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On Signed Edge Total Domination Numbers of Graphs

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Abstract Let $G=(V,E)$ be a graph. A function $f: E\to\{-1,1\}$ is said to be a signed edge total dominating function (SETDF) of G if $\sum_{e'\in N(e)} f(e')\ge 1$ holds for every edge $e\in E(G)$. The signed edge total domination number $\gamma'_{st}(G)$ of G is defined as $\gamma'_{st}(G)=\min\{\sum_{e\in E(G)} f(e)|f$ is an SETDF of $G\}$. In this paper we obtain some new lower bounds of $\gamma'_{st}(G)$.

Keywords signed edge total dominating function; signed edge total domination number; edge degree.

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1. Introduction

For the terminology and notations not defined here, we adopt those in Bondy and Murty [1] and Xu [2] and consider simple graphs only.

Let $G=(V,E)$ be a graph with vertex set $V=V(G)$ and edge set $E=E(G)$. For any vertex $v\in V$, $N_G(v)$ denotes the open neighborhood of v in G and $N_G[v]=N_G(v)\cup\{v\}$ the closed one. $d_G(v)=|N_G(v)|$ is called the degree of v in G, Δ and δ denote the maximum degree and minimum degree of G, respectively. Similarly, if $e=uv\in E$, $N_G(e)$ denotes the open edge-neighborhood of e in G and $N_G[e]=N_G(e)\cup\{e\}$ the closed one. $d_G(e)$ is called the degree of e in G, Δ_e and δ_e denote the maximum edge degree and minimum edge degree of G, respectively. If the graph is clear from the context, $N_G(v)$, $N_G[v]$, $d_G(v)$ and $N_G(e)$, $N_G[e]$, $d_G(e)$ can simply be denoted by $N(v)$, $N[v]$, $d(v)$ and $N(e)$, $N[e]$, $d(e)$.

If $d(v)$ is odd (even), then v is called an odd (even) vertex of G. Similarly, if $d(e)$ is even (odd), then e is called an even (odd) edge and $d(e)=d(u)+d(v)-2$.

In this paper, we define $E_o=\{e\in E|d(e)\text{is odd }\}$ and $E_e=\{e\in E|d(e)\text{ is even}\}$.

In recent years, several kinds of domination problems in graphs have been investigated [2–6]. Most of them belong to the vertex domination of graphs, such as signed domination [7, 8], minus domination [8], majority domination, etc. Recently, the problem has been changed from vertex domination to edge domination, and a few results have been obtained about the edge domination

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of graphs, such as signed edge domination [2], signed star domination [9], minus edge domination [10], etc. The concept of signed edge total domination was introduced in [11, 12], but there are few results that have been obtained about it and most of the bounds are on the vertex degree of graphs. In this paper, we will obtain some new bounds of $\gamma'_{st}(G)$ in terms of the edge degree of graphs.

In [11] we introduced the signed edge total domination as follows:

Definition 1 ([11]) *Let $G = (V, E)$ be a connected graph of order n ($n \geq 3$). A function $f : E \to \{-1, 1\}$ is said to be the signed edge total dominating function (SETDF) of G if $\sum_{e' \in N(e)} f(e') \geq 1$ holds for every edge $e \in E(G)$. The signed edge total domination number $\gamma'_{st}(G)$ of G is defined as $\gamma'_{st}(G) = \min\{\sum_{e \in E(G)} f(e) | f$ is an SETDF of $G\}$.*

And define $\gamma'_{st}(K_1) = 0$, $\gamma'_{st}(K_2) = 1$.

If f is such an SETDF that $\gamma'_{st}(G) = \sum_{e \in E} f(e)$, then the function f is said to be a minimum SETDF.

By the above definition, we have the following lemma.

Lemma 1 *For any two disjoint graphs G_1 and G_2, $\gamma'_{st}(G_1 \cup G_2) = \gamma'_{st}(G_1) + \gamma'_{st}(G_2)$.*

Lemma 2 *Let f be a signed edge total dominating function of G and $e \in E$. If $e \in E_o$, then $\sum_{e' \in N(e)} f(e') \geq 1$; while if $e \in E_e$, then $\sum_{e' \in N(e)} f(e') \geq 2$.*

By Lemma 1, we consider connected graphs only. In this paper, we will give some lower bounds of $\gamma'_{st}(G)$ for general connected graphs G in terms of its size m, maximum edge degree $\triangle_e$ and minimum edge degree δ_e.

2. Main results

In this section, we will give some lower bounds for the signed edge total domination number $\gamma'_{st}(G)$ of a graph G.

