Journal of Mathematical Research & Exposition
Nov., 2010, Vol. 30, No. 6, pp. 1039–1047
DOI:10.3770/j.issn:1000-341X.2010.06.012
Http://jmre.dlut.edu.cn

Existence and Exponential Stability of Almost Periodic Solution for BAM Neural Networks with Impulse

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Abstract This paper discusses a class of the bidirectional associative memories (BAM) type neural networks with impulse. By using the Banach fixed point theory and some analysis technology, we obtain the existence of almost periodic solution and stability under some sufficient conditions.

Keywords BAM neural networks; almost periodic solution; impulse; stability; Banach fixed point theory.

Document code A
MR(2000) Subject Classification 34K13; 34D23
Chinese Library Classification O175

1. Introduction

BAM neural networks have been extensively studied in past years and some important results on the stability, periodicity and almost periodicity have been reported [1–4]. In application, almost periodic oscillatory is more accordant with fact. Up to now, a few authors have considered the existence and stability of almost periodic solution for neural networks [5–8]. However, we only see that existence and exponential stability of almost periodic solution for neural networks with impulse were discussed [9, 10].

In this paper, we study the impulsive BAM-type neural networks with almost periodic coefficients, obtain some sufficient conditions ensuring existence, uniqueness and global exponential stability of almost periodic solution.

Consider the following BAM neural networks with impulse

$$\begin{cases} x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^m p_{ji}(t)f_j(y_j(t)) + c_i(t),\ t>0,\ t\neq t_k,\ k=1,2,\ldots \\ \Delta x_i(t_k) = \alpha_{ik}x_i(t_k) + I_{ik}(x_i(t_k)) + c_{ik},\ i=1,2,\ldots,n, \\ y_j'(t) = -b_j(t)y_j(t) + \sum_{i=1}^n q_{ij}(t)g_i(x_i(t)) + d_j(t),\ t>0,\ t\neq t_k,\ k=1,2,\ldots \\ \Delta y_j(t_k) = \beta_{jk}y_j(t_k) + J_{jk}(y_j(t_k)) + d_{jk},\ j=1,2,\ldots,m, \end{cases} \tag{1.1}$$

where $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$, $\Delta y_j(t_k) = y_j(t_k^+) - y_j(t_k^-)$ are impulses at moment t_k and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to+\infty} t_k = +\infty$. $x_i(t)$ and $y_j(t)$ are

Received June 25, 2008; Accepted March 16, 2009
Supported by the National Natural Science Foundation of China (Grant No. 11061031).
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the activations of the ith neuron and the jth neuron, respectively. $p_{ji}(t)$, $q_{ij}(t)$ are the connection weights at the time t, and $c_i(t)$, $d_j(t)$ denote the external inputs at the time t. $f_j(\cdot)$, $g_i(\cdot)$ are the signal functions of neurons.

Let $z(t) = z(t, t_0, z_0)$, $z = (x,\ y)^{\mathrm{T}} = (x_1, \ldots, x_n, y_1, \ldots, y_m)^{\mathrm{T}}$, $z_0 = (x_{01}, \ldots, x_{0n}, y_{01}, \ldots,\ y_{0m})^{\mathrm{T}} \in \Omega$, where Ω is a domain in R^{n+m}, $\Omega \neq \emptyset$. The system (1.1) is supplemented with initial values given by

$$z(t_0 + 0, t_0, z_0) = z(t_0). \tag{1.2}$$

Denote by $PC(J, R^{n+m})$, $J \subset R$, the space of all piecewise continuous function $z : J \to R^{n+m}$ with discontinuous points of the first kind t_k, $k = 1, 2, \ldots$ and which are continuous from the left, i.e., $z(t_k^-) = z(t_k)$.

2. Preliminaries

Let $B = \{\{t_k\} : t_k \in R, t_k < t_{k+1}, \lim_{k\to\pm\infty} t_k = \pm\infty, k = 1, 2, \ldots\}$ denote the set of all unbounded and strictly increasing sequences.

