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A New Class of Finsler Metrics with Scalar Flag Curvature

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Abstract In this paper, we study a new class of general (α,β)-metrics F defined by a Riemannian metric α, a 1-form β and $\mathcal{C}^\infty$ function $\phi(b^2,s)$. We provide the projective factor of a class of general (α,β)-metrics $F=\alpha\phi(b^2,s)$, and apply these formulae to compute its flag curvature.

Keywords scalar flag curvature; locally projectively flat; general (α,β)-metrics.

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1. Introduction

The (α,β) metrics were first introduced by Matsumoto [1]. They are Finsler metrics built from a Riemannian metric $\alpha=\sqrt{a_{ij}y^iy^j}$, 1-form $\beta=b_i(x)y^i$ and $\mathcal{C}^\infty$ function $\phi(s)$ on a manifold M. A Finsler metric of (α,β)-metrics is given by the form

$$F=\alpha\phi(s),\quad s:=\frac{\beta}{\alpha}.$$

It is known that F is positive and strongly convex on $TM\backslash\{0\}$ if and only if

$$\phi(s)>0,\ \phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0,\ |s|\le b<b_0,$$

where $b=\|\beta\|_\alpha$.

The aim of this paper is to study a new class of Finsler metrics given by [2]

$$F=\alpha\phi(b^2,s),\quad s:=\frac{\beta}{\alpha},\tag{1.1}$$

where $\phi=\phi(b^2,s)$ is a $\mathcal{C}^\infty$ positive function and $b=\|\beta\|_\alpha$.

One important example of (α,β)-metric was given by L. Berwald

$$F=\frac{(\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}.\tag{1.2}$$

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It is a projectively flat Finsler metrics on $\mathbb{B}^n \subset \mathcal{R}^n$ with flag curvature $K = 0$. Berwald's metric can be expressed in the form

$$F = \alpha\phi(b^2, s) = \alpha(\sqrt{1+b^2} + s)^2, \tag{1.3}$$

where

$$\alpha = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{1-|x|^2}, \quad \beta = \frac{\langle x, y\rangle}{(1-|x|^2)^{3/2}}, \tag{1.4}$$

$$s := \frac{\beta}{\alpha}, \quad b^2 = \frac{|x|^2}{1-|x|^2}. \tag{1.5}$$

Then we apply these formulae to discuss a class of general (α, β)-metrics $F = \alpha\phi(b^2, s)$.

Let μ be an arbitrary constant and $\Omega = \mathcal{B}^n(r_\mu)$ where $r_\mu = 1/\sqrt{-\mu}$ if $\mu < 0$ and $r_\mu = +\infty$ if $\mu \geq 0$. Let $|\cdot|$ and $\langle , \rangle$ be the standard Euclidean norm and inner product in $\mathbb{R}^n$, respectively. Define $F : T\Omega \to [0, \infty)$ by

$$\alpha(x, y) := \frac{\sqrt{(1+\mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}}{1+\mu|x|^2}, \tag{1.6}$$

$$\beta(x, y) := \frac{\lambda\langle x, y\rangle + (1+\mu|x|^2)\langle a, y\rangle - \mu\langle a, x\rangle\langle x, y\rangle}{(1+\mu|x|^2)^{3/2}}, \tag{1.7}$$

where λ is an arbitrary constant and $a \in \mathbb{R}^n$ is a constant vector. We obtain the following result:

Theorem 1.1 *Let $F = \alpha\phi(b^2, s) : T\Omega \to [0, \infty)$ be any function given in (1.6) and (1.7). Define a function $\phi(b^2, \frac{\beta}{\alpha}) = (\sqrt{1+b^2} + \frac{\beta}{\alpha})^2$. It has the following properties:*

(1) The norm of β with respect to α is given by

$$b^2 = \|\beta_\alpha\|^2 = \frac{\lambda^2}{1+\mu|x|^2}|x|^2 + \frac{2\lambda}{1+\mu|x|^2}\langle a, x\rangle + |a|^2 - \frac{\mu}{1+\mu|x|^2}\langle a, x\rangle^2. \tag{1.8}$$

