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Existence of Homoclinic Solution for a Class of Hamiltonian Systems

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Abstract We study the existence of homoclinic orbits for some Hamiltonian system. A homoclinic orbit is obtained as a limit of $2kT$-periodic solutions of a sequence of systems of differential equations.

Keywords homoclinic orbit; Hamiltonian system; critical point; Jensen inequality.

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1. Introduction and main results

In this paper, we consider the existence of homoclinic solution for the following Hamiltonian system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\Phi_p(\dot{u}(t))\big)-l(t)\Phi_p(u(t))+\nabla F(t,u(t))=f(t), \tag{HS}$$

where $p>1$, $t\in R$, $u\in R^n$, $\Phi_p(u)=|u|^{p-2}u$, $l(t)\in C(R,(0,+\infty))$, where $l(t)$ is T-periodic, $T>0$ and $F:R\times R^n\to R$ and $f:R\to R^n$ satisfy:

(A1) $F(t,0)\equiv 0$ and F is T-periodic with respect to t;

(A2) There exist functions $b\in C(R,(0,+\infty))$ and $H(t,x)\in C^1(R\times R^n,(0,+\infty))$ and the constant $\mu>p$ such that

$$F(t,x)=\frac{b(t)}{\mu}|x|^{\mu}+H(t,x),$$

where $b_0=\min_{t\in[0,T]}b(t)>0$, $b(t)$ and $H(t,x)$ are T-periodic with respect to t;

(A3) For every $t\in R$ and $x\in R^n\setminus\{0\}$,

$$0<\mu H(t,x)\le\big(\nabla H(t,x),x\big).$$

Here and subsequently, $(\cdot,\cdot):R^n\times R^n\to R$ denotes the standard inner product in R^n and $|\cdot|$ is the induced norm in R^n.

The existence of homoclinic orbits is one of the most important problems in the theory of Hamiltonian systems. Recently the existence and multiplicity of homoclinic orbits for Hamiltonian systems have been studied extensively via critical point theory, such as [1–4, 7–11] and the

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references therein. In [1], Tang considered the system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(|\dot{u}(t)|^{p-2}\dot{u}(t)\big)= \nabla F(t,u(t))+f(t), \tag{1.1}$$

and proved the following theorem:

Theorem A *Assume that F and f satisfy the following conditions:*

(F1) $F\in C^1(R\times R^n,R)$ is T-periodic with respect to t, $T>0$;

(F2) There are constants $b>0$ and $\mu>1$ such that for all $(t,x)\in[0,T]\times R^n$

$$F(t,x)\geq F(t,0)+b|x|^{\mu};$$

(F3) $f\neq 0$ is a continuous and bounded function such that $\int_R |f(t)|^{\frac{\mu}{\mu-1}}\mathrm{d}t<\infty$.

Then system (1.1) possesses a homoclinic solution $u_0\in W^{1,p}(R,R^n)$.

Our goal in this paper is to consider the different system (HS) and to use the different conditions instead of (F2), (F3). In order to obtain a homoclinic solution of (HS), we consider a sequence of systems of differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\Phi_p(\dot{u}(t))\big)-l(t)\Phi_p(u(t))+\nabla F(t,u(t))=f_k(t), \tag{HSk}$$

where for every $k\in N, f_k:R\to R^n$ is a $2kT$-periodic extension of the restriction of f to the interval $[-kT,kT]$.

For each $k\in N$, let $W^{1,p}_{2kT}(R,R^n)$, $L^p_{2kT}(R,R^n)$ and $L^{\infty}_{2kT}(R,R^n)$ denote the Banach spaces of $2kT-$periodic functions on R with values in R^n under the norms

$$\|u\|_{W^{1,p}_{2kT}}:=\Big[\int_{-kT}^{kT}\Big(|\dot{u}(t)|^p+|u(t)|^p\Big)\mathrm{d}t\Big]^{\frac{1}{p}},$$

$$\|u\|_{L^p_{2kT}}:=\Big(\int_{-kT}^{kT}|u(t)|^p\mathrm{d}t\Big)^{\frac{1}{p}}$$

and

$$\|u\|_{L^{\infty}_{2kT}}:=\operatorname{ess\,sup}\big\{|u(t)|:t\in[-kT,kT]\big\},$$

respectively. Furthermore for each $k\in N$, let

$$E_k:=\Big\{u\big|u,\dot{u}\in L^p_{2kT}(R,R^n),\int_{-kT}^{kT}\Big[|\dot{u}(t)|^p+\big(l(t)\Phi_p(u(t)),u(t)\big)\Big]\mathrm{d}t<+\infty\Big\}$$

and $\forall u\in E_k$, let

$$\|u\|_{E_k}:=\Big\{\int_{-kT}^{kT}\Big[|\dot{u}(t)|^p+\big(l(t)\Phi_p(u(t)),u(t)\big)\Big]\mathrm{d}t\Big\}^{\frac{1}{p}}. \tag{1.2}$$

Then E_k is a Hilbert space on the above norm.

