Journal of Mathematical Research & Exposition
Nov., 2010, Vol. 30, No. 6, pp. 1048–1054
DOI:10.3770/j.issn:1000-341X.2010.06.013
Http://jmre.dlut.edu.cn

On Radicals of Ideals of Ordered Semigroups

Jian TANG[1,*], **Xiang Yun XIE**[2]
1. *School of Mathematics and Computational Science, Fuyang Normal College, Anhui* 236041*, P. R. China;*
2. *School of Mathematics and Computational Science, Wuyi University, Guangdong* 529020*, P. R. China*

Abstract Let S be an ordered semigroup. In this paper, we characterize ordered semigroups in which the radical of every ideal (right ideal, bi-ideal) is an ordered subsemigroup (resp., ideal, right ideal, left ideal, bi-ideal, interior ideal) by using some binary relations on S.

Keywords ordered semigroup; ideal; archimedean (r-archimedean, t-archimedean) ordered semigroup; semilattice; radical of an ideal.

Document code A
MR(2000) Subject Classification 20M10; 06F05
Chinese Library Classification O152.7

1. Introduction

As we know, many familiar properties of the radical for rings and semigroups are also true for ordered semigroups. For example, if S is a commutative ordered semigroup, then the radical $\sqrt{I}$ of an ideal I of S is the intersection of all prime ideals containing it [7], which generalize the Hoo and Shum's prime radical theorem of an ordered semigroup [2]. In [3], Cao defined some binary relations on an ordered semigroup S and gave some necessary and sufficient conditions in order that an ordered semigroup is a semilattice of archimedean ordered subsemigroups. Xie [4] characterized ordered semigroups which is a band of weakly r-archimedean ordered subsemigroups of it. In this paper, we characterize ordered semigroups in which the radical of every ideal (right ideal, bi-ideal) is an ordered subsemigroup (resp., ideal, right ideal, left ideal, bi-ideal, interior ideal) by using some binary relations defined in [3]. As an application of some results of this paper, the corresponding results in semigroups -without order- can also be obtained by moderate modifications.

Received September 1, 2008; Accepted January 5, 2009
Supported by the National Natural Science Foundation of China (Grant No. 10961014), the Natural Science Foundation of Guangdong Province (Grant No. 0501332), the Excellent Youth Talent Foundation of Anhui Province (Grant No. 2009SQRZ149) and the Youth Foundation of Fuyang Normal College (Grant No. 2008LQ11).
* Corresponding author
E-mail address: tangjian0901@126.com (J. TANG)

2. Notations and preliminaries

Throughout this paper, we denote by Z^+ the set of all positive integers. Recall that an ordered semigroup S is a semigroup S with an order relation " $\leq$ " such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$. For $\emptyset \neq H \subseteq S$, let

$$(H] := \{t \in S | \ (\exists h \in H) \ \ t \leq h\}.$$

Lemma 2.1 ([5]) *For an ordered semigroup S, we have*

(1) $A \subseteq (A] \ \forall A \subseteq S$;

(2) If $A \subseteq B \subseteq S$, *then* $(A] \subseteq (B]$;

(3) $(A](B] \subseteq (AB] \ \forall A, B \in S$;

(4) $((A]] = (A] \ \forall A \subseteq S$;

(5) For every left (resp. right) ideal T of S, we have $(T] = T$;

(6) $(SaS]$, $(aS]$ *are an ideal and a right deal of S, $\forall a \in S$, respectively.*

By the radical of the subset A of an ordered semigroup S we mean a set $\sqrt{A}$ defined by

$$\sqrt{A} := \{x \in S | \ (\exists m \in Z^+) \ \ x^m \in A\}.$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. Then an element 0 of S is called zero element if $(\forall a \in S)$ $0 \leq a$ and $0 \cdot a = a \cdot 0 = 0$. If S is an ordered semigroup with the zero element 0, then an element a of S is a nilpotent if there exists $n \in Z^+$ such that $a^n = 0$, and we denote by $\mathrm{Nil}(S)$ the set of all nilpotents of S.