Theorem 1 *For any connected graph G of size m ($m \geq 2$),*

$$\gamma'_{st}(G) \geq \frac{3 - \delta_e + \sqrt{(\delta_e - 3)^2 + 8m(\delta_e + 1)}}{2} - m.$$

Proof Let f be a signed edge total dominating function of G such that $\gamma'_{st}(G) = \sum_{e \in E} f(e)$. Define $P = \{e \in E(G) | f(e) = 1\}$, $M = \{e \in E(G) | f(e) = -1\}$. Let $|P| = p$, $|M| = m - p$. Then $\gamma'_{st}(G) = |P| - |M| = 2p - m$.

For any edge $e \in E(G)$, by the definition of the signed edge total domination number, we can easily verify the following inequality:

$$|N(e) \cap P| \geq \lceil \frac{d(e) + 1}{2} \rceil, \quad \forall e \in E(G),$$

then

$$\sum |N(e) \cap P| \geq \frac{d(e) + 1}{2}(m - p) \geq \frac{\delta_e + 1}{2}(m - p),$$

so there exists at least one edge $e \in P$ such that e is adjacent to $\frac{(\delta_e+1)(m-p)}{2p}$ edges of M. Hence

$$p-1 \geq |N(e)\cap P| \geq 1+\frac{(\delta_e+1)(m-p)}{2p}.$$

By the above inequality, we deduce that

$$p \geq \frac{3-\delta_e+\sqrt{(\delta_e-3)^2+8m(\delta_e+1)}}{4},$$

so

$$\gamma'_{st}(G)=2p-m \geq \frac{3-\delta_e+\sqrt{(\delta_e-3)^2+8m(\delta_e+1)}}{2}-m. \quad \square$$

Theorem 2 *For any connected graph G of size m $(m \geq 2)$,*

$$\gamma'_{st}(G) \geq \left\lceil \frac{(2+\delta_e-\triangle_e)m+2m_e}{\delta_e} \right\rceil,$$

where $m_e=|E_e|$, $\triangle_e \geq \delta_e \geq 1$ and the result is the best possible.

Proof Let $s=\sum_{e\in E} d(e)$, and let f be a signed edge total dominating function of G such that $\gamma'_{st}(G)=\sum_{e\in E} f(e)$.

Define $P=\{e\in E(G)|f(e)=1\}$, $M=\{e\in E(G)|f(e)=-1\}$. By Lemma 2, we have

$$\sum_{e\in E}\sum_{e'\in N(e)} f(e')=\sum_{e\in E_o}\sum_{e'\in N(e)} f(e')+\sum_{e\in E_e}\sum_{e'\in N(e)} f(e') \geq |E_o|+2|E_e|=m+m_e. \tag{1}$$

Note that

$$\sum_{e\in P} d(e)-\sum_{e\in M} d(e)=\sum_{e\in E} d(e)-2\sum_{e\in M} d(e)=2\sum_{e\in P} d(e)-\sum_{e\in E} d(e). \tag{2}$$

On the other hand,

$$\sum_{e\in E}\sum_{e'\in N(e)} f(e')=\sum_{e\in E} d(e)f(e)=\sum_{e\in P} d(e)f(e)+\sum_{e\in M} d(e)f(e)=\sum_{e\in P} d(e)-\sum_{e\in M} d(e).$$

Since

$$\sum_{e\in E} d(e)-2\sum_{e\in M} d(e) \leq \sum_{e\in E} d(e)-2(m-|P|)\delta_e=s-2(m-|P|)\delta_e,$$

by (1) and the above inequality, we have

$$s-2(m-|P|)\delta_e \geq m+m_e. \tag{3}$$

Note that

$$2\sum_{e\in P} d(e)-\sum_{e\in E} d(e) \leq 2\triangle_e|P|-\sum_{e\in E} d(e)=2\triangle_e|P|-s,$$

we can deduce the following from the above inequality and (1):

$$2\triangle_e|P|-s \geq m+m_e.$$

By (3) and (4), we can deduce the following inequalities:

$$|P| \geq \frac{m+m_e-s}{2\delta_e}+m, |P| \geq \frac{m+m_e+s}{2\triangle_e}, \tag{4}$$

which implies that

$$|P| \geq \frac{(1+\delta_e)m + m_e}{\delta_e + \triangle_e},$$

and hence

$$\gamma'_{st}(G) = 2|P| - m \geq \frac{(2+\delta_e - \triangle_e)m + 2m_e}{\delta_e + \triangle_e}.$$

Note that $\gamma'_{st}(G)$ is an integer, this completes the proof.

Next we will show that the result is the best possible.

For a graph $G = C_n$, it is obvious that $\gamma'_{st}(C_n) = n = m = \frac{(2+\delta_e-\triangle_e)m+2m_e}{\delta_e+\triangle_e}$. □

A graph G is called k-regular if $d(v) = k$ for all $v \in V$. Similarly, a graph G is called k-edge regular if $d(e) = k$ for all $e \in E$. The following corollaries follow immediately from Theorem 2.

