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# Up-Embeddability of Graphs with New Degree-Sum of Independent Vertices

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**Abstract** Let $G$ be a $k(k \le 3)$-edge connected simple graph with minimal degree $\ge 3$, girth $g$, $r = \lfloor \frac{g-1}{2} \rfloor$. For any independent set $\{a_1, a_2, \ldots, a_{6/(4-k)}\}$ of $G$, if

$$\sum_{i=1}^{6/(4-k)} d_G(a_i) > \frac{(4-k)\nu(G) - 6(g-2r-\lfloor \frac{k}{3} \rfloor)}{(4-k)(2^r-1)(g-2r)} + \frac{6}{(4-k)}(g-2r-1),$$

then $G$ is up-embeddable.

**Keywords** up-embeddability; maximum genus; independent set.

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## 1. Introduction

Graphs considered here are all connected, finite and undirected. Terminologies and notations not defined in this paper will generally conform to [1].

Let $G = (V(G), E(G))$ be a graph, where $V(G)$, $E(G)$ are the set of vertices and edges. The cardinality of the vertex set of $G$ is denoted by $\nu(G)$. A set $S \subseteq V(G)$ is called an independent set of $G$ if all vertices in $S$ are not adjacent in $G$. The degree $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges of $G$ incident with $v$.

The distance $d_G(u,v)$ between two vertices $u$ and $v$ is the length of the shortest $(u,v)$-path of $G$. $d_G(xy, v) = \min\{d_G(x,v), d_G(y,v)\}$ is the distance between the edge $xy$ and vertex $v$. Clearly,

$$d_G(uv, u) = d_G(uv, v) = d_G(u,u) = 0.$$

For a vertex or an edge $x$ of $G$, we call $N_G^{(i)}(x) = \{v | d_G(x,v) = i, v \in V(G)\}$ the $i$-neighbor set of $x$ in $G$. The girth of $G$ is the length of a shortest cycle in $G$.

The maximum genus, $\gamma_M(G)$ of a graph $G$ is the largest integer $n$ such that there exists a cellular embedding of $G$ on the orientable surface with genus $n$. By Euler Formula, we know that

$$\gamma_M(G) \le \lfloor \frac{\beta(G)}{2} \rfloor,$$

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where $\beta(G) = |E(G)| - |V(G)| + 1$ is the Betti number of $G$. If $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$, then $G$ is called up-embeddable.

For a spanning tree $T$ of $G$, $\xi(G,T)$ denotes the number of components of $G \setminus E(T)$ with odd number of edges. $\xi(G) = \min_T \xi(G,T)$ is the Betti deficiency number of $G$, where the minimum is taken over all spanning trees of $G$.

**Theorem 1.1** ([6, 10]) *Let $G$ be a graph. Then*

*(i)* $\gamma_M(G) = \frac{\beta(G)-\xi(G)}{2}$;

*(ii)* *$G$ is up-embeddable if and only if $\xi(G) \leq 1$.*

For an edge set $A \subseteq E(G)$, $c(G \setminus A)$ denotes the number of components of $G \setminus A$, $b(G \setminus A)$ denotes the number of components of $G \setminus A$ with odd Betti number. In 1981, Nebesky [8] obtained the following combinatorial expression of $\xi(G)$.

**Theorem 1.2** ([8]) *Let $G$ be a graph. Then*

$$\xi(G) = \max_{A \subseteq E(G)} \{c(G \setminus A) + b(G \setminus A) - |A| - 1\}.$$

Let $A \subseteq E(G)$, $F_1, F_2, \ldots, F_l$ be $l$ different components of $G \setminus A$. $E(F_1, F_2, \ldots, F_l)$ denotes the set of edges whose end vertices are in two different components $F_i$ and $F_j$ $(1 \leq i < j \leq l)$. For an induced subgraph $F$ of $G$, $E(F,G)$ denotes the set of edges with one end vertex in $F$ and another not in $F$. If vertex $v \in V(F)$ is the end vertex of $i$ $(i \geq 1)$ edges of $E(F,G)$, then $v$ is called an $i$-touching vertex or touching vertex of $F$.

**Theorem 1.3** ([3]) *Let $G$ be a graph. If $G$ is not up-embeddable, i.e., $\xi(G) \geq 2$, then there exists an edge set $A \subseteq E(G)$ satisfying the following properties:*

*(i)* $c(G \setminus A) = b(G \setminus A) \geq 2$;

*(ii)* *For any component $F$ of $G \setminus A$, $F$ is an induced subgraph of $G$;*

*(iii)* *For any $l$ distinct components $F_{i_1}, \ldots, F_{i_l}$ of $G \setminus A$, $|E(F_{i_1}, \ldots, F_{i_l})| \leq 2l - 3$;*

*(iv)* $\xi(G) = 2c(G \setminus A) - |A| - 1$.