Definition 2.2 ([5]) *Let S be an ordered semigroup and I a nonempty subset of S. I is called a right ideal of S if*

(1) $IS \subseteq I$, *and*

(2) If $a \in I$, $b \leq a$ *with* $b \in S$, *then* $b \in I$.

Left ideals can be defined dually. If I is both a left ideal and a right ideal of S, then I is called an ideal of S.

Definition 2.3 ([6]) *Let S, T be two ordered semigroups. A mapping $f: \ S \to T$ is called isotone if $x, y \in S$, $x \leq y$ implies $f(x) \leq f(y)$ in T. f is called a homomorphism if it is isotone and satisfies that $f(xy) = f(x)f(y)$ for all $x, y \in S$.*

Definition 2.4 ([1]) *Let ρ be a congruence on an ordered semigroup S. Then ρ is called regular if there exists an order " $\preceq$ " on S/ρ such that:*

(1) $(S/\rho, \cdot, \preceq)$ *is an ordered semigroup (where " $\cdot$ " is the usual multiplication on S/ρ defined by* $(x)_\rho \cdot (y)_\rho := (xy)_\rho$);

(2) The mapping

$$\varphi : \ S \to S/\rho \ \ \text{with} \ \ x \to (x)_\rho$$

is isotone (Then φ is a homomorphism).

Lemma 2.5 ([1]) *Let $(S, \cdot, \leq)$ be an ordered semigroup and I an ideal of S. Then the following statements are true:*

(1) $\rho_I := (I \times I) \cup \{(x,y) \in S \times S \mid x = y\}$ *is a regular congruence of* S.

(2) $S/\rho_I = \{\{x\} \mid x \in S \setminus I\} \cup \{I\}$.

Remark It is easy to show that the element I of S/ρ_I is the zero element of the ordered semigroup S/ρ_I. Thus we may write S/ρ_I as $(S \setminus I) \cup \{0\}$.

Definition 2.6 ([3]) *An ordered semigroup* S *is called archimedean (r-archimedean, t-archimedean) if for any* $a, b \in S$, *there exists* $m \in Z^+$ *such that* $b^m \in (S^1aS^1]$ ($b^m \in (aS^1], b^m \in (aS^1a]$). *Equivalently, for any* $a, b \in S$, *there exists* $m \in Z^+$ *such that* $b^m \leq xay$ ($b^m \leq ax, b^m \leq axa$) *for some* $x, y \in S^1$. *An ordered subsemigroup* T *of* S *is called archimedean (r-archimedean, t-archimedean) if the ordered semigroup* $(T, \cdot, \leq)$ *is archimedean (r-archimedean, t-archimedean).*

In this paper, the following binary relations on S defined in [3] will be used frequently:

1) $a\tau b \Leftrightarrow (\exists x, y \in S^1)\ b \leq xay$; $a\tau_r b \Leftrightarrow (\exists y \in S^1)\ b \leq ay$;

2) $a\eta b \Leftrightarrow (\exists m \in Z^+)\ (\exists x, y \in S^1)\ b^m \leq xay$; $a\eta_r b \Leftrightarrow (\exists m \in Z^+)\ (\exists y \in S^1)\ b^m \leq ay$;

3) $a\eta_t b \Leftrightarrow (\exists m \in Z^+)\ (\exists x \in S^1)\ b^m \leq axa$.

To prove the main results of this paper, we need the following three lemmas obtained in [3]:

Lemma 2.7 *Let* S *be an ordered semigroup. Then the following statements are equivalent:*

(1) S *is a semilattice of archimedean ordered subsemigroups;*

(2) For every $a, b \in S$, $a\tau b$ *implies* $a^2\tau b^m$ *for some* $m \in Z^+$;

(3) $(\forall a, b \in S)\ (\exists n \in Z^+)\ (ab)^n \in (S^1a^2S^1]$, *i.e.,* $a\eta(ab)$;

(4) The radical of every ideal of S *is an ideal of* S.

