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# A Note on the $w$-Global Transform of Mori Domains

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**Abstract** Let $R$ be a domain and let $R^{wg}$ be the $w$-global transform of $R$. In this note it is shown that if $R$ is a Mori domain, then the $t$-dimension formula $t\text{-}\dim(R^{wg}) = t\text{-}\dim(R) - 1$ holds.

**Keywords** Mori domain; maximal $w$-ideal; the $w$-global transform.

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Throughout this paper $R$ denotes a domain with quotient field $K$. Matijevic in [6] had introduced the notion of the global transform of $R$, which is defined to be the set

$$R^g = \{x \in K \mid M_1 \cdots M_n x \subseteq R, \text{ where } M_i \in \operatorname{Max}(R)\},$$

and shown that if $R$ is Noetherian, then any ring $T$ such that $R \subseteq T \subseteq R^g$ is Noetherian. We have known that Mori domains have the ascending chain condition on divisorial ideals and strong Mori domains have the ascending chain condition on $w$-ideals. Every strong Mori domain is a Mori domain, but a Mori domain is not necessarily a strong Mori domain. Park [7] proved the $w$-analogue of Matijevic result, that is, if $R$ is a strong Mori domain, then any $w$-overring $T$ in the $w$-global transform $R^{wg}$ of $R$ is also a strong Mori domain. In this note we give the relationship of $t$-dimension of $R$ and $R^{wg}$ for a Mori domain $R$.

Let $A$ be a fractional ideal of $R$. Define $A^{-1} = \{x \in K \mid xA \subseteq R\}$ and set $A_v = (A^{-1})^{-1}$. If $A = A_v$, then $A$ is called a $v$-fractional ideal. We also define $A_t = \bigcup B_v$, where $B$ ranges over finitely generated fractional subideal of $A$. If $A_t = A$, then $A$ is called a $t$-fractional ideal. Let $J$ be a finitely generated ideal of $R$. $J$ is called a $GV$-ideal, denoted by $J \in GV(R)$, if $J^{-1} = R$. Define

$$A_w = \{x \in K \mid Jx \subseteq A \text{ for some } J \in GV(R)\}.$$

If $A_w = A$, then $A$ is called a $w$-fractional ideal, equivalently, the condition $x \in K$ and $J \in GV(R)$ with $Jx \subseteq A$ implies $x \in A$. For the discussion on $t$-ideals and $w$-ideals, readers can consult the

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literature [5] and [10]. Let $R \subseteq T$ be an extension of domains. We say that $T$ is an overring of $R$ if $T \subseteq K$. Let $T$ be an overring of $R$. Following [11] and [7], we call $T$ a $w$-overring if $T$ as an $R$-module is a $w$-module.

Let $P$ be a prime $t$-ideal of $R$. We denote by $t\text{-ht}\,P$ the supremum of the lengths $n$ of all chains $0 \subset P_n \subset P_{n-1} \subset \cdots \subset P_1 = P$, where $P_1, \ldots, P_{n-1}, P_n$ are prime $t$-ideals of $R$. Define $t\text{-dim}(R) = \sup\{t\text{-ht}\,P\}$, where $P$ ranges over all prime $t$-ideals of $R$.

Denote by $w\text{-Max}(R)$ the set of maximal $w$-ideals of $R$. Following the notation of Park [7], we denote

$$R^{wg} = \{x \in K \mid P_1 \cdots P_n x \subseteq R \text{ for some } P_1, \ldots, P_n \in w\text{-Max}(R)\}.$$

Then $R^{wg}$ is an overring of $R$ contained in $K$ and is called $w$-global transform of $R$.

Let $B$ be a fractional ideal of $R$. Define similarly the $w$-global transform of $B$ to be the set

$$B^{wg} = \{x \in K \mid P_1 \cdots P_n x \subseteq B \text{ for some } P_1, \ldots, P_n \in w\text{-Max}(R)\}.$$

**Lemma 1** *$R^{wg}$ is a $w$-overring of $R$.*

**Proof** See [7, Corollary 1.7].

**Lemma 2** *(1) Let $B_1$ and $B_2$ be fractional ideals of $R$ with $B_1 \subseteq B_2$. Then $(B_1)^{wg} \subseteq (B_2)^{wg}$.*

*(2) Let $B$ be a fractional ideal of $R$. Then $B^{wg}$ is a fractional ideal of $R^{wg}$.*