(2) F is locally projectively flat, its projective factor P is given by

$$P = \theta + c\alpha\frac{1}{\sqrt{1+b^2}}. \tag{1.9}$$

(3) F is of scalar flag curvature and its flag curvature is given by

$$K = \frac{1}{\sqrt{1+b^2}(\sqrt{1+b^2}+s)^3}(\mu + \frac{c^2}{1+b^2}), \tag{1.10}$$

where

$$\theta = \frac{\alpha_{x^k}y^k}{2\alpha} = -\frac{\mu\langle x, y\rangle}{1+\mu|x|^2},$$

$$c^2 = (1+\mu|x|^2)^{-1}(\lambda - \mu\langle a, x\rangle)^2.$$

Remark Take $\lambda = 1$, $a = 0$, $\mu = -1$ in Theorem 1.1, then $F = \alpha\phi(b^2, s)$ is the Berwald's metric, its projective factor

$$P = \frac{\langle x, y\rangle}{1-|x|^2} + \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{(\sqrt{1-|x|^2})^3},$$

and its flag curvature $K = 0$.

2. General (α, β)-metrics

Definition 2.1 *Let F be a Finsler metric on a manifold M^n. F is called a general (α, β)-metric if it can be expressed as the form $F = \alpha\phi(b^2, s)$ $(s := \frac{\beta}{\alpha})$, where $\|\beta\|_\alpha \leq b_0$ and $\phi = \phi(b^2, s)$ is a positive $\mathcal{C}^\infty$ function.*

Proposition 2.2 *Let M be an n-dimensional mannifold. A function $F = \alpha\phi(b^2, s)$ on TM is a Finsler metric on M for any Riemannian metric α and 1-form β with $\|\beta\|_\alpha < b_0$ if and only if $\phi = \phi(b^2, s)$ is a positive $\mathcal{C}^\infty$ function satisfying*

$$\phi > 0, \quad \phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \tag{2.1}$$

where s and b are arbitrary numbers with $|s| \leq b < b_0$.

Proof It is easy to verify F is a function with regularity and positive homogeneity. In the following we will verify strong covexity: The $n \times n$ Hessian matrix

$$(g_{ij}) := ([\frac{1}{2}F^2]_{y^i y^j}).$$

For the general (α, β)-metric $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$, direct computations yield

$$[F^2]_{y^i} = [\alpha^2]_{y^i}\phi^2 + 2\alpha^2\phi\phi_2 s_{y^i}, \tag{2.2}$$

$$\begin{aligned}[F^2]_{y^i y^j} =&[\alpha^2]_{y^i y^j}\phi^2 + 2[\alpha^2]_{y^i}\phi\phi_2 s_{y^j} + 2[\alpha^2]_{y^j}\phi\phi_2 s_{y^i} + 2\alpha^2[\phi_2]^2 s_{y^i} s_{y^j} + \\ &2\alpha\phi_{22}\phi_2 s_{y^i} s_{y^j} + 2\alpha[\phi_2]^2 s_{y^i y^j}.\end{aligned} \tag{2.3}$$

Direct computations yield

$$g_{ij} = \rho a_{ij} + \rho_o b_i b_j + \rho_1(b_i \alpha_{y^j} + b_j \alpha_{y^i}) - s\rho_1 \alpha_{y^i}\alpha_{y^j}, \tag{2.4}$$

where

$$\rho = \phi(\phi - s\phi_2),\ \rho_0 = \phi\phi_{22} + \phi_2\phi_2,\ \rho_1 = (\phi - s\phi_2)\phi_2 - s\phi\phi_{22}.$$

By Lemma 1.1.1 in [3], we find a formula for $\det(g_{ij})$

$$\det(g_{ij}) = \phi^{n+1}(\phi - s\phi_2)^{n-2}(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})\det(a_{ij}). \tag{2.5}$$

Assume that (2.1) is satisfied. Then by taking $b = s$ in (2.1), we see that the following inequality holds for any s with

$$\phi - s\phi_2 > 0, \quad |s| < b_0. \tag{2.6}$$

Using (2.1), (2.5) and (2.6), we get $\det(g_{ij}) > 0$, namely (g_{ij}) is positive-definite. The converse is obvious, so the proof is omitted here.