Lemma 2.8 *Let* S *be an ordered semigroup. Then the following statements are equivalent:*

(1) S *is a semilattice of r-archimedean ordered subsemigroups;*

(2) $(\forall a, b \in S)\ b\eta_r(ab)$.

Let S be an ordered semigroup. S is called weakly commutative if

$$(\forall a, b \in S)\ (\exists n \in Z^+)\ (ab)^n \in (bSa].$$

Lemma 2.9 *Let* S *be an ordered semigroup. Then the following statements are equivalent:*

(1) S *is a semilattice of t-archimedean ordered subsemigroups;*

(2) S *is weakly commutative.*

The reader is referred to [8–10] for notation and terminology not defined in this paper.

3. Main results

Theorem 3.1 *Let* S *be an ordered semigroup. Then the following statements are equivalent:*

(1) The radical of every ideal of S *is an ordered subsemigroup of* S;

(2) The set of all nilpotent elements of every homomorphic image with zero of S *forms an ordered subsemigroup;*

(3) $(\forall a, b \in S)\ (\forall k, l \in Z^+)\ a^k\eta(ab)$ *or* $b^l\eta(ab)$.

Proof (1) $\Rightarrow$ (3). Let $a, b \in S$, $k, l \in Z^+$. Since $A = (S^1\{a^k, b^l\}S^1]$ is an ideal of S and $a, b \in \sqrt{A}$, we have $ab \in \sqrt{A}$ by (1). Hence, there exists $n \in Z^+$ such that $(ab)^n \in A = (S^1\{a^k, b^l\}S^1]$. Thus, $a^k\eta(ab)$ or $b^l\eta(ab)$.

(3) $\Rightarrow$ (2). Let T be an ordered semigroup with zero element such that T is a homomorphic image of S. Then we can prove easily that the condition (3) also holds in T. For any $a, b \in \mathrm{Nil}(T)$ there exist $k, l \in Z^+$ such that $a^k = b^l = 0$, and thus $(ab)^n \in (T^1\{a^k, b^l\}T^1] = (T^1\{0, 0\}T^1] = (0]$ for some $n \in Z^+$, i.e., $(ab)^n = 0$. Therefore, $\mathrm{Nil}(T)$ is an ordered subsemigroup of T, as required.

(2) $\Rightarrow$ (1). Let A be an ideal of S. Then by Lemma 2.5 the mapping $\varphi : S \to S/\rho_A$ is a homomorphism of S onto S/ρ_A. Let $a, b \in \sqrt{A}$. Then $a^k, b^l \in A$ for some $k, l \in Z^+$, and so $\varphi(a^k) = \varphi(b^l) = 0$, i.e., $[\varphi(a)]^k = [\varphi(b)]^l = 0$, that is to say that $\varphi(a), \varphi(b) \in \mathrm{Nil}(S/\rho_A)$. Then by (2) we have $\varphi(ab) = \varphi(a)\varphi(b) \in \mathrm{Nil}(S/\rho_A)$ and so $\varphi[(ab)^n] = [\varphi(ab)]^n = 0$ for some $n \in Z^+$, i.e., $(ab)^n \in A$. Hence $ab \in \sqrt{A}$ and $\sqrt{A}$ is an ordered subsemigroup of S.

In a similar way as in the above theorem we can prove the following theorem:

Theorem 3.2 *$\sqrt{R}$ is an ordered subsemigroup of S for every right ideal R of S if and only if*

$$(\forall a, b \in S)\ (\forall k, l \in Z^+)\ a^k\eta_r(ab) \ \text{ or } \ b^l\eta_r(ab).$$

Theorem 3.3 *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) S is a semilattice of archimedean ordered subsemigroups;
(2) For every $a, b \in S$, $a\tau b$ implies $a^2\eta b$;
(3) $\sqrt{(SaS]}$ is an ideal of S, for all $a \in S$;
(4) The set of all nilpotent elements of every homomorphic image with zero of S is an ideal.