*(3) If $B$ is an ideal of $R$, then $B^{wg} = R^{wg}$ if and only if there are $P_1, \ldots, P_n \in w\text{-Max}(R)$ such that $P_1 \cdots P_n \subseteq B$. Therefore, if $Q$ is a prime ideal of $R$, then $Q^{wg} = R^{wg}$ if and only if $P \subseteq Q$ for some $P \in w\text{-Max}(R)$.*

*(4) Let $A$ be an ideal of $R^{wg}$ and let $B = A \bigcap R$. Then $A \subseteq B^{wg}$.*

**Proof** It is straightforward.

**Lemma 3** *(1) Let $Q$ be a prime ideal of $R$ such that $P \nsubseteq Q$ for any $P \in w\text{-Max}(R)$. Then $Q^{wg}$ is a prime ideal of $R^{wg}$ and $Q^{wg} \bigcap R = Q$.*

*(2) Let $Q_1$ and $Q_2$ be prime ideals of $R$ with $P \nsubseteq Q_1, Q_2$ for any $P \in w\text{-Max}(R)$. Then $(Q_1)^{wg} = (Q_2)^{wg}$ if and only if $Q_1 = Q_2$.*

*(3) Let $A$ be a prime ideal of $R^{wg}$ and let $Q = A \bigcap R$. If $P \nsubseteq Q$ for any $P \in w\text{-Max}(R)$, then $A = Q^{wg}$.*

*(4) Let $Q$ be a prime ideal of $R$ such that $P \nsubseteq Q$ for any $P \in w\text{-Max}(R)$. Then $\text{ht}\,Q^{wg} = \text{ht}\,Q$.*

**Proof** (1) By Lemma 2, $Q^{wg} \neq R^{wg}$. Let $x, y \in R^{wg}$ with $xy \in Q^{wg}$. Then there are $P_1, \ldots, P_n, P_{n+1}, \ldots, P_m \in w\text{-Max}(R)$ such that $P_1 \cdots P_n x \subseteq R$, $P_{n+1} \cdots P_m y \subseteq R$ and $P_1 \cdots P_n$ $P_{n+1} \cdots P_m xy \subseteq Q$. Hence $P_1 \cdots P_n x \subseteq Q$ or $P_{n+1}, \ldots, P_m y \subseteq Q$, that is, $x \in Q^{wg}$ or $y \in Q^{wg}$. Then $Q^{wg}$ is a prime ideal of $R^{wg}$.

It is clear that $Q \subseteq Q^{wg} \bigcap R$. Conversely, let $a \in Q^{wg} \bigcap R$. Then $P_1 \cdots P_n a \subseteq Q$ for $P_1, \ldots, P_n \in w\text{-Max}(R)$. Since $P_i \nsubseteq Q$, we have $a \in Q$. Hence $Q = Q^{wg} \bigcap R$.

(2) If $(Q_1)^{wg} = (Q_2)^{wg}$, then $Q_1 = (Q_1)^{wg} \bigcap R = (Q_2)^{wg} \bigcap R = Q_2$.

(3) By Lemma 2, $A \subseteq Q^{wg}$. Let $x \in Q^{wg}$. Then $P_1 \cdots P_n x \subseteq Q \subseteq A$ for some $P_1, \ldots, P_n \in w\text{-Max}(R)$. Because $P_i \not\subseteq A$ and $A$ is prime, we have $x \in A$. Hence $A = Q^{wg}$.

(4) It is clear by (2) that $\operatorname{ht} Q \leqslant \operatorname{ht} Q^{wg}$. Let $A_1 \subset A_2 \subset \cdots \subset A_n \subset Q^{wg}$ be a chain of prime ideals of $R^{wg}$. For each $i$, set $Q_i = A_i \bigcap R$. Then $Q_1 \subset Q_2 \subset \cdots \subset Q_n \subset Q$ is a chain of prime ideals of $R$ by (3). Hence $\operatorname{ht} Q^{wg} = \operatorname{ht} Q$. □

**Lemma 4** *(1) Let $B$ be a fractional ideal of $R$. Then, as fractional ideals of $R^{wg}$, $(B^{-1})^{wg} \subseteq (B^{wg})^{-1} \subseteq (BR^{wg})^{-1}$.*

*(2) Let $B$ be a $t$-finite type fractional ideal of $R$. Then $(B^{-1})^{wg} = (B^{wg})^{-1} = (BR^{wg})^{-1}$.*