Definition 1 ([11]) *The set of sequences $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $\{t_k\} \in B$, $k = 1, 2, \ldots$ is said to be uniformly almost periodic if for arbitrary $\epsilon > 0$ there exists relatively dense set of ϵ-almost periods common for any sequences.*

Definition 2 ([11]) *A piecewise continuous function $\varphi : R \to R^n$ with discontinuity of first kind at the points t_k is said to be almost periodic, if*

(a) the set of sequence $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $\{t_k\} \in B$, $k = 1, 2, \ldots$ is uniformly almost periodic.

(b) for any $\epsilon > 0$ there exists a real number $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuity of $\varphi(t)$ and satisfy the inequality $|t' - t''| < \delta$, then $|\varphi(t') - \varphi(t'')| < \epsilon$.

(c) for any $\epsilon > 0$ there exists a relatively dense set T such that if $\tau \in T$, then $|\varphi(t+\tau)-\varphi(t)| < \epsilon$ for a $t \in R$ satisfying the condition $|t - t_k| > \epsilon$, $k = 1, 2, \ldots$.

Together with the system (1.1), we consider the linear system

$$\begin{cases} z' = A(t)z(t) + f(t, z), & t \neq t_k \\ \Delta z = B_k z + I_k(z), & t = t_k \end{cases} \tag{2.1}$$

Introduce the following conditions:

(i) $A(t) \in C(R, R^{n+m})$ and is almost periodic.

(ii) $\det(E+B_k) \neq 0$ and the sequence $\{B_k\}$, $k = 1, 2, \ldots$ is almost periodic, $E \in R^{(n+m)\times(n+m)}$.

(iii) The set of sequences $\{t_k^j\}$, $t_k^j = t_{k+j} - t_k$, $\{t_k\} \in B$, $k = 1, 2, \ldots$ is uniformly almost periodic and there exists $\theta > 0$ such that $\inf_k t_k^1 = \theta$.

Lemma 1 ([12]) *Let $z(t)$ be any solution of (2.1) on $[t_0, \infty)$ and let $W(t,\ s)$ be the fundamental* *matrix solution of*

$$\begin{cases} z' = A(t)z(t), & t \neq t_k; \\ \Delta z = B_k z, & t = t_k. \end{cases}$$

Then $z(t)$ satisfies the integral equation for $t > t_0$

$$z(t) = W(t, t_0^+)z_0 + \int_{t_0}^{t} W(t,s)f(s,z(s))\mathrm{d}s + \sum_{t_0<t_k<t} W(t,t_k^+)I_k(z(t_k))$$

where

$$W(t,s) = \begin{cases} U_k(t,s) \quad \text{for} \;\; t,s \in (t_{k-1}, t_k]; \\ U_{k+1}(t,t_k)(E+B_k)U_k(t_k,s) \quad \text{for} \;\; t_{k-1} < s \leq t_k < t \leq t_{k+1}; \\ U_{k+1}(t,\; t_k)\prod_{j=k}^{i+1}(E+B_k)U_j(t_j,t_{j+1})U_i(t_i,s) \quad \text{for} \;\; t_{i-1} < s \leq t_i < \cdots \leq t_k < t \leq t_{k+1}. \end{cases}$$

$U_k(t,s)$ is the fundamental matrix for the system $z' = A(t)z$, $t \in (t_{k-1}, t_k]$.

Lemma 2 ([11]) *Let the condition (iii) be fulfilled. Then for each $p > 0$ there exists a positive integer N such that on each interval of length p, there are no more than N elements of the sequence $\{t_k\}$, i.e., $i(t,s) \leq N(t-s) + N$, where $i(t,s)$ is the number of the points t_k lying in the interval (s,t).*

Lemma 3 ([11]) *In addition to condition (iii), let the following conditions be fulfilled:*

(iv) The function $\varphi \in PC(R, \Omega)$, $\Omega \subset R^{n+m}$ and it is almost periodic.

Then the sequence $\{\varphi(t_k)\}$ is almost periodic.

Lemma 4 ([11]) *In addition to condition (iii) and (iv), let the following conditions be fulfilled:*

(v) $F(y)$ is uniformly continuous in Ω.

Then $F(\varphi(t))$ is almost periodic function.