Lemma 1.1 *Let $u:R\to R^n$ be a continuous mapping such that $\dot{u}\in L^p_{\mathrm{loc}}(R,R^n)$. For every $t\in R$ the following inequality holds:*

$$|u(t)|\leq 2^{\frac{p-1}{p}}\Big(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}}\big(|u(s)|^p+|\dot{u}(s)|^p\big)\mathrm{d}s\Big)^{\frac{1}{p}}. \tag{1.3}$$

Proof Fix $t \in R$. For every $\tau \in R$,

$$|u(t)| \leq |u(\tau)| + \Big| \int_\tau^t \dot{u}(s)\mathrm{d}s \Big|. \tag{1.4}$$

Integrating (1.4) over $[t-\frac{1}{2}, t+\frac{1}{2}]$ with respect to τ and using the Hölder and Jensen inequalities, we obtain

$$\begin{aligned}
|u(t)| &\leq \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \Big(|u(\tau)| + \Big| \int_\tau^t \dot{u}(s)\mathrm{d}s \Big| \Big) \mathrm{d}\tau \\
&\leq \Big(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \Big(|u(\tau)| + \Big| \int_\tau^t \dot{u}(s)\mathrm{d}s \Big| \Big)^p \mathrm{d}\tau \Big)^{\frac{1}{p}} \\
&\leq \Big(2^{p-1} \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \Big(|u(\tau)|^p + \Big| \int_\tau^t \dot{u}(s)\mathrm{d}s \Big|^p \Big) \mathrm{d}\tau \Big)^{\frac{1}{p}} \\
&\leq \Big(2^{p-1} \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \Big(|u(\tau)|^p + \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\dot{u}(s)|^p \mathrm{d}s \Big) \mathrm{d}\tau \Big)^{\frac{1}{p}} \\
&\leq 2^{\frac{p-1}{p}} \Big(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |u(\tau)|^p \mathrm{d}\tau + \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\dot{u}(s)|^p \mathrm{d}s \Big)^{\frac{1}{p}},
\end{aligned}$$

which implies that (1.3) holds. The proof is completed. □

Corollary 1.1 *Let $u \in W_{2kT}^{1,p}$. Then the following inequality holds:*

$$\|u\|_{L_{2kT}^\infty} \leq 2^{\frac{p-1}{p}} \Big(1 + \frac{1}{2T} \Big)^{\frac{1}{p}} \|u\|_{W_{2kT}^{1,p}}, \quad \forall\ k \in N. \tag{1.5}$$

Proof Integrating (1.4) over $[t-kT, t+kT]$ with respect to τ and similar to the proof of Lemma 1.1, we have

$$\begin{aligned}
2kT|u(t)| &\leq \int_{t-kT}^{t+kT} \Big(|u(\tau)| + \Big| \int_\tau^t \dot{u}(s)\mathrm{d}s \Big| \Big) \mathrm{d}\tau \\
&\leq (2kT)^{\frac{p-1}{p}} \Big(\int_{t-kT}^{t+kT} \Big(|u(\tau)| + \Big| \int_\tau^t \dot{u}(s)\mathrm{d}s \Big| \Big)^p \mathrm{d}\tau \Big)^{\frac{1}{p}} \\
&\leq (2kT)^{\frac{p-1}{p}} \Big(2^{p-1} \int_{t-kT}^{t+kT} \Big(|u(\tau)|^p + \int_{t-kT}^{t+kT} |\dot{u}(s)|^p \mathrm{d}s \Big) \mathrm{d}\tau \Big)^{\frac{1}{p}} \\
&\leq (2kT)^{\frac{p-1}{p}} 2^{\frac{p-1}{p}} \Big(\int_{t-kT}^{t+kT} |u(\tau)|^p \mathrm{d}\tau + 2kT \int_{t-kT}^{t+kT} |\dot{u}(s)|^p \mathrm{d}s \Big)^{\frac{1}{p}},
\end{aligned}$$

which implies the following

$$\|u\|_{L_{2kT}^\infty} \leq 2^{\frac{p-1}{p}} \Big(1 + \frac{1}{2kT} \Big)^{\frac{1}{p}} \|u\|_{W_{2kT}^{1,p}} \leq 2^{\frac{p-1}{p}} \Big(1 + \frac{1}{2T} \Big)^{\frac{1}{p}} \|u\|_{W_{2kT}^{1,p}}, \quad \forall k \in N.$$