By Lemma 1.1.1 in [3], we find a formula for (g^{ij})

$$g^{ij} = \rho^{-1}\left\{a^{ij} + \eta b^i b^j + \eta_0\alpha^{-1}(b^i y^j + b^j y^i) + \eta_1\alpha^{-2}y^i y^j\right\}, \tag{2.7}$$

where $(g^{ij}) = (g_{ij})^{-1}$, $(g_{ij}) = \frac{1}{2}[F^2]_{y^iy^j}$, $(a^{ij}) = (a_{ij})^{-1}$, $b^i = a^{ij}b_j$,

$$\eta = -\frac{\phi_{22}}{(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \quad \eta_0 = -\frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{\phi(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})},$$

$$\eta_1 = \frac{(s\phi + (b^2 - s^2)\phi_2)((\phi - s\phi_2)\phi_2 - s\phi\phi_{22})}{\phi^2(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}.$$

Lemma 2.3 ([2]) *Let $F = \alpha\phi(b^2, s))$ be a general (α, β)-metric on a manifold M with dimension $n \geq 2$. Then F is locally projectively flat if the following conditions hold:*

1) The function $\phi(b^2, s)$ satisfies the following partial differential equation

$$\phi_{22} = 2(\phi_1 - s\phi_{12}). \tag{2.8}$$

2) α is locally projectively flat, β is closed and conformal with respect to α.

Remark Note that ϕ_1 means the derivation of ϕ with respect to the first variable b^2. In this paper, a 1-form is called conformal with respect to a Riemannian metric if its dual vector field with respect to the Riemannian metric is conformal.

Proposition 2.4 *Suppose general (α, β)-metric $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ is a projectively flat Finsler metric, then its projectively factor P is given by*

$$P = \frac{2\alpha^{-1}(\phi - s\phi_2)G^m_\alpha y_m + \phi_2(2b_mG^m_\alpha + r_{00}) + 2\alpha\phi_1(r_0 + s_0)}{2F}, \tag{2.9}$$

where ${G_\alpha}^m$ denotes the spray coefficients of α, $r_{00} = r_{ij}y^iy^j, r_0 = b^jr_{ij}y^i, s_0 = b^js_{ij}y^i$.

Proof Recall that the spray coefficients of a Finsler metric F are given by $G^i = Py^i + Q^i$, where $P = P(x, y)$ is given by

$$P = \frac{F_{x^k}y^k}{2F}. \tag{2.10}$$

For the general (α, β)-metric $F = \alpha\phi(b^2\frac{\beta}{\alpha})$. Direct computations yield

$$F_{x^k} = \alpha_{x^k}\phi + \alpha\phi_1[b^2]_{x^k} + \alpha\phi_2 s_{x^k}, \tag{2.11}$$

$$F_{x^k}y^k = \alpha_{x^k}y^k\phi + \alpha\phi_1[b^2]_{x^k}y^k + \alpha\phi_2 s_{x^k}y^k. \tag{2.12}$$

We have

$$\alpha_{x^k}y^k = \frac{2}{\alpha}G^m_\alpha y_m, \quad s_{x^k} = \frac{1}{\alpha}b_{mk}y^m + \frac{1}{\alpha^2}\{b_m\alpha - sy_m\}\frac{\partial G^m_\alpha}{\partial y^k}, \tag{2.13}$$

$$s_{x^k}y^k = \frac{r_{00}}{\alpha} + \frac{2}{\alpha^2}\{b_m\alpha - sy_m\}{G_\alpha}^m, \quad [b^2]_{x^k}y^k = 2(r_0 + s_0). \tag{2.14}$$