The study on maximum genus of graphs was inaugurated by Nordhaus, Stewart and White [9]. From then on, various classes of graphs have been proved up-embeddable. A formerly known result [10] stated that every 4-edge connected graph is up-embeddable. But, there exists $k$ $(k \leq 3)$-edge connected graphs [5] which are not up-embeddable. Based on this, what kind of restrictions, under which a graph is up-embeddable, are studied extensively. In [4], Huang and Liu first began to consider the up-embeddability of simple graphs via degree-sum of nonadjacent vertices. Later, Chen and Liu [2] extended Huang and Liu's results. In this paper, we obtain the following result which improves the results in paper [2, 4].

**Theorem 1.4** *Let $G$ be a $k$ $(k \leq 3)$-edge connected simple graph with minimal degree $\geq 3$, girth $g$, $r = \lfloor \frac{g-1}{2} \rfloor$. For any independent set $\{a_1, a_2, \ldots, a_{6/(4-k)}\}$ of $G$, if*

$$\sum_{i=1}^{6/(4-k)} d_G(a_i) > \frac{(4-k)\nu(G) - 6(g - 2r - \lfloor \frac{k}{3} \rfloor)}{(4-k)(2^r - 1)(g - 2r)} + \frac{6}{(4-k)}(g - 2r - 1),$$

*then $G$ is up-embeddable.*

To see the lower bound presented in Theorem 1.4 is best possible, let us consider the following infinite family of graphs. Let $H$ be the complete graphs $K_{4t}$ or complete bipartite graphs $K_{2t,2t}$, $t \geq 2$. The graph $G$ is obtained by replacing each vertices of $K_{3,3}$ with $H$, then connecting the edges of $K_{3,3}$ to different vertices of $H$ such that $G$ is 3-edge connected and the girth of $G$ is equal to the girth of $H$. It is not difficult to find an independent set $\{a_1, a_2, \ldots, a_6\}$ of $G$ such that $\sum_{i=1}^{6} d_G(a_i) = \frac{\nu(G)-6(g-2r-1)}{(2^r-1)(g-2r)} + 6(g-2r-1)$. On the other hand, it is easy to check that $\xi(G) = 2$.

## 2. Characterizations of given subgraphs

In the following, we will obtain some properties on the given induced subgraphs.

**Lemma 2.1** *Let $G$ be a simple graph with minimal degree $\geq 3$, girth $g$, $r = \lfloor \frac{g-1}{2} \rfloor$. $H$ is a connected induced subgraph of $G$, $\beta(H) \geq 1$. If $\{u, v\} \subseteq V(H)$ contains all the touching vertices of $H$, then,*

*(i) When $g = 2r+2$, there exists an edge $ab \in E(H)$ such that $\min\{d_H(ab, u), d_H(ab, v)\} \geq r$;*

*(ii) When $g = 2r+1$, there exists a vertex $a \in V(H)$ such that $\min\{d_H(a, u), d_H(a, v)\} \geq r$.*

**Proof** See the proof of Proposition 1 in the paper [7]. $\square$

**Lemma 2.2** *Let $G$ be a simple graph with minimal degree $\geq 3$, girth $g$, $r = \lfloor \frac{g-1}{2} \rfloor$. $H$ is a connected induced subgraph of $G$, $\beta(H) \geq 1$. If $H$ has exactly three 1-touching vertices $u, v, w$, then,*

*(i) When $g = 2r+2$, there exists an edge $ab \in E(H)$ such that*

$$\min\{d_H(ab, u), d_H(ab, v)\} \geq r-1,\ \min\{\max\{d_H(ab, u), d_H(ab, v)\},\ d_H(ab, w)\} \geq r;$$

*(ii) When $g = 2r+1$, there exists a vertex $a \in V(H)$ such that*

$$\min\{d_H(a, u), d_H(a, v)\} \geq r-1,\ \min\{\max\{d_H(a, u), d_H(a, v)\},\ d_H(a, w)\} \geq r.$$

**Proof** See the proof of Proposition 2 in the paper [7]. $\square$

**Lemma 2.3** *Let $G$ be a simple graph with minimal degree $\geq 3$, girth $g$, $r = \lfloor \frac{g-1}{2} \rfloor$. $H$ is a connected induced subgraph of $G$, $\beta(H) \geq 1$. If $|E(H, G)| \leq 2$, then there exists a vertex $a \in V(H)$ such that*

$$d_G(a) = d_H(a) \leq \frac{\nu(H) - g + 2r}{(2^r-1)(g-2r)} + (g-2r-1).$$

**Proof** Clearly, $H$ has at most two touching vertices, assume that $\{u, v\} \subseteq V(H)$ contains all the touching vertices of $H$.

