lemma 10 *Let $s\in R$, $\frac{1}{2}<b<b'\leq 1$, $0<\delta\leq 1$. Then we have*

$$\|\theta_\delta(t)U(t)u_0\|_{X_{s,b}}\leq C\delta^{\frac{(1-2b)}{2}}\|u_0\|_{H^s}, \tag{14}$$

$$\|\theta_\delta(t)\int_0^t U(t-s)F(s)\mathrm{d}s\|_{X_{s,b}}\leq C\delta^{\frac{(1-2b)}{2}}\|F\|_{X_{s,b-1}}, \tag{15}$$

$$\|\theta_\delta(t)F\|_{X_{s,b-1}}\leq C\delta^{b'-b}\|F\|_{X_{s,b'-1}}. \tag{16}$$

Proof of Theorem 1 Let $u_0\in H^s$, $s>\frac{9}{8}$, with $\|u_0\|_{H^s}=r$. Let us define

$$B_r=\{u\in X_{s,b}:\|u\|_{X_{s,b}}\leq 2Cr\}.$$

Then B_r is a Banach space.

For $u\in B_r$, we define the mapping

$$\mathcal{T}(u)=\theta_1(t)U(t)u_0+\lambda\theta_1(t)\int_0^t U(t-s)\theta_\delta(t)[uu_{xx}](s)\mathrm{d}s.$$

We will show that $\mathcal{T}$ maps B_r into B_r and is a contraction.

We deduce from (14), (15), (16) and (12) that there exist some b, b' satisfying $\frac{1}{2}<b<b'\leq\frac{19}{32}$ such that

$$\|\mathcal{T}(u)\|_{X_{s,b}}\leq\|\theta_1(t)U(t)u_0\|_{X_{s,b}}+\left\|\theta_1(t)\int_0^t U(t-s)\theta_\delta(t)[uu_{xx}](s)\mathrm{d}s\right\|_{X_{s,b}}$$

$$\begin{aligned}&\le C\|u_0\|_{H^s}+C\|\theta_\delta(t)uu_{xx}\|_{X_{s,b-1}}\\&\le C\|u_0\|_{H^s}+C\delta^{b'-b}\|uu_{xx}\|_{X_{s,b'-1}}\\&\le C\|u_0\|_{H^s}+C\delta^{b'-b}\|u\|^2_{X_{s,b}}.\end{aligned}$$

Hence, if we take δ small enough, $\mathcal{T}$ maps B_r into B_r.

Using the same argument as above, we obtain

$$\|\mathcal{T}(u)-\mathcal{T}(v)\|_{X_{s,b}}<\frac{1}{2}\|u-v\|_{X_{s,b}}.$$

Thus, $\mathcal{T}$ is contraction.

Finally, by Banach fixed point theorem, we deduce that there is a unique solution u of the map $\mathcal{T}$ on the ball B_r for $0<t\le 1$ satisfying the integral equations:

$$u=U(t)u_0+\lambda\int_0^t U(t-s)[uu_{xx}](s)\mathrm{d}s,$$

which is equivalent to (1). □

Remark For $s>\frac{9}{8}$, and b and b' as in Theorem 2,

$$\|u_xv_{xx}\|_{X_{s,b-1}}\le C\|u\|_{X_{s,b'}}\|v\|_{X_{s,b'}}$$

also holds, which implies the well-posedness of IVP (2).

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