Journal of Mathematical Research with Applications
Mar., 2014, Vol. 34, No. 2, pp. 127–136
DOI:10.3770/j.issn:2095-2651.2014.02.001
Http://jmre.dlut.edu.cn

The Signless Laplacian Spectral Radii and Spread of Bicyclic Graphs

Fengmei SUN, Ligong WANG*
Department of Applied Mathematics, School of Science, Northwestern Polytechnical University, Shaanxi 710072, *P. R. China*

Abstract The signless Laplacian spread of a graph is defined to be the difference between the largest eigenvalue and the smallest eigenvalue of its signless Laplacian matrix. In this paper, we determine the first to 11th largest signless Laplacian spectral radii in the class of bicyclic graphs with n vertices. Moreover, the unique bicyclic graph with the largest or the second largest signless Laplacian spread among the class of connected bicyclic graphs of order n is determined, respectively.

Keywords bicyclic graph; signless Laplacian; spread; spectral radius.

MR(2010) Subject Classification 05C50; 15A18

1. Introduction

In this paper, let $G=(V,E)$ be a simple graph with n vertices and m edges. If G is connected with $m=n+c-1$, then G is called a c-cyclic graph. Especially, if $c=0,1,2$ or 3, then G is called a tree, a unicyclic graph, a bicyclic graph or a tricyclic graph, respectively. Let $\mathcal{B}_n$ be the class of bicyclic graphs with n vertices. Let $d(u)=d_G(u)$ denote the degree of the vertex u of G. Let $\Delta=\Delta(G)$ and $\delta=\delta(G)$ be the maximum degree and minimum degree of G, respectively. If $d(u)=1$, we call u a pendant vertex of G. The neighbor set of a vertex v is denoted by $N(v)=N_G(v)$. The adjacency matrix of G is the $n\times n$ matrix $A(G)=(a_{ij})$, where $a_{ij}=1$ if and only if v_i and v_j are adjacent in G and $a_{ij}=0$ otherwise. Since $A(G)$ is symmetric, the eigenvalues of $A(G)$ can be arranged as follows: $\lambda_1(G)\geq\cdots\geq\lambda_{n-1}(G)\geq\lambda_n(G)$. The adjacency spread of the graph G is defined as [14]: $SA(G)=\lambda_1(G)-\lambda_n(G)$. The spread of a graph has received much attention. In [24], Petrović determined all connected graphs with adjacency spread at most 4. In [14, 17], Gregory, Liu et al. presented some lower and upper bounds for the spread of a graph. In [16], Li et al. determined the unique graph with maximal spread among all unicyclic graphs on $n(\geq 18)$ vertices with a maximum matching of cardinality k. After then, in [12] and [23], Fan and Petrović et al. determined the maximal spreads among all unicyclic graphs and all bicyclic graphs of given order n, respectively.

Received April 10, 2013; Accepted October 12, 2013
Supported by the National Natural Science Foundation of China (Grant No. 11171273) and Graduate Starting Seed Fund of Northwestern Polytechnical University (Grant No. Z2014173).
* Corresponding author
E-mail address: sunfm402@163.com (Fengmei SUN); lgwangmath@163.com (Ligong WANG)

Let $D(G) = \mathrm{diag}(d(v_1, d(v_2), \ldots, d(v_n)))$ denote the diagonal matrix of vertex degrees of a graph G of order n. The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$. Since $L(G)$ is positive semidefinite, its eigenvalues can be arranged as follows: $\mu_1(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$. The Laplacian spread of the graph G, denoted by $SL(G)$, is defined to be $SL(G) = \mu_1(G) - \mu_{n-1}(G)$. In [13], Fan et al. identified the maximal and minimal Laplacian spreads among all trees of given order. Bao et al. [1] and Li et al. [15] determined the maximal Laplacian spread among all unicyclic graphs of given order by using different methods, respectively. In [26], You and Liu determined the minimal Laplacian spread among all unicyclic graphs on n vertices. In [20], by using different methods from [1, 13, 15, 26], Liu determined the four trees (resp., the three unicyclic graphs), which share the second to fourth (resp., the second to fourth) largest Laplacian spreads among all the trees (resp., connected unicyclic graphs) of given order. In [11], Fan et al. proved that there exist exactly two bicyclic graphs with maximal Laplacian spread among all bicyclic graphs of given order. In [9], Chen and Wang proved that there exist exactly five types of tricyclic graphs with maximal Laplacian spread among all tricyclic graphs of given order.