**Case 1** $g = 2r+1$. By Lemma 2.1, there exists a vertex $a \in V(H)$ such that

$$\min\{d_H(a, u), d_H(a, v)\} \geq r.$$

Clearly, $a$ is not the touching vertex of $H$, so $d_G(a) = d_H(a)$.

As the girth of $G$ is $g$, for any $x,y \in N_H^{(i)}(a)$, $x \neq y$, $0 \leq i \leq r-1$, we have

$$xy \notin E(H),\ (N_H^{(i+1)}(a) \cap N_H^{(1)}(x)) \cap (N_H^{(i+1)}(a) \cap N_H^{(1)}(y)) = \emptyset.$$

Or else, the girth of $H$ will be less than $g$. Hence,

$$|N_H^{(0)}(a)| = 1,\ |N_H^{(i)}(a)| \geq d_H(a) \cdot 2^{i-1}, \quad 1 \leq i \leq r.$$

So, we get

$$\nu(H) \geq |\bigcup_{i=0}^{r} N_H^{(i)}(a)| = \sum_{i=0}^{r} |N_H^{(i)}(a)| \geq 1 + \sum_{i=1}^{r} d_H(a) \cdot 2^{i-1} = 1 + d_H(a)(2^r - 1).$$

Combining $g = 2r+1$, by simple calculation, we have

$$d_G(a) = d_H(a) \leq \frac{\nu(H)-1}{2^r-1} = \frac{\nu(H)-g+2r}{(2^r-1)(g-2r)} + (g-2r-1).$$

**Case 2** $g = 2r+2$. By Lemma 2.1, there exists an edge $ab \in E(H)$ such that

$$\min\{d_H(ab,u), d_H(ab,v)\} \geq r.$$

As the girth of $G$ is $g$, for any $x,y \in N_H^{(i)}(ab), x \neq y, 0 \leq i \leq r-1$, we have

$$xy \notin E(H),\ (N_H^{(i+1)}(ab) \cap N_H^{(1)}(x)) \cap (N_H^{(i+1)}(ab) \cap N_H^{(1)}(y)) = \emptyset.$$

Or else, the girth of $H$ will be less than $g$. Hence,

$$|N_H^{(0)}(ab)| = 2,\ |N_H^{(i)}(ab)| \geq (d_H(a) + d_H(b) - 2) \cdot 2^{i-1}, \quad 1 \leq i \leq r.$$

Without loss of generality, let

$$d_H(a) = \min\{d_H(a), d_H(b)\}.$$

So, we obtain

$$\nu(H) \geq \sum_{i=0}^{r} |N_H^{(i)}(ab)| \geq 2 + (d_H(a) + d_H(b) - 2)(2^r - 1) \geq 2 + (2d_H(a) - 2)(2^r - 1).$$

As $a$ is not the touching vertex of $H$, combining $g = 2r+2$, we have

$$d_G(a) = d_H(a) \leq \frac{\nu(H)-2}{2(2^r-1)} + 1 = \frac{\nu(H)-g+2r}{(2^r-1)(g-2r)} + (g-2r-1). \quad \square$$

**Lemma 2.4** *Let $G$ be a simple graph with minimal degree $\geq 3$, girth $g \geq 4$, $r = \lfloor \frac{g-1}{2} \rfloor$. $H$ is a connected induced subgraph of $G$, $\beta(H) \geq 1$. If $|E(H,G)| = 3$, then there exists a vertex $a \in V(H)$ such that*

$$d_G(a) = d_H(a) \leq \frac{\nu(H)-g+2r+1}{(2^r-1)(g-2r)} + (g-2r-1).$$

**Proof** First, when $H$ has at most two touching vertices, from the proof of Lemma 2.3, this result holds.

Second, assume that $H$ has exactly three 1-touching vertices $u, v, w$.