Proof (2) $\Rightarrow$ (1). It follows from Lemma 2.7.

(1) $\Rightarrow$ (3). Since $(SaS]$ is an ideal of S for all $a \in S$, that $\sqrt{(SaS]}$ is an ideal of S follows from Lemma 2.7.

(3) $\Rightarrow$ (4). Let T be an ordered semigroup with zero element such that T is a homomorphic image of S. Then we can easily show, by the condition (3), that $\sqrt{(TbT]}$ is an ideal of T for all $b \in T$. For any $a \in \mathrm{Nil}(T)$ and $b \in T$, then there exists $m \in Z^+$ such that $a^m = 0$, where "0" is a zero element of T. Clearly, $a \in \sqrt{(Ta^mT]}$ and so we have $ab \in \sqrt{(Ta^mT]}$. Then there exists $n \in Z^+$ such that $(ab)^n \in (Ta^mT] = (T\{0\}T] = (0]$, that is, $(ab)^n = 0$. Hence, $ab \in \mathrm{Nil}(T)$. Similarly, $ba \in \mathrm{Nil}(T)$. If $b \le a \in \mathrm{Nil}(T)$, then $b^k \le a^k = \{0\}$ which implies $b^k = 0$. Thus, $b \in \mathrm{Nil}(T)$. We have thus shown that $\mathrm{Nil}(T)$ is an ideal of T.

(4) $\Rightarrow$ (2). For any $a, b \in S$, let $a\tau b$ and $A = (S^1a^2S^1]$. Then by Lemma 2.5 the mapping $\varphi : S \to S/\rho_A$ is a homomorphism of S onto S/ρ_A. Clearly, there exists $n \in Z^+$ such that $a^n \in A$, and so we have $[\varphi(a)]^n = \varphi(a^n) = 0$, which implies $\varphi(a) \in \mathrm{Nil}(S/\rho_A)$. From $a\tau b$ we have $b \le xay$ for some $x, y \in S^1$. By assumption, $\mathrm{Nil}(S/\rho_A)$ is an ideal of S/ρ_A, and so we have

$$\varphi(b) \le \varphi(xay) = \varphi(x)\varphi(a)\varphi(y) \in \mathrm{Nil}(S/\rho_A).$$

Thus, $\varphi(b) \in \mathrm{Nil}(S/\rho_A)$. It follows that $b^m \in A = (S^1a^2S^1]$ for some $m \in Z^+$. Therefore, $a^2\eta b$, as required.

Theorem 3.4 *The radical of every right ideal of an ordered semigroup S is a bi-ideal of S if* *and only if*

$$(*) \quad (\forall a,b,c \in S)\ (\forall k,l \in Z^+)\ a^k \eta_r(abc) \text{ or } c^l \eta_r(abc).$$

Proof For any $a,b,c \in S$ and $k,l \in Z^+$, let $R = (\{a^k, c^l\}S^1]$. Then R is a right ideal of S. Since $a,c \in \sqrt{R}$ and $\sqrt{R}$ is a bi-ideal of S, we have $(abc)^n \in R = (\{a^k, c^l\}S^1]$. Thus, $a^k\eta_r(abc)$ or $c^l\eta_r(abc)$.

Conversely, let R be a right ideal of S. Let $a,c \in \sqrt{R}$ and $b \in S$. Then there exist $k,l \in Z^+$ such that $a^k, c^l \in R$. Now by $(*)$ we have that

$$(abc)^n \in (\{a^k, c^l\}S^1] \subseteq (RS^1] \subseteq (R] = R,$$

for some $n \in Z^+$. Hence, $abc \in \sqrt{R}$. If $b \le a \in \sqrt{R}$, then $b^k \le a^k \in R$ which implies $b^k \in R$. Thus, $b \in \sqrt{R}$. Therefore, $\sqrt{R}$ is a bi-ideal of S.