The signless Laplacian matrix of G is defined as $Q(G) = D(G) + A(G)$. It is well known that $Q(G)$ is also positive semidefinite, and its eigenvalues can be arranged as follows: $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)$. The signless Laplacian spread of the graph G, denoted by $SQ(G)$, is defined to be $SQ(G) = q_1(G) - q_n(G)$. Let $q(G)$ be the signless Laplacian spectral radius of G, namely, $q(G) = q_1(G)$. Let $\Phi(G, x)$ be the signless Laplacian characteristic polynomial of G, that is, $\Phi(G, x) = \det(xI - Q(G))$, simply Q-polynomial of G. Research on signless Laplacian matrices has become popular recently. In [2, 3, 5–8], some properties of signless Laplacian spectra of graphs were studied. In [18], Liu and Liu presented some upper and lower bounds for $SQ(G)$ and determined the unique unicyclic graph with maximal signless Laplacian spread among the class of connected unicyclic graphs of order n. In [21], Liu gave the second to fourth largest signless Laplacian spectral radii and the second to fourth largest signless Laplacian spreads of unicyclic graphs with n vertices. In this paper, we determine the first to 11th largest signless Laplacian spectral radii in the class of bicyclic graphs with n vertices. Moreover, the unique bicyclic graph with the largest or the second largest signless Laplacian spread among the class of connected bicyclic graphs of order n is determined, respectively.

2. The signless Laplacian spectral radii of bicyclic graphs

In this section, we will determine the first to 11th largest signless Laplacian spectral radii in the class of bicyclic graphs with n vertices. We first introduce some preliminary results.

Lemma 2.1 ([22, 25]) *If G is a graph with at least one edge, then $q(G) \geq \mu_1(G) \geq \Delta + 1$. If G is connected, then the first equality holds if and only if G is bipartite, and the second equality holds if and only if $\Delta = n - 1$.*

Lemma 2.2 ([10]) *Let G be a connected graph on $n(n \geq 2)$ vertices. Then $q(G) \leq \max\{d(v) + m(v), v \in V(G)\}$, where $m(v) = \sum_{u \in N(v)} d(u)/d(v)$.*

Lemma 2.3 ([21]) *Suppose $c \geq 1$ and G is a c-cyclic graph on n vertices with $\Delta \leq n-3$. If $n \geq 2c+5$, then $q(G) \leq n-1$.*

When $c = 2$, let G be a bicyclic graph with n vertices and $\Delta \leq n-3$. If $n \geq 9$, by Lemma 2.3, then $q(G) \leq n-1$. From Lemma 2.1, we can obtain that if $\Delta = n-2$, then $q(G) > n-1$. If $\Delta = n-1$, then $q(G) \geq n$. Thus all bicyclic graphs on n vertices with $\Delta = n-2$ or $\Delta = n-1$ are shown in Figure 1. Next we will prove bicyclic graphs with the first to 11th largest signless Laplacian spectral radii in the class of bicyclic graphs with n vertices only are the graphs shown in Figure 1.