Substituting them into (2.10), by a direct computation we can obtain

$$P = \frac{2\alpha^{-1}(\phi - s\phi_2)G^m_\alpha y_m + \phi_2(2b_mG^m_\alpha + r_{00}) + 2\alpha\phi_1(r_0 + s_0)}{2F}. \tag{2.15}$$

Example 2.4 Consider the Funk metric $F = \alpha\phi(b^2, s)$ on the unit ball $\mathcal{B}^n \subset \mathcal{R}^n$,

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2} + \frac{\langle x, y\rangle}{1 - |x|^2}.$$

Funk metric can be expressed in the form

$$F = \alpha\phi(b^2, s) = \alpha\frac{s + \sqrt{1 - (b^2 - s^2)}}{1 - b^2},$$

where

$$\alpha = |y|,\ \beta = \langle x, y\rangle\,, \quad b^2 = |x|^2.$$

By a direct computation, one obtains

$$\phi_{22} = 2(\phi_1 - s\phi_{12}).$$

α has constant sectional curvature $K = 0$, β is closed and conformal with respect to α. So Funk metric satisfies two conditions of Lemma 2.3. Namely, it is a projectively flat Finsler metrics with $G^i = Py^i$, where

$$P = \frac{1}{2}\Big\{\frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2} + \frac{\langle x, y\rangle}{1 - |x|^2}\Big\},$$

and its flag curvature $K = -\frac{1}{4}$.

3. Proof of Theorem 1.1

A Finsler metric $F = F(x, y)$ on an open domain $\mathcal{U} \subset \mathbb{R}^n$ is said to be projectively flat in $\mathcal{U}$ if all geodesics are straight lines. This is equivalent to $G^i = P(x, y)y^i$, where $G^i = G^i(x, y)$ are the spray coefficients of F, which are given by

$$G^i = \frac{1}{4}g^{il}\left\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\right\}. \tag{3.1}$$

In this case the flag curvature K is a scalar function on $T\mathcal{U}$ given by

$$K = \frac{P^2 - P_{x^m}y^m}{F^2}. \tag{3.2}$$

Set

$$\omega = 1 + \mu|x|^2, \quad \alpha^2 = a_{ij}y^iy^j, \quad \beta = b_iy^i. \tag{3.3}$$

Then

$$a_{ij} = \frac{\delta_{ij}}{\omega} - \frac{\mu x^i x^j}{\omega^2}, \quad b_i = \frac{\lambda x^i + (1 + \mu|x|^2)a^i - \mu\,\langle a, x\rangle\, x^i}{(1 + \mu|x|^2)^{\frac{3}{2}}}. \tag{3.4}$$

By a simple calculation, we get

$$\alpha_{x^k}y^k = -\frac{2\mu\,\langle x, y\rangle}{\omega}\alpha, \tag{3.5}$$

and

$$\alpha_{x^k y^l}y^k - \alpha_l = (\alpha_{x^k}y^k)_{y_l} - 2\alpha_l = 0. \tag{3.6}$$

By G. Hamel Theorem [4], we get α is a projectively flat Finsler metric, and its projectively factor is given by

$$\theta = \frac{\alpha_{x^k}y^k}{2\alpha} = -\frac{\mu\,\langle x, y\rangle}{1 + \mu|x|^2},$$

and sectional curvature of α

$$^{\alpha}K = \frac{\theta^2 - \theta_{x^m} y^m}{\alpha^2} = \frac{\mu\alpha^2}{\alpha^2} = \mu.$$