*(3) Let $R$ be a Mori domain and let $B$ be a fractional ideal of $R$. Then, as fractional ideals of $R^{wg}$, $(B^{wg})_v = (BR^{wg})_v = (B_v)^{wg}$. Therefore, if $B$ is a $v$-ideal of $R$, then $B^{wg}$ is a $v$-ideal of $R^{wg}$.*

*(4) Let $R$ be a Mori domain and let $A$ be an ideal of $R^{wg}$. Then $A_v = (B_v)^{wg}$, where $B = A \bigcap R$. Therefore, if $A$ is a $v$-ideal of $R^{wg}$, then $B = A \bigcap R$ is a $v$-ideal of $R$ and $A = B^{wg} = (BR^{wg})_v$.*

*(5) Let $R$ be a Mori domain and let $B$ be an ideal of $R$. Then $(B^{wg})^{-1} = R^{wg}$ if and only if there are $P_1, \ldots, P_n \in w\text{-Max}(R)$ such that $P_1 \cdots P_n \subseteq B_v$. Therefore, $(PR^{wg})^{-1} = R^{wg}$ for any $P \in w\text{-Max}(R)$.*

*(6) Let $R$ be a Mori domain and let $A$ be an ideal of $R^{wg}$. Then $A_v = R^{wg}$ if and only if there are $P_1, \ldots, P_n \in w\text{-Max}(R)$ such that $P_1 \cdots P_n \subseteq B_v$, where $B = A \bigcap R$.*

**Proof** (1) Let $x \in (B^{-1})_S$. There are $P_1, \ldots, P_n \in w\text{-Max}(R)$ such that $P_1 \cdots P_n x \subseteq B^{-1}$. For any $y \in B^{wg}$, take $P_{n+1}, \ldots, P_m \in w\text{-Max}(R)$ such that $P_{n+1} \cdots P_m y \subseteq B$. Thus $P_1 \cdots P_m xy \subseteq B^{-1}B \subseteq R$. Hence $xy \in R^{wg}$. Thus $x \in (B^{wg})^{-1}$, whence, $(B^{-1})^{wg} \subseteq (B^{wg})^{-1}$. From $BR^{wg} \subseteq B^{wg}$, we have $(B^{wg})^{-1} \subseteq (BR^{wg})^{-1}$.

(2) It suffices by (1) to show that $(BR^{wg})^{-1} \subseteq (B^{-1})^{wg}$. Let $x \in (BR^{wg})^{-1}$. Since $B$ is of $t$-finite type, there is a finitely generated fractional subideal $J$ of $B$ such that $B_v = J_v$, therefore, $J^{-1} = B^{-1}$. Because $xJ \subseteq xB \subseteq R^{wg}$ and $J$ is finitely generated, there are $P_1, \ldots, P_n \in w\text{-Max}(R)$ such that $P_1 \cdots P_n Jx \subseteq R$. Then $P_1 \cdots P_n x \in J^{-1} = B^{-1}$. Hence $x \in (B^{-1})^{wg}$. Thus we have $(BR^{wg})^{-1} \subseteq (B^{-1})^{wg}$.

(3) This follows from (2) since $B^{-1}$ is also of $t$-finite type in a Mori domain.

(4) Since $BR^{wg} \subseteq A \subseteq B^{wg}$ by Lemma 2 (4), we have $(BR^{wg})_v \subseteq A_v \subseteq (B^{wg})_v$. Hence $A = (B_v)^{wg}$ by (3).

Suppose $A$ is a $v$-ideal of $R^{wg}$. Since $BR^{wg} \subseteq A \subseteq B^{wg}$, we have $(BR^{wg})_v \subseteq A \subseteq (B_v)^{wg}$. Hence $A = (BR^{wg})_v = (B_v)^{wg}$. Then $B_v \subseteq A \bigcap R = B$, that is, $B = B_v$. Hence $A = B^{wg} = (BR^{wg})_v$.

(5) From (3), $(B^{wg})^{-1} = R^{wg}$ if and only if $(B_v)^{wg} = R^{wg}$, if and only if there are $P_1, \ldots, P_n \in w\text{-Max}(R)$ such that $P_1 \cdots P_n \subseteq B_v$ by Lemma 2.