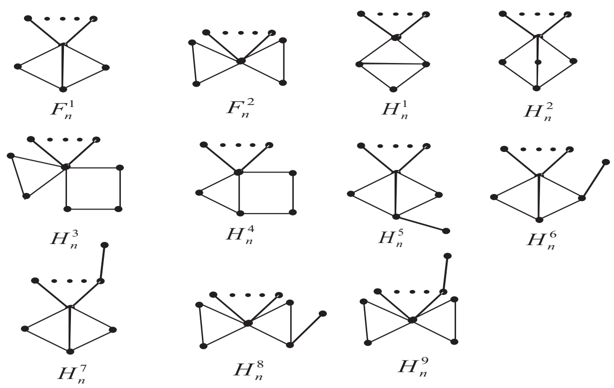

Figure 1 The bicyclic graphs with the first to 11th largest signless Laplacian spectral radii in the class of bicyclic graphs with n vertices

Theorem 2.4 *Let B be a bicyclic graph in $\mathcal{B}_n$ and $n \geq 9$. Then we have the following results.*

(1) If $B \in \mathcal{B}_n$, then $q(B) \leq q(F_n^1)$, where the equality holds if and only if $B \cong F_n^1$.

(2) If $B \in \mathcal{B}_n \backslash \{F_n^1\}$, then $q(B) \leq q(F_n^2)$, where the equality holds if and only if $B \cong F_n^2$.

(3) If $B \in \mathcal{B}_n \backslash \{F_n^1, F_n^2\}$, then $q(B) \leq q(H_n^5)$, where the equality holds if and only if $B \cong H_n^5$.

(4) If $B \in \mathcal{B}_n \backslash \{F_n^1, F_n^2, H_n^5\}$, then $q(B) \leq q(H_n^8)$, where the equality holds if and only if $B \cong H_n^8$.

(5) If $B \in \mathcal{B}_n \backslash \{F_n^1, F_n^2, H_n^5, H_n^8\}$, then $q(B) \leq \min\{q(H_n^9), q(H_n^7)\} \leq \max\{q(H_n^9), q(H_n^7)\}$, where $\begin{cases} q(H_n^7) > q(H_n^9), & \text{if } 9 \leq n \leq 12, \\ q(H_n^7) < q(H_n^9), & \text{if } n > 12. \end{cases}$

(6) If $B \in \mathcal{B}_n \backslash \{F_n^1, F_n^2, H_n^5, H_n^8, H_n^9, H_n^7\}$, then $q(B) \leq q(H_n^1)$, where the equality holds if and only if $B \cong H_n^1$.

(7) If $B \in \mathcal{B}_n \backslash \{F_n^1, F_n^2, H_n^5, H_n^8, H_n^9, H_n^7, H_n^1\}$, then $q(B) \leq q(H_n^4)$, where the equality holds if and only if $B \cong H_n^4$.

(8) If $B \in \mathcal{B}_n \backslash \{F_n^1, F_n^2, H_n^5, H_n^8, H_n^9, H_n^7, H_n^1, H_n^4\}$, then $q(B) \leq q(H_n^3)$, where the equality holds if and only if $B \cong H_n^3$.

(9) If $B \in \mathcal{B}_n \backslash \{F_n^1, F_n^2, H_n^5, H_n^8, H_n^9, H_n^7, H_n^1, H_n^4, H_n^3\}$, then $q(B) \leq q(H_n^2)$, where the equality holds if and only if $B \cong H_n^2$.

(10) If $B \in \mathcal{B}_n \backslash \{F_n^1, F_n^2, H_n^5, H_n^8, H_n^9, H_n^7, H_n^1, H_n^4, H_n^3, H_n^2\}$, then $q(B) \leq q(H_n^6)$, where the equality holds if and only if $B \cong H_n^6$.

In order to prove Theorem 2.4, the following Lemma is needed.