In a similar way as in the proof of Theorem 3.4 we can prove the next theorem

Theorem 3.5 *The radical of every right ideal of an ordered semigroup S is an interior ideal of S if and only if*

$$(\forall a,b,c \in S)\ (\forall k \in Z^+)\ b^k \eta_r(abc).$$

Theorem 3.6 *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) S is a semilattice of r-archimedean ordered subsemigroups;

(2) $(\forall a,b \in S)\ (\forall k \in Z^+)\ b^k\eta_r(ab)$;

(3) The radical of every right ideal of S is a left ideal of S.

Proof (1) $\Rightarrow$ (2). Let S be a semilattice Y of r-archimedean ordered subsemigroups S_α $(\alpha \in Y)$ of S. Then for any $a \in S_\alpha, b \in S_\beta$ $(\alpha, \beta \in Y)$ we have that $ab, b^k a \in S_{\alpha\beta}$ for all $k \in Z^+$, and so there exists $n \in Z^+$ such that $(ab)^n \in (b^k a S^1_{\alpha\beta}] \subseteq (b^k S^1]$. Thus, $b^k\eta_r(ab)$.

(2) $\Rightarrow$ (1). It follows from Lemma 2.8.

(2) $\Rightarrow$ (3). Let R be a right ideal of S. Assume that $a \in S, b \in \sqrt{R}$. Then $b^k \in R$ for some $k \in Z^+$, so by (2) $(ab)^n \in (b^k S^1] \subseteq (RS^1] \subseteq (R] = R$ for some $n \in Z^+$. Thus, $ab \in \sqrt{R}$. If $a \le b \in \sqrt{R}$, then $a^k \le b^k \in R$ which implies $a^k \in R$. Hence $a \in \sqrt{R}$, and so $\sqrt{R}$ is a left ideal of S.

(3) $\Rightarrow$ (2). For any $a,b \in S$ and $k \in Z^+$, let $R = (b^k S^1]$. Then R is a right ideal of S and $b \in \sqrt{R}$. Since $\sqrt{R}$ is a left ideal of S, we have that $ab \in \sqrt{R}$, i.e., there exists $n \in Z^+$ such that $(ab)^n \in R = (b^k S^1]$. Thus, $b^k\eta_r(ab)$, as required.

Lemma 3.7 *Let S be weakly commutative ordered semigroup. Then we have*

$$(\forall a,b \in S)\ (\forall k \in Z^+)\ (\exists m,n \in Z^+)\ (ab)^m, (ba)^n \in (a^k bSba^k].$$

Proof We shall prove the assertion by induction on k. If $k = 1$ it is true. Indeed: Let $a,b \in S$. Since $(ab)^2 = (aba)b$ and S is weakly commutative, and so there exists $m \in Z^+$ such that $(ab)^{2m} = ((aba)b]^m \in (bS(aba)] \subseteq (Sba]$, we thus have $(ab)^{2m+1} = (ab)(ab)^{2m} \in (ab](Sba] \subseteq (abSba]$. Similarly, we have $(\exists n \in Z^+)\ (ba)^{2n+1} \in (abSba]$. Assume that the assertion is true for less than k. We claim that

$$(\forall a, b \in S)\ (\exists m, n \in Z^{+})\ (ab)^{m}, (ba)^{n} \in (a^{k}bSba^{k}].$$