Lemma 5 ([11]) *Let $g(t) \in PC(R, \Omega)$ and the sequence $\{g_k\}$, $k = 1, 2, \ldots$ be almost periodic. Then there exists a positive constant C_1 such that*

$$\max\{\sup_{t\in R}\|g(t)\|, \sup_{k=1,2,\ldots}\|g_k\|\} \leq C_1.$$

Throughout this paper, we introduce the following conditions:

(H_1) $a_i(t)$, $b_j(t)$, $p_{ji}(t)$, $q_{ij}(t)$, $c_i(t)$, $d_j(t)$ are almost periodic functions, and denote

$$\underline{a}_i = \inf_{t\in R}\{a_i(t)\}, \quad \underline{b}_j = \inf_{t\in R}\{b_j(t)\}, \quad \overline{p}_{ji} = \sup_{t\in R}\{p_{ji}(t)\},$$

$$\overline{q}_{ij} = \sup_{t\in R}\{q_{ij}(t)\}, \quad \overline{c}_i = \sup_{t\in R}\{c_i(t)\}, \quad \overline{d}_j = \sup_{t\in R}\{d_j(t)\}.$$

(H_2) The condition (iii) holds.

(H_3) The function f_j, g_i are uniformly continuous functions defined in Ω with

$$0 < \sup_{t\in R}\{f_j(y_j(t)),\; g_i(x_i(t))\} < +\infty, \quad f_j(0) = g_i(0) = 0$$

and there exist positive constants L_j^f, L_i^g such that for u, $v \in \Omega$,

$$|f_j(u) - f_j(v)| < L_j^f|u-v|, \;\; |g_i(u) - g_i(v)| < L_i^g|u-v|.$$

(H_4) $\{c_i(t)\}$, $\{c_{ik}\}$, $\{d_j(t)\}$, $\{d_{jk}\}$ are almost periodic sequences and from Lemma 5, there exist strictly positive constants M_0, M_1, M_2, where $M_0 = \max\{M_1, M_2\}$, such that

$$\max\{\max_i\{|c_i(t)|\}, \max_{i,k}\{|c_{ik}|\}\} \le M_1, \quad \max\{\max_j\{|d_j(t)|\}, \max_{j,k}\{|d_{jk}|\}\} \le M_2.$$

(H_5) The sequences of functions $I_{ik}(x_i(t_k))$, $J_{jk}(y_j(t_k))$ are almost periodic uniformly with respect to $x \in \Omega$ and there exists positive constants L_i^I, L_j^J such that for u, $v \in \Omega$, $k = 1, 2, \ldots$,

$$|I_{ik}(u) - I_{ik}(v)| \le L_i^I |u - v|, \ |J_{jk}(u) - J_{jk}(v)| \le L_j^J |u - v|.$$

Lemma 6 ([10]) *In addition to conditions (H_1), (H_2), (H_4) are fulfilled.*

(vi) For the fundamental matrix $W(t, s)$ of the system (2.1), there exist positive constants K and λ such that

$$|W(t, s)| \le Ke^{-\lambda(t-s)}, \quad t \ge s, t, s \in R,$$

$$K = \max\{\max_i \xi_i, \max_j \xi_j\}, \quad \lambda = \min\{\min_i \lambda_i, \min_j \lambda_j\},$$

$$\xi_i = \exp\{N \ln(1 + \max_k\{|\alpha_{ik}|\})\}, \quad \xi_j = \exp\{N \ln(1 + \max_k\{|\beta_{jk}|\})\},$$

$$\lambda_i = \underline{a}_i - N \ln(1 + \max_k\{|\alpha_{ik}|\}), \quad \lambda_j = \underline{b}_j - N \ln(1 + \max_k\{|\beta_{jk}|\}).$$

Then for any $\epsilon > 0$, t, $s \in R$, $t \ge s$, $|t - t_k| > \epsilon$, $|s - t_k| > \epsilon$, $k = 1, 2, \ldots$ there exists a relatively dense set T of ϵ-almost periods of the matrix $P(t)$ and a positive constant Γ such that for $\tau \in T$ it follows

$$|W(t + \tau, s + \tau) - W(t, s)| \le \epsilon\Gamma e^{\frac{\lambda}{2}(t-s)}.$$