Lemma 2.5 ([19, 21]) *Let G be a graph on $n-k$ $(1 \le k \le n-2)$ vertices with $V(G) = \{v_n, v_{n-1}, \ldots, v_{k+1}\}$. If G' is obtained from G by attaching k new pendant vertices, say, $v_1, v_2, \ldots, v_k$ to v_{k+1}, then $\Phi(Q(G'), x) = (x-1)^k \cdot \det(xI_{n-k} - Q(G) - M_{n-k})$, where $a_{11}(Q(G))$ is corresponding to the vertex v_{k+1}, and $M_{n-k} = \mathrm{diag}\{k + (k/(x-1)), 0, \ldots, 0\}$.*

Example Let F_n^1 be the bicyclic graph as shown in Figure 1. By Lemma 2.5, we have

$$\Phi(Q(F_n^1), x) = (x-1)^{n-4}\det(M),$$

where $M = \begin{bmatrix} x-(n-1)-\frac{n-4}{x-1} & -1 & -1 & -1 \\ -1 & x-2 & -1 & 0 \\ -1 & -1 & x-3 & -1 \\ -1 & 0 & -1 & x-2 \end{bmatrix}$.

By using "Matlab", it easily follows that $\Phi(Q(F_n^1), x) = (x-1)^{n-4}(x-2)(x^3-(n+4)x^2+4nx-8)$. With the similar method, by Lemma 2.5 and using "Matlab", we have the following results.

$$\begin{aligned}
\Phi(Q(F_n^2), x) =& (x-1)^{n-4}(x-3)(x^3-(n+3)x^2+3nx-8). \\
\Phi(Q(H_n^1), x) =& (x-1)^{n-5}(x-2)(x^4-(n+5)x^3+6nx^2-6nx+8). \\
\Phi(Q(H_n^2), x) =& (x-1)^{n-6}(x-2)^2x(x^3-(n+4)x^2+(5n-2)x-3n). \\
\Phi(Q(H_n^3), x) =& (x-1)^{n-6}(x-2)(x^5-(n+6)x^4+(7n+7)x^3-(14n-2)x^2+ \\
& (16+6n)x-8). \\
\Phi(Q(H_n^4), x) =& (x-1)^{n-6}(x^6-(n+8)x^5+(9n+18)x^4-(27n+10)x^3+(31n+ \\
& 10)x^2-(11n+32)x+16). \\
\Phi(Q(H_n^5), x) =& (x-1)^{n-6}(x-2)(x^5-(n+6)x^4+(7n+4)x^3-(11n+2)x^2+ \\
& (4n+16)x-8). \\
\Phi(Q(H_n^6), x) =& (x-1)^{n-5}(x^5-(n+6)x^4+(7n+7)x^3-(13n+3)x^2+(4n+ \\
& 24)x-8). \\
\Phi(Q(H_n^7), x) =& (x-1)^{n-4}(x-2)(x^3-(n+3)x^2+(4n-4)x-8). \\
\Phi(Q(H_n^8), x) =& (x-1)^{n-6}(x^6-(n+8)x^5+(9n+18)x^4-(27n+12)x^3+(30n+ \\
& 29)x^2-(9n+64)x+24). \\
\Phi(Q(H_n^9), x) =& (x-1)^{n-6}(x-3)(x^5-(n+5)x^4+(6n+4)x^3-(10n+2)x^2+ \\
& (3n+24)x-8).
\end{aligned}$$

Because the Q-polynomial of a graph has only real roots, we only deal with the polynomials with real roots in this paper. If $f(x)$ is a polynomial in the variable x, the degree of $f(x)$ is denoted by $\partial(f)$, and the maximum root of the equation $f(x) = 0$ by $q_1(f)$. The following Lemma provides an effective method to compare the largest roots of two polynomials.

Lemma 2.6 ([4]) *Let $f(x)$, $g(x)$ be two monic polynomials with real roots, and $\partial(f) \ge \partial(g)$. If*

$f(x) = q(x)g(x) + r(x)$, where $q(x)$ is also a monic polynomial, and $\partial(r) \le \partial(g)$, $q_1(g) > q_1(q)$, then

(1) When $r(x) = 0$, then $q_1(f) = q_1(g)$;

(2) When $r(x) > 0$ for any x satisfying $x \ge q_1(g)$, then $q_1(f) < q_1(g)$;

(3) When $r(q_1(g)) < 0$, then $q_1(f) > q_1(g)$.