**Case 1** $g = 2r+1 \geq 5$. By Lemma 2.2, there exists a vertex $a \in V(H)$ such that

$$\min\{d_H(a,u), d_H(a,v)\} \geq r-1,\ \min\{\max\{d_H(a,u), d_H(a,v)\},\ d_H(a,w)\} \geq r.$$

Similarly, we have

$$|N_H^{(r)}(a)| \geq d_H(a) \cdot 2^{r-1} - 1,\ |N_H^{(i)}(a)| \geq d_H(a) \cdot 2^{i-1},\quad 1 \leq i \leq r-1.$$

Hence,

$$\nu(H) \geq \sum_{i=0}^{r} |N_H^{(i)}(a)| \geq 1 + \sum_{i=1}^{r} d_H(a) \cdot 2^{i-1} - 1 = d_H(a)(2^r - 1).$$

As $a$ is not the touching vertex of $H$, combining $g = 2r+1$, we obtain

$$d_G(a) = d_H(a) \leq \frac{\nu(H)}{2^r - 1} = \frac{\nu(H) - g + 2r + 1}{(2^r - 1)(g - 2r)} + (g - 2r - 1).$$

**Case 2** $g = 2r + 2$. By Lemma 2.2, there exists an edge $ab \in E(H)$ such that

$$\min\{d_H(ab,u), d_H(ab,v)\} \geq r - 1,\ \ \min\{\max\{d_H(ab,u), d_H(ab,v)\}, d_H(ab,w)\} \geq r.$$

**Subcase 2.1** $g = 2r + 2 \geq 6$. Clearly, $a$, $b$ are not the touching vertex of $H$. Without loss of generality, let

$$d_G(a) \leq d_G(b).$$

Similarly, we have

$$|N_H^{(r)}(ab)| \geq (d_H(a) + d_H(b) - 2) \cdot 2^{r-1} - 1 \geq (2d_G(a) - 2) \cdot 2^{r-1} - 1,$$

$$|N_H^{(i)}(ab)| \geq (d_H(a) + d_H(b) - 2) \cdot 2^{i-1} \geq (2d_G(a) - 2) \cdot 2^{i-1},\quad 1 \leq i \leq r-1.$$

Hence,

$$\nu(H) \geq \sum_{i=0}^{r} |N_H^{(i)}(ab)| \geq 2 + \sum_{i=1}^{r} (2d_G(a) - 2) \cdot 2^{i-1} - 1 = (2d_G(a) - 2)(2^r - 1) + 1.$$

As $g = 2r + 2$, then

$$d_G(a) = d_H(a) \leq \frac{\nu(H) - 1}{2(2^r - 1)} + 1 = \frac{\nu(H) - g + 2r + 1}{(2^r - 1)(g - 2r)} + (g - 2r - 1).$$

**Subcase 2.2** $g = 2r + 2 = 4$. Clearly, we can assume that $a$ is not the touching vertex of $H$.

First, if $d_G(a) > d_G(b)$, then $d_G(a) \geq 4$. Hence, there exists a vertex $a' \in N_H^{(1)}(a) \setminus \{u, v, w\}$ such that

$$\min\{d_H(aa', u), d_H(aa', v), d_H(aa', w)\} \geq 1.$$

Now, without loss of generality, assume that $d_G(a') = \min\{d_G(a'), d_G(a)\}$. So, we have

$$|N_H^{(1)}(aa')| = d_H(a') + d_H(a) - 2 = d_G(a') + d_G(a) - 2 \geq 2d_G(a') - 2.$$

Hence,

$$\nu(H) \geq |N_H^{(0)}(aa')| + |N_H^{(1)}(aa')| \geq 2d_G(a').$$

As $g = 2r + 2 = 4$, we have

$$d_G(a') = d_H(a') \leq \frac{\nu(H)}{2} \leq \frac{\nu(H) - g + 2r + 1}{(2^r - 1)(g - 2r)} + (g - 2r - 1).$$

Secondly, if $d_G(a) \leq d_G(b)$, as $u, v, w$ are 1-touching vertices of $H$, we have

$$|N_H^{(1)}(ab)| = d_H(a) + d_H(b) - 2 \geq d_G(a) + d_G(b) - 3 \geq 2d_G(a) - 3.$$

Hence, we have

$$\nu(H) \geq |N_H^{(0)}(ab)| + |N_H^{(1)}(ab)| \geq 2d_G(a) - 1.$$

As $g = 2r + 2 = 4$, it follows

$$d_G(a) = d_H(a) \leq \frac{\nu(H)+1}{2} = \frac{\nu(H) - g + 2r + 1}{(2^r - 1)(g - 2r)} + (g - 2r - 1). \quad \square$$