Corollary 1 *For any k-edge regular graph G of size m,* $\gamma'_{st}(G) \geq \begin{cases} \frac{2m}{k}, & k \text{ is even,} \\ \frac{m}{k}, & k \text{ is odd.} \end{cases}$

Corollary 2 *For any k-regular graph G of order n,* $\gamma'_{st}(G) \geq \frac{kn}{2k-2}$.

Theorem 3 *For any connected graph G of size m $(m \geq 2)$,*

$$\gamma'_{st}(G) \geq \Big(\frac{\lceil(\delta_e-1)/2\rceil - \lfloor(\triangle_e-1)/2\rfloor + 1}{\lceil(\delta_e-1)/2\rceil + \lfloor(\triangle_e-1)/2\rfloor + 1}\Big)m.$$

Proof Let f be a signed edge total dominating function of G such that $\gamma'_{st}(G) = \sum_{e\in E} f(e)$. By Lemma 2, we have

$$\sum_{e\in E}\sum_{e'\in N(e)} f(e') = \sum_{e\in E_o}\sum_{e'\in N(e)} f(e') + \sum_{e\in E_e}\sum_{e'\in N(e)} f(e') \geq |E_o| + 2|E_e| = m + m_e.$$

We now write E as the disjoint union of six sets. Let $P = P_{\triangle_e} \bigcup P_{\delta_e} \bigcup P_{\lambda_e}$, where $P_{\triangle_e}$ and P_{δ_e} are sets of all edges of P with edge degree equal to $\triangle_e$ and δ_e, respectively, and P_{λ_e} contains all other edges in P. If possible, let $M = M_{\triangle_e} \bigcup M_{\delta_e} \bigcup M_{\lambda_e}$, where $M_{\triangle_e}$,M_{δ_e} and M_{λ_e} are defined similarly. Further, for $i \in \{\triangle_e, \delta_e, \lambda_e\}$, let E_i be defined by $E_i = P_i \bigcup M_i$. Thus $m = |E_{\triangle_e}| + |E_{\delta_e}| + |E_{\lambda_e}|$.

If $e \in E_{\lambda_e}$, then $\delta_e + 1 \leq d(e) \leq \triangle_e - 1$. Therefore,

$$\sum_{e\in E}\sum_{e'\in N(e)} f(e') = \sum_{e\in E} d(e)f(e) \geq m + m_e. \tag{5}$$

Writing the sum in Eq. (5) as sum of six summations and replacing $f(e)$ with the corresponding value $+1$ or -1 yields

$$\sum_{e\in E} d(e)f(e) = \sum_{e\in P_{\triangle_e}} \triangle_e + \sum_{e\in P_{\delta_e}} \delta_e + \sum_{e\in P_{\lambda_e}} (\triangle_e - 1) - \sum_{e\in M_{\triangle_e}} \triangle_e - \sum_{e\in M_{\delta_e}} \delta_e - \sum_{e\in M_{\delta_e}} (\delta_e + 1) \geq m + m_e, \tag{6}$$

and replacing $|P_i|$ with $|E_i| - |M_i|$ for $i \in \{\triangle_e, \delta_e, \lambda_e\}$ yields

$$\triangle_e(|E_{\triangle_e}| - |M_{\triangle_e}|) + \delta_e(|E_{\delta_e}| - |M_{\delta_e}|) + (\triangle - 1)(|E_{\lambda_e}| - |M_{\lambda_e}|) - \triangle_e|M_{\triangle_e}| - \delta_e|M_{\delta_e}| - (\delta_e + 1)|M_{\lambda_e}|$$

$$= \triangle_e|E_{\triangle_e}| - 2\triangle_e|M_{\triangle_e}| + \delta_e|E_{\delta_e}| - 2\delta_e|M_{\delta_e}| + (\triangle_e - 1)|E_{\delta_e}| - (\triangle_e + \delta_e)|M_{\lambda_e}|$$
$$\geq m + m_e. \tag{7}$$

We now simplify the left-hand side of (7) as follows. Replacing $|E_{\delta_e}|$ with $|P_{\delta_e}| + |M_{\delta_e}|$, and $|M_{\delta_e}| + |M_{\lambda_e}|$ with $|M| - |M_{\lambda_e}|$, we have

$$\delta_e|E_{\delta_e}| - 2\delta_e|M_{\delta_e}| - \delta_e|M_{\lambda_e}| = \delta_e|P_{\delta_e}| - \delta_e(|M| - |M_{\triangle_e}|). \tag{8}$$