The proof is completed. □

Corollary 1.2 *Let $u \in E_k$. Then the following inequality holds:*

$$\|u\|_{L_{2kT}^\infty} \leq 2^{\frac{p-1}{p}} \max\{1, \frac{1}{l^*}\} \Big(1 + \frac{1}{2T} \Big)^{\frac{1}{p}} \|u\|_{E_k}, \tag{1.6}$$

where $l^* = \min_{t \in [0,T]} l(t)$.

Proof For each $u \in E_k$, we have by (1.2) and (1.5),

$$\|u\|_{E_k}^p \geq \int_{-kT}^{kT} \Big[|\dot{u}(t)|^p + l^*|u(t)|^p\Big]\mathrm{d}t \geq \min\{1, l^*\}\|u\|_{W_{2kT}^{1,p}}^p$$
$$\geq \min\{1, l^*\}\Big[2^{\frac{p-1}{p}}\Big(1+\frac{1}{2T}\Big)^{\frac{1}{p}}\Big]^{-p}\|u\|_{L_{2kT}^\infty}^p,$$

so (1.6) holds.

Set $M := \sup\{H(t,u) : t \in [0,T], |u| = 1\}$, $b_1 = \max_{t\in[0,T]} b(t)$, $C_1 = 2^{\frac{p-1}{p}} \max\{1, \frac{1}{l^*}\}(1 + \frac{1}{2T})^{\frac{1}{p}}$ and suppose that:

(A4) $\Big(\int_R |f(t)|^{\frac{p}{p-1}}\mathrm{d}t\Big)^{1-\frac{1}{p}} := M_1 < \frac{1}{C_1^{p-1}}\Big[\frac{1}{p}\delta^{p-1} - \Big(\frac{b_1}{\mu} + M\Big)\delta^{\mu-1}\Big]$,

where $\delta = \Big[\frac{p-1}{p(\mu-1)\left(\frac{b_1}{\mu}+M\right)}\Big]^{\frac{1}{\mu-p}}$.

Remark 1.1 For $f(x) = \frac{1}{p}x^{p-1} - (\frac{b_1}{\mu} + M)x^{\mu-1}$, $x \in [0,1]$ and δ defined above, by simple calculation, we can obtain that δ is the maximum point of $f(x)$ and $f(\delta) = \frac{1}{p}\delta^{p-1} - (\frac{b_1}{\mu} + M)\delta^{\mu-1} > 0$.

In this paper, the main result is the following theorem.

Theorem 1.1 *If the conditions (A1)–(A4) are satisfied, then system (HS) possesses a nontrivial homoclinic solution u_0.*

2. Preliminaries

Let $I_k : E_k \to R$ be defined by

$$I_k(u) = \frac{1}{p}\int_{-kT}^{kT}\Big[|\dot{u}(t)|^p + \big(l(t)\Phi_p(u(t)), u(t)\big)\Big]\mathrm{d}t - \int_{-kT}^{kT} F(t,u(t))\mathrm{d}t + \int_{-kT}^{kT}\big(f_k(t), u(t)\big)\mathrm{d}t. \quad (2.1)$$

Then $I_k \in C^1(E_k, R)$ and one can easily check that

$$I_k'(u)v = \int_{-kT}^{kT}\Big[\big(\Phi_p(\dot{u}(t)), \dot{v}(t)\big) + \big(l(t)\Phi_p(u(t)), v(t)\big) - \big(\nabla F(t,u(t)), v(t)\big) + \big(f_k(t), v(t)\big)\Big]\mathrm{d}t \quad (2.2)$$

for all $u, v \in E_k$, where $I_k'(u)$ means the Frechet derivative. Furthermore, the critical points of I_k are classical $2kT-$periodic solutions of ($\mathrm{HS_k}$). We will obtain a critical point of I_k by using a standard version of the Mountain Pass Theorem, therefore we state the definition of PS-condition and this theorem precisely.