Lemma 7 ([11]) *Assume the condition (H_1–H_4) hold. Then for each $\epsilon > 0$ there exist ϵ_1, $0 < \epsilon_1 < \epsilon$ and relatively dense sets T of real numbers and Q of whole numbers, such that the following relations are fulfilled:*

(a) $|A(t + \tau) - A(t)| < \epsilon$, $t \in R$, $\tau \in T$, $|t - t_k| > \epsilon$;

(b) $|B_{k+q} - B_k| < \epsilon$, $q \in Q$, $k = 1, 2, \ldots$;

(c) $|c_i(t + \tau) - c_i(t)| < \epsilon$, $|d_j(t + \tau) - d_j(t)| < \epsilon$, $t \in R$, $\tau \in T$, $|t - t_k| > \epsilon$, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$;

(d) $|c_{ik+q} - c_{ik}| < \epsilon$, $|d_{jk+q} - d_{jk}| < \epsilon$, $q \in Q$, $k = 1, 2, \ldots$; $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$.

3. Main results

Theorem 1 *In addition to (H_1)–(H_5), further assume that*

(H_6) $r = \max\{r_1, r_2\} < 1$, $r_1 = \max_i \lambda_i^{-1} \xi_i \sum_{j=1}^m \overline{p}_{ji} L_j^f + \max_i (1 - e^{-\lambda_i})^{-1} \xi_i L_i^I$,

$$r_2 = \max_j \lambda_j^{-1} \xi_j \sum_{i=1}^n \overline{q}_{ij} L_i^g + \max_j (1 - e^{-\lambda_j})^{-1} \xi_j L_j^J.$$

Then the system (1.1) has a unique almost periodic solution $z^*(t)$.

Proof Let

$$D = z(t) = (\varphi(t), \psi(t))^{\mathrm{T}} = (\varphi_1(t), \ldots, \varphi_n(t), \psi_1(t), \ldots, \psi_m(t))^{\mathrm{T}} \in PC(R, R^{n+m})$$

be almost periodic with $\|z\| \le \bar{K}$, where

$$\|z\| = \sup_{t\in R} \max\{\max_{1\le i\le n}\{|\varphi_i(t)|\}, \max_{1\le j\le m}\{|\psi_j(t)|\}\}, \quad \bar{K} = KM_0(\frac{1}{\lambda} + \frac{1}{1-e^{-\lambda}}).$$

Obviously, $D \in PC(R, R^{n+m})$. Set

$$H(t, z(t)) = (F_1(t, y(t)), \ldots, F_n(t, y(t)), \quad G_1(t, x(t)), \ldots, G_m(t, x(t)))^{\mathrm{T}},$$

$$F_i(t, y(t)) = \sum_{j=1}^{m} p_{ji}(t) f_j(y_j(t)), \quad G_j(t, x(t)) = \sum_{i=1}^{n} q_{ij}(t) g_i(x_i(t)),$$

$$H_k(t, z(t_k)) = (I_{1k}(x_1(t_k)), \ldots, I_{1k}(x_n(t_k)), \quad J_{1k}(y_1(t_k)), \ldots, J_{mk}(y_m(t_k)))^{\mathrm{T}},$$

$$z(t) = (x(t), y(t))^{\mathrm{T}} = (x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_m(t))^{\mathrm{T}},$$

$$C(t) = (c_1(t), \ldots, c_m(t), d_1(t), \ldots, d_m(t))^{\mathrm{T}},$$

$$C_k = (c_{1k}, \ldots, c_{mk}, d_{1k}, \ldots, d_{mk})^{\mathrm{T}}.$$

Define a mapping Φ in D

$$\Phi z = \int_{-\infty}^{t} W(t,s)[H(t, z(t)) + C(s)]\mathrm{d}s + \sum_{t<t_k} W(t, t_k)[H_k(t, z(t)) + C_k]. \tag{3.1}$$

From Lemma 1, it is easy to check that Φz is a solution of (1.1). Take subset $D^* \subset D$, $D^* = \{z \in D \mid \|z - z_0\| < \frac{rK}{1-r}\}$, where

$$z_0 = \int_{-\infty}^{t} W(t,s)C(s)\mathrm{d}s + \sum_{t<t_k} W(t, t_k)C_k. \tag{3.2}$$