Proof of Theorem 2.4 Note that F_n^1 and F_n^2 are the only two bicyclic graphs with $\Delta = n-1$, and $H_n^1, H_n^2, H_n^3, H_n^4, H_n^5, H_n^6, H_n^7, H_n^8, H_n^9$ are all the bicyclic graphs with $\Delta = n-2$. Assume that $B \in \mathcal{B}_n \backslash \{F_n^1, F_n^2, H_n^1, H_n^2, H_n^3, H_n^4, H_n^5, H_n^6, H_n^7, H_n^8, H_n^9\}$. By Lemmas 2.1 and 2.3, we have $\max\{q(F_n^1), q(F_n^2)\} \ge n > n-1 \ge q(B)$, because $\Delta(B) \le n-3$.

In order to finish the proof of Theorem 2.4, we only need to prove that

$$\begin{aligned} q(F_n^1) > q(F_n^2) > q(H_n^5) > q(H_n^8) > \max\{q(H_n^7), q(H_n^9)\} > \min\{q(H_n^7), q(H_n^9)\} \\ > q(H_n^1) > q(H_n^4) > q(H_n^3) > q(H_n^2) > q(H_n^6), \end{aligned}$$

where

$$\begin{cases} q(H_n^7) > q(H_n^9), & \text{if } 9 \le n \le 12, \\ q(H_n^7) < q(H_n^9), & \text{if } n > 12. \end{cases}$$

Claim 1 $q(F_n^1) > q(F_n^2)$.

For the Q-polynomials of F_n^1 and F_n^2, by Lemma 2.1, we have $q_1(F_n^1) \ge n, q_1(F_n^2) \ge n$. Let $g(x) = x^3 - (n+4)x^2 + 4nx - 8$ and $f(x) = x^3 - (n+3)x^2 + 3nx - 8$. It is easy to see that

$$f(x) = g(x) + x(x-n).$$

Let $r(x) = x - n$. We can obtain $q_1(F_n^1) \ne n$ and $q_1(F_n^2) \ne n$. So when $x \ge q_1(g) > n$ and $n \ge 9$, $r(x) > 0$. By Lemma 2.6, we obtain $q_1(F_n^2) < q_1(F_n^1)$.

Claim 2 $q(F_n^2) > q(H_n^5) > q(H_n^8)$.

For the Q-polynomials of F_n^2 and H_n^5, let $g(x) = x^3 - (n+3)x^2 + 3nx - 8$ and $f(x) = x^5 - (n+6)x^4 + (7n+4)x^3 - (11n+2)x^2 + (4n+16)x - 8$. It is easy to see that

$$f(x) = g(x)(x-1)(x-2) + (n-7)x^3 + 12x^2 - (2n+8)x + 8.$$

Let $r(x) = (n-7)x^3 + 12x^2 - (2n+8)x + 8$. If $x \ge q_1(g) > n-1$, $n \ge 9$, then $r'(x) = 3(n-7)x^2 + 24x - (2n+8) > 0$. Since $r(n-1) > 0$, $r(x) > 0$. By Lemma 2.6, we obtain $q_1(H_n^5) < q_1(F_n^2)$.

For the Q-polynomials of F_n^2 and H_n^8, let $g(x) = x^3 - (n+3)x^2 + 3nx - 8$ and $f(x) = x^6 - (n+8)x^5 + (9n+18)x^4 - (27n+12)x^3 + (30n+29)x^2 - (9n+64)x + 24$. It is easy to see that

$$f(x) = g(x)(x-1)^2(x-3) + x((n-4)x^3 - (5n-20)x^2 + (6n-20)x - 8).$$

Let $r(x) = (n-4)x^3 - (5n-20)x^2 + (6n-20)x - 8$. If $x \ge q_1(g) > n-1$ and $n \ge 9$, then $r(x) > 0$. By Lemma 2.6, we deduce that $q_1(H_n^8) < q_1(F_n^2)$.