Write

$$(a^{ij}) = (a_{ij})^{-1}.$$

By Lemma 1.1.1 in [3] we have

$$a^{ij} = \omega(\delta^{ij} + \mu x^i x^j). \tag{3.7}$$

Using (3.4) and (3.7), we get

$$\begin{aligned} b^2 =& \|\beta\|^2{}_{\alpha} = a^{ij} b_i b_j \\ =& \omega(\delta^{ij} + \mu x^i x^j) \frac{\lambda x^i + (1+\mu|x|^2) a^i - \mu \langle a, x\rangle x^i}{(1+\mu|x|^2)^{\frac{3}{2}}} \cdot \\ & \frac{\lambda x^j + (1+\mu|x|^2) a^j - \mu \langle a, x\rangle x^j}{(1+\mu|x|^2)^{\frac{3}{2}}} \\ =& \frac{\lambda^2}{1+\mu|x|^2}|x|^2 + \frac{2\lambda}{1+\mu|x|^2}\langle a, x\rangle + |a|^2 - \frac{\mu}{1+\mu|x|^2}\langle a, x\rangle^2 . \end{aligned} \tag{3.8}$$

By approximate evaluation, if $\mu \geq 0$,

$$b^2 \leq \frac{\lambda^2}{1+\mu|x|^2}|x|^2 + \frac{2\lambda}{1+\mu|x|^2}\langle a, x\rangle + |a|^2. \tag{3.9}$$

Using $1+\mu|x|^2 > 1$, $< a, x > \leq |a||x|$, we obtain

$$b^2 \leq \left(\frac{|\lambda|}{\sqrt{\mu}} + |a|\right)^2.$$

If $\mu < 0$, then

$$b^2 \leq \left(\frac{|\lambda|}{\sqrt{-\mu}} + |a|\right)^2 + |a|^2\sqrt{-\mu}.$$

So b^2 has upper bound. We get $F = \alpha\phi(b^2, s)$ satisfies two conditions of Proposition 2.2 by the above equalities, therefore F is Finsler metric.

Let $b_{i|j}$ denote the coefficients of the covariant derivative of β with respect to α. Let

$$\begin{aligned} r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}), \quad r_{00} = r_{ij} y^i y^j, \quad s^i{}_j = a^{ik} s_{kj}, \\ s^i{}_0 = s^i{}_j y^j, \quad r_i = b^j r_{ij}, \quad s_i = b^j s_{ij}, \quad r_0 = r_i y^i, \\ s_0 = s_i y^i, \quad r^i = a^{ij} r_j, \quad s^i = a^{ij} s_j, \quad r = b^i r_i. \end{aligned} \tag{3.10}$$

It is easy to see that β is closed if and only if $s_{ij} = 0$. We have

$$b_{i|j} = \frac{\partial b_i}{\partial x^j} - b_k \Gamma^k{}_{ij}. \tag{3.11}$$

From (3.3), we get $\frac{\partial\omega}{\partial x^i} = 2\mu x^i$. Together with (3.7) we have

$$\frac{\partial a_{ij}}{\partial x^i} = -\frac{\mu}{\omega^2}(\delta_{il} x_i + \delta_{il} x_j + 2\delta_{ij} x_l) + \frac{4\mu^2}{\omega^3} x_i x_j x_k. \tag{3.12}$$

By (3.7) and (3.12), we get the Christoffel symbols of α.

$$\Gamma^k{}_{ij} = \frac{1}{2}a^{kl}\left(\frac{\partial a_{il}}{\partial x^j} + \frac{\partial a_{jl}}{\partial x^i} - \frac{\partial a_{ij}}{\partial x^l}\right) = -\frac{\mu}{\omega}(x^i\delta_{jk} + x^j\delta_{ik}).$$

Note that b_i satisfies (3.7), we have

$$b_k\Gamma^k{}_{ij} = -\frac{2\mu x^i x^j}{\omega^{5/2}}(\lambda - \mu\langle a, x\rangle) - \frac{\mu\omega}{\omega^{5/2}}(a_i x_j + a_j x_i),$$

$$\frac{\partial b_i}{\partial x^j} = \frac{(\lambda - \mu\langle a, x\rangle)\delta_{ij} - \mu(a_i x_j + a_j x_i)}{\omega^{3/2}} - 3\mu x^i x^j\frac{\lambda - \mu\langle a, x\rangle}{\omega^{5/2}}. \tag{3.13}$$