From (3.2), it follows from Lemma 6 that

$$\begin{aligned}
\|z_0\| =& \sup_{t\in R} \max\{\max_{1\le i\le n} \int_{-\infty}^{t} |W(t,s)||c_i(s)|\mathrm{d}s + \max_{1\le i\le n} \sum_{t<t_k} |W(t,\ t_k)||c_{ik}|, \\
& \max_{1\le j\le m} \int_{-\infty}^{t} |W(t,s)||d_j(s)|\mathrm{d}s + \max_{1\le j\le m} \sum_{t<t_k} |W(t, t_k)||d_{jk}|\} \\
\le& \sup_{t\in R} \max\{\max_{1\le i\le n} \int_{-\infty}^{t} Ke^{-\lambda(t-s)}|c_i(s)|\mathrm{d}s + \max_{1\le i\le n} \sum_{t<t_k} Ke^{-\lambda(t-t_k)}|c_{ik}|, \\
& \max_{1\le j\le m} \int_{-\infty}^{t} Ke^{-\lambda(t-s)}d_j(s)|\mathrm{d}s + \max_{1\le j\le m} \sum_{t<t_k} Ke^{-\lambda(t-t_k)}|d_{jk}|\} \\
\le& \max\{\frac{M_1}{\lambda}K + \frac{M_1}{1-e^{-\lambda}}, \frac{M_2}{\lambda}K + \frac{M_2}{1-e^{-\lambda}}\} \\
\le& \frac{M_0}{\lambda}K + \frac{M_0}{1-e^{-\lambda}} := \bar{K}.
\end{aligned} \tag{3.3}$$

Then for arbitrary $z \in D^*$, it follows from (3.1)–(3.3) that

$$\|z\| \le \|z - z_0\| + \|z_0\| \le \frac{r}{1-r}\bar{K} + \bar{K} = \frac{1}{1-r}\bar{K}. \tag{3.4}$$

Now we prove that Φ is self-mapping from $D^* \to D^*$. In fact,

$$\|\Phi z - z_0\| = \sup_{t\in R} \max\{\max_{i} |\Phi\varphi_i - \Phi\varphi_{0i}|, \max_{j} |\Phi\psi_j - \Phi\psi_{0j}|\}, \quad \forall z \in D^*.$$

We have

$$
\begin{aligned}
&\sup_{t\in R}\max_i|\Phi\varphi_i-\Phi\varphi_{0i}|\\
&=\sup_{t\in R}\{\max_i\int_{-\infty}^{t}|W(t,s)||\sum_{j=1}^{m}p_{ji}(s)f_j(\psi_j(s))|\mathrm{d}s+\max_i\sum_{t<t_k}|W(t,t_k)||I_{ik}(\varphi_i(t_k))|\}\\
&\le\sup_{t\in R}\{\max_i\int_{-\infty}^{t}\xi_ie^{-\lambda_i(t-s)}|\sum_{j=1}^{m}p_{ji}(s)f_j(\psi_j(s))|\mathrm{d}s+\max_i\sum_{t<t_k}\xi_ie^{-\lambda_i(t-t_k)}|I_{ik}(\varphi_i(t_k))|\}\\
&\le\sup_{t\in R}\{\max_i\int_{-\infty}^{t}\xi_ie^{-\lambda_i(t-s)}|\sum_{j=1}^{m}\bar p_{ji}L_j^f|\psi_j(s)||\mathrm{d}s+\max_i\sum_{t<t_k}\xi_ie^{-\lambda_i(t-t_k)}L_i^I|\varphi_i(t_k)|\}\\
&\le\{\max_i\lambda_i^{-1}\xi_i\sum_{j=1}^{m}\bar p_{ji}L_j^f+\max_i(1-e^{-\lambda_i})^{-1}\xi_iL_i^I\}\|z\|\\
&=r_1\|z\|.
\end{aligned}\tag{3.5}
$$

Similarly to (3.5),

$$\sup_{t\in R}\max_j|\Phi\psi_j-\Phi\psi_{0j}|\le[\max_j\lambda_j^{-1}\xi_j\sum_{i=1}^{n}\overline{q}_{ij}L_i^g+\max_j(1-e^{-\lambda_j})^{-1}\xi_jL_j^J]\|z\|=r_2\|z\|.$$