Further, replacing $|E_{\triangle_e}|$ with $m - |E_{\lambda_e}| - |E_{\delta_e}|$, we have

$$\triangle_e|E_{\triangle_e}| + \triangle_e|E_{\lambda_e}| - 2\triangle_e|M_{\triangle_e}| - \triangle_e|M_{\lambda_e}| = \triangle_e m - \triangle_e|M| - \triangle_e|P_{\triangle_e}| - \triangle_e|M_{\triangle_e}|. \tag{9}$$

We can deduce the following from (7), (8) and (9)

$$\delta_e|P_{\delta_e}| - \delta_e(|M| - |M_{\triangle_e}|) + \triangle_e m - \triangle_e|M| - \triangle_e|P_{\triangle_e}| - \triangle_e|M_{\triangle_e}| - |E_{\lambda_e}|$$
$$= \triangle_e m - (\triangle_e - \delta_e)|P_{\delta_e}| - (\triangle_e + \delta_e)|M| - (\triangle_e - \delta_e)|M_{\triangle_e}| - |E_{\lambda_e}|$$
$$\geq m + m_e.$$

That is

$$(\triangle_e - 1)m \geq (\triangle_e - \delta_e)|P_{\delta_e}| + (\triangle_e + \delta_e)|M| - (\triangle_e - \delta_e)|M_{\triangle_e} + |E_{\lambda_e}| + m_e. \tag{10}$$

We now consider four possibilities depending on the parity of $\triangle_e$ and δ_e. In each case, we obtain an upper bound on $|M|$ in terms of $\triangle_e$, δ_e and m. Since $\gamma'_{st}(G) = m - 2|M|$, this upper bound provides the desired lower bound on $\gamma'_{st}(G)$ in terms of $\triangle_e$, δ_e and m.

Case 1 $\triangle_e$, δ_e are odd.

By Eq. (10), $(\triangle_e - 1)m \geq (\triangle_e + \delta_e)|M|$, and so $|M| \leq \frac{(\triangle_e - 1)m}{\triangle_e + \delta_e}$. Therefore,

$$\gamma'_{st}(G) = m - 2|M| \geq \frac{\delta_e - \triangle_e + 2}{\triangle_e + \delta_e} m.$$

Case 2 $\triangle_e$, δ_e are even.

Then $|E_e| \geq |E_{\triangle_e}| + |E_{\delta_e}|$. Thus, by Eq. (10), $(\triangle_e - 1)m \geq (\triangle_e + \delta_e)|M| + m$, and so $|M| \leq \dfrac{(\triangle_e - 2)m}{\triangle_e + \delta_e}$, then $\gamma'_{st}(G) = m - 2|M| \geq \frac{\delta_e - \triangle_e + 4}{\triangle_e + \delta_e} m$.

Case 3 δ_e is even and $\triangle_e$ is odd.

Then $\triangle_e - \delta_e \geq 1$, $|E_e| \geq |E_{\delta_e}|$. Thus, by Eq. (10),

$$(\triangle_e - 1)m \geq |P_{\delta_e}| + (\triangle_e + \delta_e)|M| + |M_{\triangle_e}| + |E_{\lambda_e}| + m_e$$
$$\geq |P_{\delta_e}| + |P_{\lambda_e}| + |P_{\delta_e}| + (\triangle_e + \delta_e)|M| + |M_{\triangle_e}| + |M_{\lambda_e}| + |M_{\delta_e}|$$
$$\geq (\triangle_e + \delta_e + 1)|M|.$$

And so $|M| \leq \frac{(\triangle_e - 1)m}{\triangle_e + \delta_e + 1}$. Therefore, $\gamma'_{st}(G) = m - 2|M| \geq \frac{\delta_e - \triangle_e + 3}{\triangle_e + \delta_e + 1} m$.

Case 4 δ_e is odd and $\triangle_e$ is even.

Then $\triangle_e - \delta_e \geq 1$, $|E_e| \geq |E_{\triangle_e}|$. Thus, by Eq. (10),

$$(\triangle_e - 1)m \geq |P_{\delta_e}| + (\triangle_e + \delta_e)|M| + |M_{\triangle_e}| + |E_{\lambda_e}| + m_e$$

$$\geq |P_{\delta_e}| + |P_{\lambda_e}| + |P_{\triangle_e}| + (\triangle_e + \delta_e)|M| + |M_{\triangle_e}| + |M_{\lambda_e}| + |M_{\triangle_e}|$$
$$\geq |P| + (\triangle_e + \delta_e)|M| = m - |M| + (\triangle_e + \delta_e)|M|.$$

And so $|M| \leq \frac{(\triangle_e - 2)m}{\triangle_e + \delta_e - 1}$. Therefore, $\gamma'_{st}(G) = m - 2|M| \geq \frac{\delta_e - \triangle_e + 3}{\triangle_e + \delta_e - 1}m$.

Combining the above four cases, we have completed the proof of Theorem 5. □

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