For the graphs H_n^5 and H_n^8. If $x \ge n-1$ and $n \ge 9$, then

$$\begin{aligned} \Phi(Q(H_n^8), x) - \Phi(Q(H_n^5), x) =& (x-1)^{n-6}(2x^4 - (2n-2)x^3 + (4n+9)x^2 - \\ & (24+n)x + 8) > 0. \end{aligned}$$

We obtain $q_1(H_n^5) > q_1(H_n^8)$.

Claim 3 $q(H_n^8) > \max\{q(H_n^7), q(H_n^9)\} > \min\{q(H_n^7), q(H_n^9)\} > q(H_n^1)$, where

$$\begin{cases} q(H_n^7) > q(H_n^9), & 9 \le n \le 12, \\ q(H_n^7) < q(H_n^9), & n > 12. \end{cases}$$

For the graphs H_n^8 and H_n^7, if $x \ge n-1$ and $n \ge 9$, then

$$\begin{aligned}\Phi(Q(H_n^7), x) - \Phi(Q(H_n^8), x) =& (x-1)^{n-6}(x^4 - (n+4)x^3 + (6n-9)x^2 - \\ & (7n-16)x + 8) > 0.\end{aligned}$$

We obtain $q_1(H_n^8) > q_1(H_n^7)$.

For the Q-polynomials of H_n^9 and H_n^8, let $g(x) = x^5 - (n+5)x^4 + (6n+4)x^3 - (10n+2)x^2 + (3n+24)x - 8$ and $f(x) = x^6 - (n+8)x^5 + (9n+18)x^4 - (27n+12)x^3 + (30n+29)x^2 - (9n+64)x + 24$. It is easy to see that

$$f(x) = g(x)(x-3) + x(-x^3 + (n+2)x^2 - (3n+1)x + 16).$$

Let $r(x) = -x^3 + (n+2)x^2 - (3n+1)x + 16$. If $x \ge q_1(g) > n-1$ and $n \ge 9$, then $r(q_1(g)) < 0$. By Lemma 2.6, we obtain $q_1(H_n^9) < q_1(H_n^8)$.

For the Q-polynomials of H_n^1 and H_n^9, let $g(x) = x^4 - (n+5)x^3 + 6nx^2 - 6nx + 8$ and $f(x) = x^5 - (n+5)x^4 + (6n+4)x^3 - (10n+2)x^2 + (3n+24)x - 8$. It is easy to see that

$$f(x) = g(x)(x-1) + x(x^3 - (n+1)x^2 + (2n-2)x - 3n + 16).$$

Let $r(x) = x^3 - (n+1)x^2 + (2n-2)x - 3n + 16$. If $x \ge q_1(g) > n-1$ and $n \ge 9$, then $r(x) > 0$. By Lemma 2.6, we obtain $q_1(H_n^9) > q_1(H_n^1)$.

For the graphs H_n^1 and H_n^7. If $x \ge n-1$ and $n \ge 9$, then

$$\Phi(Q(H_n^7), x) - \Phi(Q(H_n^1), x) = (x-1)^{n-5}(x-1)x((n+1)x^2 + (n+1)x + 2n - 4) > 0.$$

We obtain $q_1(H_n^7) > q_1(H_n^1)$.

For the Q-polynomials of H_n^7 and H_n^9, let $g(x) = x^3 - (n+3)x^2 + (4n-4)x - 8$ and $f(x) = x^5 - (n+5)x^4 + (6n+4)x^3 - (10n+2)x^2 + (3n+24)x - 8$. It is easy to see that

$$f(x) = g(x)(x-1)^2 + x(x^2 - (n-1)x - (n-12)).$$

Let $r(x) = x^2 - (n-1)x - (n-12)$. If $x > q_1(g) > n-1$ and $9 \le n \le 12$, then $r(x) > 0$. By Lemma 2.6, we obtain $q_1(H_n^7) > q_1(H_n^9)$. If $n > 12$, then $r(n-1) < 0$. We can calculate the largest root of $g(x)$ between $n-1$ and the largest root of $r(x)$, then $r(q_1(g(x))) < 0$. By Lemma 2.6, $q_1(H_n^9) \ge q_1(H_n^7)$.