Hence

$$\|\Phi z-z_0\|=\max\{r_1\|z\|,r_2\|z\|\}=r\|z\|\le\frac{r}{1-r}\bar K.$$

Secondly, we shall prove that Φz is almost periodic. In fact, let $\tau\in T$, $q\in Q$, where sets T and Q are determined in Lemma 7. By Lemmas 6 and 7, we have

$$\|\Phi z(t+\tau)-\Phi z(t)\|=\sup_{t\in R}\max\{\max_i|\Phi\varphi_i(t+\tau)-\Phi\varphi_i(t)|,\max_j|\Phi\psi_j(t+\tau)-\Phi\psi_j(t)|\},$$

where

$$
\begin{aligned}
&\sup_{t\in R}\max_i|\Phi\varphi_i(t+\tau)-\Phi\varphi_i(t)|\\
&\le\sup_{t\in R}\{\int_{-\infty}^{t}|W(t+\tau,s+\tau)-W(t,s)|\max_i[|\sum_{j=1}^{m}p_{ji}(s+\tau)f_j(\psi_j(s+\tau))+c_i(s+\tau)|]\mathrm{d}s+\\
&\quad\int_{-\infty}^{t}|W(t,s)|\max_i[|\sum_{j=1}^{m}p_{ji}(s+\tau)f_j(\psi_j(t+\tau))-\sum_{j=1}^{m}p_{ji}(s)f_j(\psi_j(s))|+|c_i(s+\tau)-c_i(s)|]\mathrm{d}s+\\
&\quad\sum_{t<t_k}|W(t+\tau,t_{k+q})-W(t,t_k)|\max_i[|I_{ik+q}(\varphi_i(t_{k+q}))|+|c_{ik+q}|]+\\
&\quad\sum_{t<t_k}|W(t,\ t_k)|\max_i[|I_{ik+q}(\varphi_i(t_{k+q}))-I_{ik}(\varphi_i(t_k))|+|c_{ik+q}-c_{ik}|]\}\le\epsilon\bar M_1,
\end{aligned}\tag{3.6}
$$

where

$$\bar M_1=\frac{1}{\lambda}\{\max_i[\sum_{j=1}^{m}((2\Gamma+K)\bar p_{ji}+K)L_j^f]+K\}+\frac{\Gamma N}{1-e^{-\lambda}}(\max_i L_i^I+1).$$

Similarly to (3.6)

$$\sup_{t\in R}\max_j|\Phi\psi_j(t+\tau)-\Phi\psi_j(t)|\le\frac{1}{\lambda}\{\max_j[\sum_{i=1}^{n}((2\Gamma+K)\bar q_{ij}+K)L_i^g]+K\}+\frac{\Gamma N}{1-e^{-\lambda}}(\max_j L_j^J+1)$$

$$= \epsilon \bar{M}_2.$$

Hence,

$$\|\Phi z(t+\tau) - \Phi z(t)\| = \max\{\epsilon \bar{M}_1, \epsilon \bar{M}_2\} = \epsilon \bar{M}, \tag{3.7}$$

where $\bar{M} = \max\{\bar{M}_1, \bar{M}_2\}$. It follows from (3.4) and (3.7) that $\Phi z \in D^*$, and hence the mapping Φ is a self-mapping. For arbitrary $z_1, z_2 \in D^*$, we note

$$z_1 = (\varphi_1^1, \ldots, \varphi_n^1, \psi_1^1, \ldots, \psi_m^1)^{\mathrm{T}}, \quad z_2 = (\varphi_1^2, \ldots, \varphi_n^2, \psi_1^2, \ldots, \psi_m^2)^{\mathrm{T}}.$$