(6) It is direct from (4) and (5). □

**Proposition 5** *Let $R$ be a Mori domain and let $A$ be a $w$-ideal of $R^{wg}$. Then $B = A \bigcap R$ is a*

*$w$-ideal of $R$ and $A = B^{wg} = (BR^{wg})_w$.*

**Proof** By [11, Lemma 3.1], $B$ is a $w$-ideal of $R$. Since $B \subseteq A$, we have $BR^{wg} \subseteq A \subseteq B^{wg}$. Hence $(BR^{wg})_w \subseteq A \subseteq B^{wg}$. Let $x \in B^{wg}$. Then there are $P_1, \ldots, P_n \in w\text{-Max}(R)$ such that $P_1 \cdots P_n x \subseteq B$. Let $I_i$ be a finitely generated subideal of $P_i$ such that $P_i = (I_i)_v$ for $i = 1, \ldots, n$. Thus $I_1 \cdots I_n x \subseteq B$. By Lemma 4, $I_i R^{wg} \in GV(R^{wg})$. Then $x \in (BR^{wg})_w$, and hence $A = (BR^{wg})_w = B^{wg}$. □

**Proposition 6** *(1) Let $R$ be a Mori domain. Then $R^{wg}$ is also a Mori domain.*

*(2) Let $R$ be a strong Mori domain. Then $R^{wg}$ is also a strong Mori domain.*

**Proof** (1) It follows from Lemma 4. Also see [8, Théorème 2].

(2) It follows from Proposition 5. Also see [7, Theorem 1.5 & Corollary 1.7]. □

**Theorem 7** *Let $R$ be a Mori domain. Let $A$ be a maximal $v$-ideal of $R^{wg}$ and set $B = A \bigcap R$. Then, for any $P \in w\text{-Max}(R)$, $P \nsubseteq B$, and $B$ is a maximal prime $v$-subideal of $P$ for any maximal $v$-ideal $P$ of $R$ with $B \subseteq P$.*

**Proof** For any $P \in w\text{-Max}(R)$, then $P$ is a $v$-ideal because $R$ is a H-domain by [5]. Write $P = J_v$, where $J$ is a finitely generated subideal of $P$. By Lemma 4(6), $JR^{wg} \in GV(R^{wg})$. Hence $P \nsubseteq B$.

By Lemma 4, $B$ is a prime $v$-ideal of $R$ and $A = B^{wg}$. Let $P$ be a maximal $w$-ideal of $R$ with $B \subseteq P$ and let $Q$ be a prime $v$-ideal of $R$ with $B \subseteq Q \subseteq P$. If $Q \neq P$, then $Q^{wg}$ is a prime $v$-ideal of $R^{wg}$ by Lemma 3 and Lemma 4. Hence $A = Q^{wg}$ by the maximality of $A$. Then $B = Q$ by Lemma 3 again. □

**Theorem 8** *Let $R$ be a Mori domain (but not a field). Then $t\text{-}\dim(R^{wg}) = t\text{-}\dim(R) - 1$.*

**Proof** Let $A_n \subset A_{n-1} \subset \cdots \subset A_1 \subset A_0$ be a chain of prime $v$-ideals of $R^{wg}$. Set $B_i = A_i \bigcap R$ for $i = 0, 1, \ldots, n$. Then $B_i$ is a prime $v$-ideal of $R$ by Lemmas 3 and 4, and $B_n \subset B_{n-1} \subset \cdots \subset B_1 \subset B_0$ be a chain of prime $v$-ideals of $R$. By Theorem 7, $B_0$ is not a maximal $t$-ideal of $R$. Hence $t\text{-dim}(R^{wg}) \leqslant t\text{-}\dim(R) - 1$. Conversely, let $B_n \subset B_{n-1} \subset \cdots \subset B_1 \subset B_0$ be a chain of prime $v$-ideals of $R$ such that $B_0$ is not maximal $v$-ideal of $R$. By Lemma 3, $B_n^{wg} \subset B_{n-1}^{wg} \subset \cdots \subset B_1^{wg} \subset B_0^{wg}$ is a chain of prime $v$-ideals of $R^{wg}$. Hence $t\text{-dim}(R^{wg}) \geqslant t\text{-}\dim(R) - 1$. □

**Corollary 9** *Let $R$ be a Mori domain. If $t\text{-}\dim(R) = 1$, then $R^{wg} = K$.*

**Proof** Since $t\text{-}\dim(R) = 1$, we have $t\text{-}\dim(R^{wg}) = 0$ by Theorem 8. Hence $R^{wg}$ is a field, that is, $R^{wg} = K$. □

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