Definition 2.1 ([5]) *The function $\varphi \in C^1(X,R)$ satisfies the Palais-Smale condition (PS) if every sequence $\{u_n\}$ in X satisfies that $\{\varphi(u_n)\}$ is bounded and*

$$\varphi'(u_n) \to 0 \quad (n \to \infty)$$

contains a convergent subsequence.

Lemma 2.1 *([5]) Let E be a real Banach space and $I \in C^1(E,R)$ satisfy the Palais-Smale condition. If further I satisfies the following conditions:*

(i) $I(0) = 0$;

(ii) *There exist constants* $\rho, \alpha > 0$ *such that* $I|_{\partial B_\rho(0)} \geq \alpha$*;*

(iii) *There exists* $e \in E \setminus \overline{B}_\rho(0)$ *such that* $I(e) \leq 0$*,*

then I *possesses a critical value* $c \geq \alpha$ *given by*

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where $B_\rho(0)$ *is an open ball in* E *of radius* ρ *about* 0*, and*

$$\Gamma = \big\{g \in C([0,1], E) : g(0) = 0, g(1) = e\big\}.$$

Lemma 2.2 ([2]) *Assume that (A3) holds, then for every* $t \in [0, T]$*, the following inequalities hold:*

$$H(t,u) \leq H\big(t, \frac{u}{|u|}\big)|u|^\mu, \quad 0 < |u| \leq 1; \tag{2.3}$$

$$H(t,u) \geq H\big(t, \frac{u}{|u|}\big)|u|^\mu, \quad |u| \geq 1. \tag{2.4}$$

Meanwhile, we set $m := \inf\{H(t,u) : t \in [0,T], |u| = 1\}$*. Then for every* $\zeta \in R \setminus \{0\}$ *and* $u \in E_k \setminus \{0\}$*, one can find from [2]*

$$\int_{-kT}^{kT} H(t, \zeta u(t))\mathrm{d}t \geq m|\zeta|^\mu \int_{-kT}^{kT} |u(t)|^\mu \mathrm{d}t - 2kTm. \tag{2.5}$$

Lemma 2.3 ([2]) *Let* $u : R \to R^n$ *be a continuous mapping. If a weak derivative* $\dot{u} : R \to R^n$ *is continuous at* t_0*, then*

$$\lim_{t\to t_0} \frac{u(t) - u(t_0)}{t - t_0} = \dot{u}(t_0).$$

3. Proof of theorem

We will divide the proof of Theorem 1.1 into a series of lemmas.

Lemma 3.1 *Under the conditions of Theorem 1.1, for every* $k \in N$*, system (HS*$_\mathrm{k}$*) possesses a* $2kT$*-periodic solution.*

Proof It is clear that $I_k(0) = 0$. We firstly show that I_k satisfies the Palais-Smale condition. Assume that $\{u_j\}_{j\in N} \subset E_k$ is a sequence such that $\{I_k(u_j)\}_{j\in N}$ is bounded and $I_k'(u_j) \to 0$ $(j \to +\infty)$. From (2.1), (2.2) and (A2), we have

$$\mu I_k(u_j) = \frac{\mu}{p}\|u_j\|_{E_k}^p - \int_{-kT}^{kT} \Big[b(t)|u_j(t)|^\mu + \mu H(t, u_j(t))\Big]\mathrm{d}t + \mu \int_{-kT}^{kT} \big(f_k(t), u_j(t)\big)\mathrm{d}t, \tag{3.1}$$

and

$$I_k'(u_j)u_j = \|u_j\|_{E_k}^p - \int_{-kT}^{kT} \Big[b(t)|u_j(t)|^\mu + \big(\nabla H(t, u_j(t)), u_j(t)\big)\Big]\mathrm{d}t + \int_{-kT}^{kT} \big(f_k(t), u_j(t)\big)\mathrm{d}t. \tag{3.2}$$