We have

$$\|\Phi z_1 - \Phi z_2\| = \max\{\max_i |\Phi\varphi_i^1 - \Phi\varphi_i^2|, \max_j |\Phi\psi_j^1 - \Phi\psi_j^2|\},$$

where

$$\begin{aligned}
\sup_{t\in R}\max_i |\Phi\varphi_i^1 - \Phi\varphi_i^2| \leq & \sup_{t\in R}\{\max_i \int_{-\infty}^{t} \xi_i e^{-\lambda_i(t-s)} \sum_{j=1}^{m} |p_{ji}(s)||(f_j(\psi_j^1(s)) - f_j(\psi_j^2(s)))|\mathrm{d}s + \\
& \max_i \sum_{t<t_k} \xi_i e^{-\lambda_i(t-t_k)} |I_{ik}(\varphi_i^1(t_k)) - I_{ik}(\varphi_i^2(t_k))|\} \\
\leq & [\max_i \lambda_i^{-1} \xi_i \sum_{j=1}^{m} \overline{p}_{ji} L_j^f + \max_i (1 - e^{-\lambda_i})^{-1} \xi_i L_i^I] \|z_1 - z_2\| \\
= & r_1 \|z_1 - z_2\|.
\end{aligned} \tag{3.8}$$

Similarly to (3.8),

$$\begin{aligned}
\sup_{t\in R}\max_j |\Phi\psi_j^1 - \Phi\psi_j^2| &\leq [\max_j \lambda_j^{-1} \xi_j \sum_{i=1}^{n} \overline{q}_{ij} L_i^g + \max_j (1 - e^{-\lambda_j})^{-1} \xi_j L_j^J] \|z_1 - z_2\| \\
&= r_2 \|z_1 - z_2\|.
\end{aligned}$$

Hence, it follows from (H_6), that

$$\|\Phi z_1 - \Phi z_2\| = \max\{r_1\|z_1 - z_2\|, r_2\|z_1 - z_2\|\} = r\|z_1 - z_2\|. \tag{3.9}$$

Then from (3.9), it follows that Φ is a contraction mapping in D^*. Thus, by applying the Banach fixed point Theorem, the mapping Φ has a unique fixed point $z^* \in D^*$, such that $\Phi z^* = z^*$. So there exists a unique almost periodic solution z^* of (1.1). The proof is completed. □

Theorem 2 *In addition to (H_1)–(H_5), further assume that*

(H_7) $\lambda - \max\{\max_i \xi_i \sum_{j=1}^{m} \overline{p}_{ji} L_j^f, \max_j \xi_j \sum_{i=1}^{n} \overline{q}_{ij} L_i^g\} + \ln(1 + \max\{\max_i \xi_i L_i^I, \max_j \xi_j L_j^J\}) > 0$.

Then the solution $z^*(t)$ is exponentially stable.

Proof Let $z(t) = (x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_m(t))^{\mathrm{T}}$ be arbitrary solution of (1.1) with the initial condition (1.2), and $z^*(t) = (x^*(t), y^*(t))^{\mathrm{T}}$ be the unique almost periodic solution of (1.1) with the initial condition $z^*(t_0 + 0, t_0, z_0^*) = z_0^*$. Then from Lemma 1, we have

$$z(t) - z^*(t) = W(t, t_0)(z - z_0^*) + \int_{t_0}^{t} W(t, s)[H(s, z(s)) - H(s, z^*(s))]\mathrm{d}s +$$

$$\sum_{t_0<t_k<t} W(t,t_k)[H_k(t,z(t_k)) - H_k(t,z^*(t_k))],$$

$$\|z(t) - z^*(t)\| = \sup_{t\in R} \max\{\max_i |x_i(t) - x_i^*(t)|, \max_j |y_j(t) - y_j^*(t)|\},$$

where

$$\begin{aligned}
&\sup_{t\in R}\max_i |x_i(t) - x_i^*(t)| \\
&\quad \le Ke^{-\lambda}(t-t_0)\sup_{t\in R}\max_i |x_{0i} - x_{0i}^*| + \\
&\qquad \sup_{t\in R}\{\max_i \int_{t_0}^t \xi_i e^{-\lambda_i(t-s)}[\sum_{j=1}^m |p_{ji}(s)||f_j(y_j(s)) - f_j(y_j^*(s))|]\mathrm{d}s + \\
&\qquad\qquad \max_i \sum_{t_0<t_k<t} \xi_i e^{-\lambda_i(t-t_k)}|I_{ik}(x_i(t_k)) - I_{ik}(x_i^*(t_k))|\} \\
&\quad \le Ke^{-\lambda}(t-t_0)\|z_0 - z_0^*\| + \max_i \xi_i \sum_{j=1}^m \overline{p}_{ji}L_j^f \int_{t_0}^t e^{-\lambda(t-s)}\|z(s) - z^*(s)\|\mathrm{d}s + \\
&\qquad\qquad \max_i \xi_i L_i^I \sum_{t_0<t_k<t} e^{-\lambda(t-t_k)}\|z(t_k) - z^*(t_k)\|.
\end{aligned} \tag{3.10}$$