## 3. The proof of Theorem 1.4

**Proof of Theorem 1.4** Suppose that graph $G$ is not up-embeddable. There exists an edge set $A \subseteq E(G)$ satisfying the properties (1)–(4) of Theorem 1.3. Define $C(G \setminus A)$ to be the set of components of $G \setminus A$, and

$$B_4 = \{F \mid |E(F,G)| \geq 4, F \in C(G \setminus A)\},$$
$$B_i = \{F \mid |E(F,G)| = i, F \in C(G \setminus A)\}, \quad i = 1,2,3.$$

Obviously,

$$c(G \setminus A) = |B_1| + |B_2| + |B_3| + |B_4|. \tag{1}$$

For each edge $e \in A$, the end vertices of $e$ must belong to two distinct components of $G \setminus A$, because any component $F \in C(G \setminus A)$ is an induced subgraph of $G$, which means that there exist just two components $F_1, F_2 \in C(G \setminus A)$ such that $e \in E(F_1, G)$ and $e \in E(F_2, G)$. On the other hand, each edge $e \in E(F,G)$ must belong to $A$. Thus

$$A = \cup_{F \in C(G \setminus A)} |E(F,G)|$$

and

$$|A| = \frac{1}{2} \sum_{F \in C(G \setminus A)} |E(F,G)| \geq 2|B_4| + \frac{3}{2}|B_3| + |B_2| + \frac{1}{2}|B_1|. \tag{2}$$

Combining Theorem 1.3, Equations (1) and (2), we have

$$\begin{aligned} \xi(G) &= 2c(G \setminus A) - |A| - 1 \\ &\leq 2(|B_4| + |B_3| + |B_2| + |B_1|) - (2|B_4| + \frac{3}{2}|B_3| + |B_2| + \frac{1}{2}|B_1|) - 1 \\ &= \frac{1}{2}|B_3| + |B_2| + \frac{3}{2}|B_1| - 1. \end{aligned}$$

As $G$ is not up-embeddable, i.e., $\xi(G) \geq 2$, we have

$$\frac{1}{2}|B_3| + |B_2| + \frac{3}{2}|B_1| \geq 3. \tag{3}$$

Since $|B_i| = 0$ for $i < k$, simple calculation gives

$$|B_3| + |B_2| + |B_1| \geq \frac{6}{4-k}. \tag{4}$$

Without loss of generality, let

$$|E(F_i, G)| \leq 3, \; F_i \in C(G \setminus A), \quad 1 \leq i \leq 6/(4-k).$$

When $g = 3$ and $k = 3$, $6/(4-k) = 6$. First, assume that each vertex in $F_i$ $(1 \leq i \leq 6)$ is a touching vertex of $F_i$. Since $|E(F_i, G)| = 3$, $V(F_i)$ contains exactly three 1-touching vertices, denoted by $\{x_i, y_i, z_i\}$. Furthermore, suppose $\{x_6z_5, y_6z_4, z_6x_3\} = E(F_6, G)$. As the vertex $z_3$ connects at most one vertex in $V(F_1) \cup V(F_2)$, there are at least 2 vertices in $F_1$ and $F_2$, denoted by $\{z_1, y_1\}$ and $\{z_2, y_2\}$, respectively, which are not adjacent with $z_3$. But, as $|E(F_1, F_2)| \leq 1$, we can assume that $z_1$ and $z_2$ are not adjacent. Now, the vertices set $\{z_1, z_2, \ldots, z_6\}$ is clearly an independent set of $G$. Secondly, if there exists one vertex $u_i$ in some $F_i$ $(1 \leq i \leq 6)$ which is not the touching vertex of $F_i$, then by replacing $z_i$ with $u_i$, we also obtain an independent set of $G$ with 6 vertices. For $k = 1, 2$, by similar discussions, there exist vertices $a_i \in V(F_i)$ $(1 \leq i \leq 6/(4-k))$, where $a_i$ is at most a 1-touching vertex of $F_i$, such that $\{a_1, a_2, \ldots, a_{6/(4-k)}\}$ is an independent set of $G$.