By (3.11) and (3.13) we get

$$b_{i|j} = \frac{\delta_{ij}}{\omega^{3/2}}(\lambda - \mu\langle a, x\rangle) - \frac{\mu x^i x^j}{\omega^{5/2}}(\lambda - \mu\langle a, x\rangle). \tag{3.14}$$

The last equality implies

$$s_{ij} = 0, \qquad r_{ij} = \omega^{-\frac{1}{2}}(\lambda - \mu\langle a, x\rangle)a_{ij}.$$

So β is closed and conformal with respect to α with conformal factor $c(x) = \omega^{-\frac{1}{2}}(\lambda - \mu\langle a, x\rangle)$.

By a direct calculation, differentiating $F = \alpha(\sqrt{1+b^2} + s)^2$ with respect to b^2, s yields

$$\phi_1 - s\phi_{12} = \phi_{22}. \tag{3.15}$$

We know ϕ satisfies two conditions of Lemma 2.3 by the above equalities. So F is locally projectively flat. It is obvious that

$$r_{00} = c\alpha^2, \quad r_0 = c\beta, \quad r = cb^2,$$

$$r^i = cb^i s^i{}_0, \quad s_0 = 0, \quad s^i = 0. \tag{3.16}$$

Substituting (3.16) into projective factor P in Proposition 2.4 gives

$$\begin{aligned} P =& \frac{2\alpha^{-1}(\phi - s\phi_2)G^m_\alpha y_m + \phi_2(2b_m G^m_\alpha + r_{00}) + 2\alpha\phi_1(r_0 + s_0)}{2F} \\ =& \frac{2\theta\alpha(\phi - s\phi_2) + \phi_2(2\theta\beta + c\alpha^2) + 2c\alpha\phi_1\beta}{2F} \\ =& \frac{2\theta\alpha\phi + c\alpha^2(2\phi_1 s + \phi_2)}{2F} \\ =& \theta + c\alpha\frac{1}{\sqrt{1+b^2}}, \end{aligned} \tag{3.17}$$

where

$$\theta = \frac{\alpha_{x^k}y^k}{2\alpha} = -\frac{\mu\langle x, y\rangle}{1 + \mu|x|^2}. \tag{3.18}$$

By a direct computation, we get

$$P_{x^k}y^k = \theta_{x^k}y^k + c_{x^k}y^k\alpha\frac{1}{\sqrt{1+b^2}} + \alpha_{x^k}y^k c\frac{1}{\sqrt{1+b^2}} - \frac{c\alpha[b^2]_{x^k}y^k}{2(1+b^2)^{\frac{3}{2}}}, \tag{3.19}$$

$$P^2 - P_{x^k}y^k = (\theta^2 - \theta_{x^k}y^k) + (2\alpha\theta c - \alpha_{x^k}y^k c - c_{x^k}y^k\alpha)\frac{1}{\sqrt{1+b^2}} + \frac{c^2\alpha^2}{1+b^2} + \frac{c\alpha[b^2]_{x^k}y^k}{2(1+b^2)^{\frac{3}{2}}}, \tag{3.20}$$

$$c_{x^k}y^k = -\mu\beta. \tag{3.21}$$

By (3.2), (3.20) and (3.21), we get

$$K = \frac{P^2 - P_{x^k}y^k}{F^2} = \frac{1}{\alpha^2\phi^2}\Big\{\alpha^2\mu + \frac{\mu\alpha\beta}{\sqrt{1+b^2}} + (\sqrt{1+b^2}+s)\frac{c^2\alpha^2}{(1+b^2)^{3/2}}\Big\}$$
$$= \frac{1}{\sqrt{1+b^2}(\sqrt{1+b^2}+s)^3}\big(\mu + \frac{c^2}{1+b^2}\big). \quad \square \tag{3.22}$$

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