In fact: By hypothesis, there exist $h, l \in Z^{+}$ such that $(ab)^{h}, (ba)^{l} \in (a^{k-1}bSba^{k-1}]$. Then we have

$$(ab)^{l+1} = a(ba)^{l}b \in (a](a^{k-1}bSba^{k-1}](b] \subseteq (a^{k}bS] \tag{1}$$

and

$$(ba)^{h+1} = b(ab)^{h}a \in (b](a^{k-1}bSba^{k-1}](a] \subseteq (Sba^{k}], \tag{2}$$

i.e., $(ab)^{l+1} \leq a^{k}bx$ and $(ba)^{h+1} \leq yba^{k}$ for some $x, y \in S$. So $(ab)^{h+2} = a(ba)^{h+1}b \leq (ayba^{k})b$ and $(ba)^{l+2} = b(ab)^{l+1}a \leq b(a^{k}bxa)$. Since S is weakly commutative, we have

$$(ab)^{h_1(h+2)} \leq [(ayba^{k})b]^{h_1} \in (bSayba^{k}] \subseteq (Sba^{k}]$$

and

$$(ba)^{l_1(l+2)} \leq [b(a^{k}bxa]^{l_1} \in (a^{k}bxaSb] \subseteq (a^{k}bS]$$

for some $h_1, l_1 \in Z^{+}$. Since $(Sba^{k}]$ is a left ideal of S and $(a^{k}bS]$ is a right ideal of S, respectively, we have

$$(ab)^{h_1(h+2)} \in (Sba^{k}] \text{ and } (ba)^{l_1(l+2)} \in (a^{k}bS]. \tag{3}$$

It follows from (1), (2) and (3) that

$$(ab)^{m} = (ab)^{(l+1)}(ab)^{h_1(h+2)} \in (a^{k}bS](Sba^{k}] \subseteq (a^{k}bSba^{k}]$$

and

$$(ba)^{n} = (ba)^{l_1(l+2)}(ba)^{(h+1)} \in (a^{k}bS](Sba^{k}] \subseteq (a^{k}bSba^{k}],$$

where $m = (l+1) + h_1(h+2), n = l_1(l+2) + (h+1) \in Z^{+}$. Hence we complete the proof. □

Theorem 3.8 *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) S is a semilattice of t-archimedean ordered subsemigroups;

(2) $(\forall a, b \in S)\ (\exists n \in Z^{+})\ (ab)^{n} \in (bSa]$;

(3) The radical of every bi-ideal of S is an ideal of S.

Proof (1) ⇔ (2). It follows from Lemma 2.9.

(2) ⇒ (3). Let B be a bi-ideal of S and let $a \in \sqrt{B}, b \in S$. Then $a^{k} \in B$ for some $k \in Z^{+}$, so by (2) and Lemma 3.7 we have

$$(ab)^{m}, (ba)^{n} \in (a^{k}bSba^{k}] \subseteq (BSB] \subseteq (B] = B,$$

for some $m, n \in Z^{+}$. Thus $ab, ba \in \sqrt{B}$. If $b \leq a \in \sqrt{B}$, then $b^{k} \leq a^{k} \in B$ which implies $b^{k} \in B$. Hence $b \in \sqrt{B}$ and $\sqrt{B}$ is an ideal of S.

(3) ⇒ (2). Let $a, b \in S$. Assume that $A = (aSa]$ and $B = (bSb]$. Clearly, A and B are two bi-ideals of S. It is easy to see that $a \in \sqrt{A}$ and $b \in \sqrt{B}$. By (3), $\sqrt{A}$ and $\sqrt{B}$ are two ideals of S, and we have $ab \in \sqrt{A} \cap \sqrt{B}$, i.e., there exist $m, n \in Z^{+}$ such that $(ab)^{m} \in A = (aSa]$ and $(ab)^{n} \in B = (bSb]$. Thus we have that $(ab)^{n+m} \in (bSb](aSa] \subseteq (bSbaSa] \subseteq (bSa]$, as required.