Claim 4 $q(H_n^1) > q(H_n^4) > q(H_n^3)$.

For the Q-polynomials of H_n^1 and H_n^4, let $g(x) = x^4 - (n+5)x^3 + 6nx^2 - 6nx + 8$ and $f(x) = x^6 - (n+8)x^5 + (9n+18)x^4 - (27n+10)x^3 + (31n+10)x^2 - (11n+32)x + 16$. It is easy to see that

$$f(x) = g(x)(x-2)(x-1) + x(x^3 - nx^2 + (n+2)x + n - 8).$$

Let $r(x) = x^3 - nx^2 + (n+2)x + n - 8$. If $x \geq q_1(g) > n-1$ and $n \geq 9$, then $r(x) > 0$. By Lemma 2.6, we obtain $q_1(H_n^1) > q_1(H_n^4)$.

For the Q-polynomials of H_n^4 and H_n^3, let $g(x) = x^6 - (n+8)x^5 + (9n+18)x^4 - (27n+10)x^3 + (31n+10)x^2 - (11n+32)x + 16$ and $f(x) = (x-2)(x^5 - (n+6)x^4 + (7n+7)x^3 - (14n-2)x^2 + (6n+16)x - 8)$. It is easy to see that

$$f(x) = g(x) + x(x^3 + (n+2)x^2 - (6n+4)x + n + 8).$$

Let $r(x) = x^3 + (n+2)x^2 - (6n+4)x + n + 8$. If $x \geq q_1(g) > n-1$ and $n \geq 9$, then $r(x) > 0$. By Lemma 2.6, we obtain $q_1(H_n^4) > q_1(H_n^3)$.

Claim 5 $q(H_n^3) > q(H_n^2) > q(H_n^6)$.

For the Q-polynomials of H_n^3 and H_n^6, let $g(x) = x^5 - (n+6)x^4 + (7n+7)x^3 - (14n-2)x^2 + (6n+16)x - 8$ and $f(x) = x^5 - (n+6)x^4 + (7n+7)x^3 - (13n+3)x^2 + (4n+24)x - 8)$. It is easy to see that

$$f(x) = g(x) + x((n-5)x - 2n + 8).$$

Let $r(x) = (n-5)x - 2n + 8$. If $x \geq q_1(g) > n-1$ and $n \geq 9$, then $r(x) > 0$. By Lemma 2.6, we obtain $q_1(H_n^3) > q_1(H_n^6)$.

For the Q-polynomials of H_n^2 and H_n^3, let $g(x) = x^3 - (n+4)x^2 + (5n-2)x - 3n$ and $f(x) = x^5 - (n+6)x^4 + (7n+7)x^3 - (14n-2)x^2 + (6n+16)x - 8$. It is easy to see that

$$f(x) = g(x)x(x-2) + x^3 - (n+2)x^2 + 16x - 8.$$

Let $r(x) = x^3 - (n+2)x^2 + 16x - 8$. For the graph H_n^2, if $n > q_1(g) > n-1$ and $n \geq 9$, then $r(q_1(g)) < 0$. By Lemma 2.6, we obtain $q_1(H_n^3) > q_1(H_n^2)$.