So we obtain from (3.1), (3.2), (A3) and (A4),

$$\Big(\frac{\mu}{p} - 1\Big)\|u_j\|_{E_k}^p = \mu I_k(u_j) - I_k'(u_j)u_j + \int_{-kT}^{kT} \Big[\mu H(t, u_j(t)) - \big(\nabla H(t, u_j(t)), u_j(t)\big)\Big]\mathrm{d}t -$$

$$
\begin{aligned}
&(\mu-1)\int_{-kT}^{kT}\big(f_k(t),u_j(t)\big)\mathrm{d}t\\
\le&\mu I_k(u_j)-I_k'(u_j)u_j+(\mu-1)\Big(\int_{-kT}^{kT}\big|f_k(t)\big|^{\frac{p}{p-1}}\mathrm{d}t\Big)^{1-\frac{1}{p}}\Big(\int_{-kT}^{kT}\big|u_j(t)\big|^{p}\mathrm{d}t\Big)^{\frac{1}{p}}\\
\le&\mu I_k(u_j)-I_k'(u_j)u_j+(\mu-1)M_1\|u_j\|_{E_k}. \qquad (3.3)
\end{aligned}
$$

Since $\frac{\mu}{p}-1>0$, $\{I_k(u_j)\}$ is bounded and $I_k'(u_j)\to 0$ $(j\to+\infty)$, (3.3) shows that $\{u_j\}_{j\in N}$ is bounded in E_k.

In a similar way to [6], we can prove that $\{u_j\}_{j\in N}$ has a convergent subsequence in E_k. Hence, I_k satisfies the Palais-Smale condition.

We now show that there exist constants ρ, $\alpha>0$ independent of k such that I_k satisfies condition (ii) of Lemma 2.1 with these constants. If $\|u\|_{E_k}=\frac{\delta}{C_1}:=\rho$, then it follows from Corollary 1.2 that $|u(t)|\le\delta\le 1$ for $t\in[-kT,kT]$. From (2.3), we have

$$
\begin{aligned}
\int_{-kT}^{kT}H(t,u(t))\mathrm{d}t\le&\int_{\{t\in[-kT,kT]|u(t)\neq 0\}}H\Big(t,\frac{u(t)}{|u(t)|}\Big)|u(t)|^{\mu}\mathrm{d}t\\
\le&M\delta^{\mu-p}\int_{-kT}^{kT}|u(t)|^{p}\mathrm{d}t, \qquad (3.4)
\end{aligned}
$$

where $M=\sup\{H(t,u):t\in[0,T],|u|=1\}$. According to (2.1), (A2), (3.4) and (A4), if $\|u\|_{E_k}=\frac{\delta}{C_1}$, then

$$
\begin{aligned}
I_k(u)=&\frac{1}{p}\|u\|_{E_k}^{p}-\frac{1}{\mu}\int_{-kT}^{kT}b(t)|u(t)|^{\mu}\mathrm{d}t-\int_{-kT}^{kT}H(t,u(t))\mathrm{d}t+\int_{-kT}^{kT}\big(f_k(t),u(t)\big)\mathrm{d}t\\
\ge&\frac{1}{p}\|u\|_{E_k}^{p}-\Big(\frac{b_1}{\mu}+M\Big)\delta^{\mu-p}\int_{-kT}^{kT}|u(t)|^{p}\mathrm{d}t-M_1\|u\|_{E_k}\\
\ge&\frac{1}{p}\|u\|_{E_k}^{p}-\Big(\frac{b_1}{\mu}+M\Big)\delta^{\mu-p}\|u\|_{E_k}^{p}-M_1\|u\|_{E_k}\\
=&\frac{\delta}{C_1}\Big[\frac{1}{C_1^{p-1}}\frac{1}{p}\delta^{p-1}-\frac{1}{C_1^{p-1}}\Big(\frac{b_1}{\mu}+M\Big)\delta^{\mu-1}-M_1\Big]\\
:=&\alpha>0, \qquad (3.5)
\end{aligned}
$$

where b_1, M_1, and M are given previously. Then inequality (3.5) shows that $\|u\|_{E_k}=\frac{\delta}{C_1}=\rho$ which implies that $I_k(u)\ge\alpha$.

Finally, it remains to show that I_k satisfies condition (iii) of Lemma 2.1. From (A2), (2.1) and (2.5), we have

$$
I_k(u)\le\frac{1}{p}\|u\|_{E_k}^{p}-\frac{b_0}{\mu}\int_{-kT}^{kT}|u(t)|^{\mu}\mathrm{d}t-m\int_{-kT}^{kT}|u(t)|^{\mu}\mathrm{d}t+\int_{-kT}^{kT}\big(f_k(t),u(t)\big)\mathrm{d}t+2kTm, \quad (3.6)
$$

where $b_0=\min_{t\in[0,T]}b(t)>0$ as defined in (A2). From (3.6), we have for every $\zeta\in R^{+}$ and $q\in E_1$,

$$
\begin{aligned}
I_1(\zeta q)\le&\frac{1}{p}\zeta^{p}\|q\|_{E_1}^{p}-\frac{b_0}{\mu}\zeta^{\mu}\int_{-T}^{T}|q(t)|^{\mu}\mathrm{d}t-m\zeta^{\mu}\int_{-T}^{T}|q(t)|^{\mu}\mathrm{d}t+\\
&\zeta\|f_1\|_{L^1_{2T}}\|q\|_{L^1_{2T}}+2Tm. \qquad (3.7)
\end{aligned}
$$