Theorem 3.9 *The radical of every ideal of an ordered semigroup S is a bi-ideal of S if and only if*

$$(\forall a,b,c \in S)\ (\forall k,l \in Z^{+})\ a^{k}\eta(abc) \ \text{ or } \ c^{l}\eta(abc).$$

Proof For any $a,b,c \in S$ and $k,l \in Z^{+}$, let $A = (S^{1}\{a^{k},c^{l}\}S^{1}]$. Clearly, $a,c \in \sqrt{A}$ and $\sqrt{A}$ is a bi-ideal of S. Hence $abc \in \sqrt{A}S\sqrt{A} \subseteq \sqrt{A}$, and so there exists $n \in Z^{+}$ such that $(abc)^{n} \in A = (S^{1}\{a^{k},c^{l}\}S^{1}]$. Thus, $a^{k}\eta(abc)$ or $c^{l}\eta(abc)$.

Conversely, let A be an ideal of S. For any $a,c \in \sqrt{A}, b \in S$, then there exist $k,l \in Z^{+}$ such that $a^{k},c^{l} \in A$. Thus we have

$$(abc)^{n} \in (S^{1}\{a^{k},c^{l}\}S^{1}] \subseteq (S^{1}AS^{1}] \subseteq (A] = A,$$

for some $n \in Z^{+}$. Hence, $abc \in \sqrt{A}$. If $b \leq a \in \sqrt{A}$, then $b^{k} \leq a^{k} \in A$ which implies $b^{k} \in A$. Therefore, $b \in \sqrt{A}$ and $\sqrt{A}$ is a bi-ideal of S.

The proofs of the following three theorems follow the same line as the proof for Theorem 3.9, and we omit them.

Theorem 3.10 *The radical of every ideal of an ordered semigroup S is an interior ideal of S if and only if*

$$(\forall a,b,c \in S)\ (\forall k \in Z^{+})\ b^{k}\eta(abc).$$

Theorem 3.11 *The radical of every bi-ideal of an ordered semigroup S is a bi-ideal of S if and only if*

$$(\forall a,b,c \in S)\ (\forall k,l \in Z^{+})\ (\exists n \in Z^{+})\ (abc)^{n} \in (\{a^{k},c^{l}\}S\{a^{k},c^{l}\}].$$

Theorem 3.12 *The radical of every bi-ideal of an ordered semigroup S is an ordered subsemigroup of S if and only if*

$$(\forall a,b \in S)\ (\forall k,l \in Z^{+})\ (\exists n \in Z^{+})\ (ab)^{n} \in (\{a^{k},b^{l}\}S\{a^{k},b^{l}\}].$$

References

[1] XIE Xiangyun. *On regular, strongly regular congruences on ordered semigroups* [J]. Semigroup Forum, 2000, **61**(2): 159–178.

[2] SHUM K P, HOO C S. *Prime radical theorem on ordered semigroups* [J]. Semigroup Forum, 1980, **19**(1): 87–94.

[3] CAO Yonglin. *On weak commutativity of po-semigroups and their semilattice decompositions* [J]. Semigroup Forum, 1999, **58**(3): 386–394.

[4] XIE Xiangyun. *Bands of weakly r-Archimedean ordered semigroups* [J]. Semigroup Forum, 2001, **63**(2): 180–190.

[5] KEHAYOPULU N. *On weakly prime ideals of ordered semigroups* [J]. Math. Japon., 1990, **35**(6): 1051–1056.

[6] XIE Xiangyun, WU Mingfen. *On congruences on ordered semigroups* [J]. Math. Japon., 1997, **45**(1): 81–84.

[7] WU Mingfen, XIE Xiangyun. *Prime radical theorems on ordered semigroups* [J]. J. Algebra Number Theory Appl., 2001, **1**(1): 1–9.

[8] HOWIE J M. *Introduction to Semigroups* [M]. Academic Press, London-New York, 1976.

[9] BOGDANOVIĆ S. *Semigroups with a System of Subsemigroups* [M]. University of Novi Sad, Institute of Mathematics, Faculty of Science, Novi Sad, 1985.

[10] XIE Xiangyun. *An Introduction to Ordered Semigroup Theory* [M]. Kexue Press, Beijing, 2001.