Similarly to (3.10)

$$\begin{aligned}
&\sup_{t\in R}\max_j |y_j(t) - y_j^*(t)| \\
&\quad \le Ke^{-\lambda}(t-t_0)\|z_0 - z_0^*\| + \max_j \xi_j \sum_{i=1}^n \overline{q}_{ij}L_i^g \int_{t_0}^t e^{-\lambda(t-s)}\|z(s) - z^*(s)\|\mathrm{d}s + \\
&\qquad \max_j \xi_j L_j^J \sum_{t_0<t_k<t} e^{-\lambda(t-t_k)}\|z(t_k) - z^*(t_k)\|.
\end{aligned}$$

Hence, From Gronwall-Bellman's Lemma [13], we have

$$\begin{aligned}
\|z(t) - z^*(t)\| \le& K\|z_0 - z_0^*\|(1 + \max\{\max_i \xi_i L_i^I, \max_j \xi_j L_j^J\})^{i(t_0,t)} \times \\
&\exp\{[-\lambda + \max\{\max_i \xi_i \sum_{j=1}^m \overline{p}_{ji}L_j^f, \max_j \xi_j \sum_{i=1}^n \overline{q}_{ij}L_i^g\}](t-t_0)\}.
\end{aligned}$$

By Lemma 2, then one has

$$\begin{aligned}
\|z(t) - z^*(t)\| \le& K\exp\{N(1 + \max\{\max_i \xi_i L_i^I,\ \ \max_j \xi_j L_j^J\})\|z_0 - z_0^*\|\} \times \\
&\exp\{-[\lambda - \max\{\max_i \xi_i \sum_{j=1}^m \overline{p}_{ji}L_j^f, \max_j \xi_j \sum_{i=1}^n \overline{q}_{ij}L_i^g\} + \\
&\ln(1 + \max\{\max_i \xi_i L_i^I,\ \ \max_j \xi_j L_j^J\})](t-t_0)\}.
\end{aligned}$$

From the assumption (H_7), the solution $z^*(t)$ is exponentially stable. The proof is completed. □

4. An illustrative example

Consider the following simple BAM neural networks with impulse

$$\begin{cases} x'(t) = -(9 - \cos t)x(t) + \sin t\ f(y(t)) + \frac{1}{2}\cos t, \quad t \neq t_k,\ k = 1, 2, \ldots \\ \Delta x_i(t_k) = (-1)^{k+1}\frac{1}{2}\exp(\cos(x(t_k))), \\ y'(t) = -(11 + \sin t)y(t) + \cos t\ g(x(t)) + \sin t,\ t \neq t_k, \quad k = 1, 2, \ldots \\ \Delta y(t_k) = (-1)^k \frac{1}{2}\exp(\sin(y(t_k))), \end{cases} \tag{4.1}$$

where $f(u) = g(u) = \frac{1}{2}(|u+1| - |u-1|)$, $u \in \Omega$. We have $L^f = L^g = 1$, and get

$$r_1 = \lambda_1^{-1}\xi_1\overline{p}L^f + (1 - e^{-\lambda_1})^{-1}\xi_1 L^I = 0.6337032,$$

$$r_2 = \lambda_2^{-1}\xi_2\overline{q}L^g + (1 - e^{-\lambda_2})^{-1}\xi_2 L^J = 0.700227.$$

Hence, $r = \max\{r_1,\ r_2\} < 1$. Moreover,

$$\lambda - \max\{\xi_1\overline{p}L^f,\ \xi_2\overline{q}L^g\} + \ln(1 + \max\{\xi_1 L^I, \xi_2 L^J\}) = 7 + \ln 1.5 > 0.$$

Thus, it follows from Theorems 1 and 2 that the almost periodic solution (4.1) is exponentially stable.

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