Take $Q \in E_1$ such that $Q(\pm T) = 0$. Since $1 < p < \mu$, (3.7) implies that there exists $\zeta_1 \in R^+$ such that $\|\zeta_1 Q\|_{E_1} > \rho$ and $I_1(\zeta_1 Q) < 0$. Set $e_1(t) = \zeta_1 Q(t)$ and

$$e_k(t) = \begin{cases} e_1(t), & |t| < T, \\ 0, & T \le |t| \le kT, \end{cases} \tag{3.8}$$

then $e_k \in E_k$, $\|e_k\|_{E_k} = \|e_1\|_{E_1} > \rho$ and $I_k(e_k) = I_1(e_1) < 0$. By Lemma 2.1, I_k possesses a critical value $c_k \ge \alpha$ given by

$$c_k = \inf_{g\in\Gamma_k} \max_{s\in[0,1]} I_k(g(s)), \tag{3.9}$$

where $\Gamma_k = \{g \in C([0,1], E_k) : g(0) = 0, g(1) = e_k\}$. Hence, $\forall k \in N$, there exists $u_k \in E_k$ such that

$$I_k(u_k) = c_k, \quad I_k'(u_k) = 0. \tag{3.10}$$

Since $c_k > 0$, u_k is a nontrivial solution even if $f_k(t) = 0$. The proof is completed. □

Lemma 3.2 *Let $u_k \in E_k$ be the $2kT$-periodic solution of (HS$_\mathrm{k}$) which satisfies (3.10). Then there exists a positive constant M_2 independent of k such that*

$$\|u_k\|_{L^\infty_{2kT}} \le M_2, \quad \forall k \in N. \tag{3.11}$$

Proof For $\forall k \in N$, let $g_k : [0,1] \to E_k$ be a curve given by $g_k(s) = se_k$, where e_k is defined as in (3.8). Then $g_k \in \Gamma_k$ and $I_k(g_k(s)) = I_1(g_1(s))$ $(k > 1)$ and $s \in [0,1]$. Therefore by (3.9),

$$c_k \le \max_{s\in[0,1]} I_1(g_1(s)) = \widetilde{M}_2,$$

where $\widetilde{M}_2$ is independent of k. Hence, we have

$$I_k(u_k) \le \widetilde{M}_2, \quad I_k'(u_k) = 0.$$

Similarly to (3.3), we obtain

$$\Big(\frac{\mu}{p} - 1\Big)\|u_k\|^p_{E_k} \le \mu\widetilde{M}_2 + (\mu - 1)M_1\|u_k\|_{E_k},$$

so there exists a constant $\overline{M}_2 > 0$ such that

$$\|u_k\|_{E_k} \le \overline{M}_2, \quad \forall k \in N. \tag{3.12}$$

Therefore from (1.6), we get

$$\|u_k\|_{L^\infty_{2kT}} \le C_1\|u_k\|_{E_k} \le C_1\overline{M}_2 := M_2,$$

where C_1 is given in the condition (A$_4$). The proof is completed. □

Lemma 3.3 *Let $u_k \in E_k$ be the $2kT$-periodic solution of system (HS$_\mathrm{k}$) which satisfies (3.11). Then there exists a subsequence $\{u_{kj}\}$ of $\{u_k\}$ convergent to a certain $u_0 \in C^1(R, R^n)$ in $C^1_{loc}(R, R^n)$.*

Proof By (3.11), we know that $\{u_k\}$ is a uniformly bounded sequence. Since $u_k(t)$ is a $2kT$-periodic solution of (HS$_\mathrm{k}$), for every $t \in [-kT, kT]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(|\dot{u}_k(t)|^{p-2}\dot{u}_k(t)\big) = l(t)\Phi_p(u_k(t)) - \nabla F(t, u_k(t)) + f_k(t),$$

hence,

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}\Big(|\dot{u}_k(t)|^{p-2}\dot{u}_k(t)\Big)\Big| \le \max_{t\in[0,T]}|l(t)|M_2^{p-1} + \sup_{t\in[0,T],|x|\le M_2}\big|\nabla F(t,x)\big| + \sup_{t\in R}|f(t)|$$
$$:= M_3, \quad t\in[-kT,kT]. \tag{3.13}$$