Hence, for $k \leq 3$ and $g = 2r + 1 = 3$, there exists vertex $a_i \in F_i$ $(1 \leq i \leq 6/(4-k))$ such that $\{a_1, a_2, \ldots, a_{6/(4-k)}\}$ is an independent set of $G$, and

$$d_G(a_i) \leq d_{F_i}(a_i) + 1 \leq \nu(F_i) = \frac{\nu(F_i) - g + 2r + 1}{(2^r - 1)(g - 2r)} + (g - 2r - 1), \quad 1 \leq i \leq 6/(4-k). \tag{5}$$

**Case 1** $c(G \setminus A) = 6/(4-k)$. First, when $k \leq 2$, $6/(4-k) = k + 1$. By Theorem 1.3, it is easy to know that

$$|E(F_i, G)| \leq 2, \quad 1 \leq i \leq k + 1.$$

So, by Lemma 2.3, there exist vertices $a_i \in F_i$ $(1 \leq i \leq k+1)$ such that $\{a_1, \ldots, a_{k+1}\}$ is an independent set of $G$, and

$$d_G(a_i) \leq \frac{\nu(F_i) - (g - 2r)}{(2^r - 1)(g - 2r)} + (g - 2r - 1), \quad 1 \leq i \leq k + 1.$$

Hence, we have

$$\begin{aligned}\sum_{i=1}^{k+1} d_G(a_i) &\leq \frac{\sum_{i=1}^{k+1} \nu(F_i) - (k+1)(g - 2r)}{(2^r - 1)(g - 2r)} + (k+1)(g - 2r - 1) \\ &= \frac{\nu(G) - (k+1)(g - 2r)}{(2^r - 1)(g - 2r)} + (k+1)(g - 2r - 1).\end{aligned}$$

But, this contradicts the condition.

Secondly, when $k = 3$, $6/(4-k) = 6$. Combining equation (5) and Lemma 2.4, there exist vertices $a_i \in F_i$ $(1 \leq i \leq 6)$ such that $\{a_1, a_2, \ldots, a_6\}$ is an independent set of $G$, and

$$d_G(a_i) \leq \frac{\nu(F_i) - (g - 2r - 1)}{(2^r - 1)(g - 2r)} + (g - 2r - 1), \quad 1 \leq i \leq 6.$$

Hence, we have

$$\begin{aligned}\sum_{i=1}^{6} d_G(a_i) &\leq \frac{\sum_{i=1}^{6} \nu(F_i) - 6(g - 2r - 1)}{(2^r - 1)(g - 2r)} + 6(g - 2r - 1) \\ &= \frac{\nu(G) - 6(g - 2r - 1)}{(2^r - 1)(g - 2r)} + 6(g - 2r - 1).\end{aligned}$$

But, this also contradicts the condition.

**Case 2** $c(G \setminus A) > 6/(4-k)$. Combining equation (5) and Lemma 2.4, there exist vertices $a_i \in F_i$ $(1 \le i \le 6/(4-k))$ such that $\{a_1, a_2, \dots, a_{6/(4-k)}\}$ is an independent set of $G$, and

$$d_G(a_i) \le \frac{\nu(F_i) - (g-2r-1)}{(2^r-1)(g-2r)} + (g-2r-1), \quad 1 \le i \le 6/(4-k).$$

As $c(G \setminus A) > 6/(4-k)$ and the order of each component of $G \setminus A$ is at least 3, we have

$$\sum_{i=1}^{6/(4-k)} \nu(F_i) \le \nu(G) - 3.$$

Thus,

$$\begin{aligned}\sum_{i=1}^{6/(4-k)} d_G(a_i) &\le \sum_{i=1}^{6/(4-k)} \frac{\nu(F_i) - (g-2r-1)}{(2^r-1)(g-2r)} + \frac{6(g-2r-1)}{4-k} \\ &= \frac{\sum_{i=1}^{6/(4-k)} \nu(F_i) - \frac{6}{4-k}(g-2r-1)}{(2^r-1)(g-2r)} + \frac{6(g-2r-1)}{4-k} \\ &\le \frac{(\nu(G)-3) - \frac{6}{4-k}(g-2r-1)}{(2^r-1)(g-2r)} + \frac{6(g-2r-1)}{4-k}.\end{aligned}$$

But, this also contradicts the condition. Hence, $G$ is up-embeddable. This completes the proof. $\square$

**Corollary 3.1** *Let $G$ be a $k$ $(k \le 3)$-edge connected simple graph with girth $g$, $r = \lfloor \frac{g-1}{2} \rfloor$. If minimal degree $\delta(G) \ge 3$ and*

$$\delta(G) > \frac{(4-k)\nu(G) - 6(g-2r-\lfloor \frac{k}{3} \rfloor)}{6(2^r-1)(g-2r)} + (g-2r-1),$$

*then $G$ is up-embeddable.*

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