Thus we can claim that $\{\dot{u}_k\}$ is uniformly and equicontinuous in a similar way to [1]. So $\{u_k\}$ and $\{\dot{u}_k\}$ are both uniformly bounded and equicontinuous, and we can obtain the existence of subsequence $\{u_{k_j}\}$ convergent to a certain u_0 in $C^1_{loc}(R,R^n)$ by using the Arzelà-Ascoli theorem.

Proof of Theorem 1.1 Step 1. We show that u_0 is a solution of (HS). By Lemmas 3.1 and 3.3, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Phi_p\big(\dot{u}_{k_j}(t)\big)\Big) - l(t)\Phi_p\big(u_{k_j}(t)\big) + \nabla F(t,u_{k_j}(t)) = f_{k_j}(t), \quad t\in[-k_jT,k_jT].$$

Take a,b such that $a<b$. Then there exists $j_0\in N$ such that for all $j>j_0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Phi_p\big(\dot{u}_{k_j}(t)\big)\Big) - l(t)\Phi_p\big(u_{k_j}(t)\big) + \nabla F(t,u_{k_j}(t)) = f(t), \quad t\in[a,b].$$

From Lemma 3.3, it follows that $\dot{u}_{k_j}\to\dot{u}_0$ uniformly on $[a,b]$, and we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Phi_p\big(\dot{u}_0(t)\big)\Big) - l(t)\Phi_p\big(u_0(t)\big) + \nabla F(t,u_0(t)) = f(t), \quad t\in[a,b].$$

Since a and b are arbitrary, we conclude that u_0 satisfies (HS).

Step 2. We prove that $u_0(t)\to 0$ $(t\to\pm\infty)$. From (3.12), we obtain

$$\int_{-k_jT}^{k_jT}\Big[\big|\dot{u}_{k_j}(t)\big|^p + \big(l(t)\Phi_p(u_{k_j}(t)), u_{k_j}(t)\big)\Big]\mathrm{d}t \le \overline{M}_2^p := M_4.$$

For every $l\in N$, there exists $j_1\in N$ such that for $j>j_1$,

$$\int_{-lT}^{lT}\Big[\big|\dot{u}_{k_j}(t)\big|^p + \big(l(t)\Phi_p(u_{k_j}(t)), u_{k_j}(t)\big)\Big]\mathrm{d}t \le M_4.$$

It follows that for each $l\in N$,

$$\int_{-lT}^{lT}\Big[\big|\dot{u}_0(t)\big|^p + \big(l(t)\Phi_p(u_0(t)), u_0(t)\big)\Big]\mathrm{d}t \le M_4.$$

Letting $l\to\infty$ yields

$$\int_{-\infty}^{\infty}\Big[\big|\dot{u}_0(t)\big|^p + \big(l(t)\Phi_p(u_0(t)), u_0(t)\big)\Big]\mathrm{d}t \le M_4.$$

Hence

$$\int_{|t|\ge r}\Big[\big|\dot{u}_0(t)\big|^p + \big(l(t)\Phi_p(u_0(t)), u_0(t)\big)\Big]\mathrm{d}t \to 0, \quad r\to\infty,$$

then we have

$$\int_{|t|\ge r}\Big[\big|\dot{u}_0(t)\big|^p + \big|u_0(t)\big|^p\Big]\mathrm{d}t \to 0, \quad r\to\infty. \tag{3.14}$$

Combining (3.14) with (1.3), we get $u_0(t)\to 0$ as $t\to\pm\infty$.

Step 3. We prove

$$\dot{u}_0(t)\to 0, \quad t\to\pm\infty. \tag{3.15}$$

Since $u_0(t) \to 0$ $(t \to \pm\infty)$, there holds

$$|u_0(t)| \leq M_2 \quad \text{for} \quad t \in R.$$

From this, (HS) and (A4), similarly to (3.13), we have

$$\begin{aligned}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\Big(|\dot{u}_0(t)|^{p-2}\dot{u}_0(t)\Big)\Big| &\leq \max_{t\in[0,T]}|l(t)|M_2^p + \sup_{(t,x)\in[0,T]\times[-M_2,M_2]}\big|\nabla F(t,x)\big| + \sup_{t\in R}\big|f(t)\big|.\\ &:= M_5.\end{aligned}$$

If (3.15) does not hold, then there exist $\varepsilon_0 \in (0, \frac{1}{2})$ and a sequence $\{t_k\}$ such that

$$|t_1| < |t_2| < |t_3| < \cdots, |t_k| + 1 < |t_{k+1}|, \quad k = 1, 2, \ldots,$$

and

$$|\dot{u}_0(t_k)| \geq (2\varepsilon_0)^{\frac{1}{p-1}}, \quad k = 1, 2, \ldots.$$

From this we have for $t \in [t_k, t_k + \frac{\varepsilon_0}{1+M_5}]$

$$\begin{aligned}|\dot{u}_0(t)|^{p-1} &= \Big||\dot{u}_0(t_k)|^{p-2}\dot{u}_0(t_k) + \int_{t_k}^{t}\frac{\mathrm{d}}{\mathrm{d}s}\Big(|\dot{u}_0(s)|^{p-2}\dot{u}_0(s)\Big)\mathrm{d}s\Big|\\ &\geq |\dot{u}_0(t_k)|^{p-1} - \int_{t_k}^{t}\Big|\frac{\mathrm{d}}{\mathrm{d}s}\Big(|\dot{u}_0(s)|^{p-2}\dot{u}_0(s)\Big)\Big|\mathrm{d}s \geq \varepsilon_0.\end{aligned}$$

It follows that

$$\int_{-\infty}^{\infty}|\dot{u}_0(t)|^p\mathrm{d}t \geq \sum_{k=1}^{\infty}\int_{t_k}^{t_k+\frac{\varepsilon_0}{1+M_5}}|\dot{u}_0(t)|^p\mathrm{d}t = \infty,$$

which contradicts (3.14) and so (3.15) holds.

Step 4. In the end, we have to show that if $f \equiv 0$, then $u_0 \not\equiv 0$. Let $Y : [0, +\infty) \to [0, +\infty)$ be given as follows: $Y(0) = 0$ and

$$Y(s) = \max_{t\in[0,T], 0<|u|\leq s}\frac{|(u, \nabla F(t,u))|}{|u|^p} \quad \text{for} \quad s > 0.$$

Then Y is continuous, nondecreasing and $Y(s) \geq 0$ for $s \geq 0$. It folows from (1.6)

$$\int_{-k_jT}^{k_jT}\big(u_{k_j}(t), \nabla F(t, u_{k_j}(t))\big)\mathrm{d}t \leq C_1 Y\big(\|u_{k_j}\|_{L^\infty_{2kT}}\big)\|u_{k_j}\|^p_{E_{k_j}}, \quad \forall\, j \in N. \tag{3.16}$$

Since $I'_{k_j}(u_{k_j})u_{k_j} = 0$, we have

$$\begin{aligned}\int_{-k_jT}^{k_jT}\Big(u_{k_j}(t), \nabla F(t, u_{k_j}(t))\Big)\mathrm{d}t &= \int_{-k_jT}^{k_jT}\Big[\Big(\Phi_p(\dot{u}_{k_j}(t)), \dot{u}_{k_j}(t)\Big) + \Big(l(t)\Phi_p(u_{k_j}(t)), u_{k_j}(t)\Big)\Big]\mathrm{d}t\\ &= \|u_{k_j}\|^p_{E_{k_j}}.\end{aligned} \tag{3.17}$$

By (3.16) and (3.17), it follows that

$$Y\big(\|u_{k_j}\|_{L^\infty_{2k_jT}}\big) \geq \frac{1}{C_1} > 0.$$

If $\|u_{k_j}\|_{L^\infty_{2k_jT}} \to 0$ as $j \to \infty$, we would have $Y(0) \geq \frac{1}{C_1} > 0$, which is a contradiction. Thus there is $\gamma > 0$ such that

$$\|u_{k_j}\|_{L^\infty_{2k_jT}} \geq \gamma, \quad \forall j \in N. \tag{3.18}$$

We can assume that the maximum of u_{k_j} occurs in $[-T,T]$. If $u_0(t)\equiv 0$, then

$$\|u_{k_j}\|_{L^\infty_{2k_jT}}=\max_{t\in[-T,T]}|u_{k_j}(t)|\to 0,$$

which contradicts (3.18). The proof is completed. □

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