For the Q-polynomials of H_n^6 and H_n^2, let $g(x) = x^5 - (n+6)x^4 + (7n+7)x^3 - (13n+3)x^2 + (4n+24)x - 8)$ and $f(x) = (x-2)x(x^3 - (n+4)x^2 + (5n-2)x - 3n)$. It is easy to see that

$$f(x) = g(x) - x^3 + 7x^2 + (2n-24)x + 8.$$

Let $r(x) = -x^3 + 7x^2 + (2n-24)x + 8$. If $x \geq q_1(g) > n-1$ and $n \geq 9$, then $r(q_1(g)) < 0$. By Lemma 2.6, we obtain $q_1(H_n^2) > q_1(H_n^6)$.

Hence, by Claims 1–5, Theorem 2.4 is proved. $\square$

3. The (second) largest signless Laplacian spreads of bicyclic graphs

In this section, we determine the unique bicyclic graph with the largest or the second largest signless Laplacian spreads among the class of connected bicyclic graphs of order n, respectively.

Theorem 3.1 *If $n \geq 9$ and $B \in \mathcal{B}_n \backslash \{F_n^1, F_n^2\}$, then $SQ(F_n^1) > SQ(F_n^2) > SQ(B)$.*

To prove Theorem 3.1, we need to introduce a lemma as follows.

Lemma 3.2 *Suppose B is a bicyclic graph on n vertices with $\Delta \leq n-3$. If $n \geq 9$, then $SQ(B) \leq n-1$.*

Proof Note that $q_n(B) \geq 0$ and $SQ(B) = q_1(B) - q_n(B) \leq q_1(B)$. We only need to prove

$q_1(B) \le \max\{d(v)+m(v) : v \in V\} \le n-1$. By Lemma 2.2, suppose $d(u)+m(u) = \max\{d(v)+m(v) : v \in V\}$. We consider the following three cases.

Case 1 If $d(u) = 1$, suppose $v \in N(u)$. Then, $d(u)+m(u) = 1+d(v) \le 1+\Delta \le n-2 < n-1$.

Case 2 If $d(u) = 2$, suppose $N(u) = \{w, v\}$. Note that B is a bicyclic graph. Then, $|N(v) \cap N(w)| \le 3$ and $|N(v) \cup N(w)| \le n$. Therefore,

$$d(u)+m(u) = 2+\frac{d(v)+d(w)}{2} \le 2+\frac{n+3}{2} \le n-1.$$

Case 3 Suppose $3 \le d(u) \le n-3$. Note that B has $n+1$ edges. So we have

$$d(u)+m(u) \le d(u)+\frac{2(n+1)-d(u)-2}{d(u)} = d(u)-1+\frac{2n}{d(u)}.$$

Next we will prove that $d(u)-1+\frac{2n}{d(u)} \le n-1$, equivalently, $d(u)(n-d(u)) \ge 2n$. Let $g(x) = (n-x)x$, where $3 \le x \le n-3$. Since $g'(x) = n-2x$ and $3 \le x \le n-3$, we have $g(x) \ge g(3) = g(n-3) = 3n-9 \ge 2n$. Then the result follows. $\square$

Proof of Theorem 3.1 Note that F_n^1 and F_n^2 are the only two bicyclic graphs with $\Delta = n-1$, and $H_n^1, H_n^2, H_n^3, H_n^4, H_n^5, H_n^6, H_n^7, H_n^8, H_n^9$ are all the bicyclic graphs with $\Delta = n-2$. If $B \in \mathcal{B}_n$ is a bicyclic graph on $n(\ge 9)$ vertices with $\Delta(B) \le n-3$, by Lemma 3.2, then $SQ(B) \le n-1$. In order to finish the proof of Theorem 3.1, we will prove firstly the following six claims.

Claim 1 If $n \ge 9$, then $SQ(F_n^1) > n-0.3 > n-1$.

Indeed by Lemma 2.5, we have

$$\Phi(Q(F_n^1), x) = (x-1)^{n-4}(x-2)\varphi_1(x),$$

where $\varphi_1(x) = x^3-(n+4)x^2+4nx-8$. Since $n \ge 9$, $\varphi_1(0) = -8 < 0, \varphi_1(0.3) = 1.11n-8.333 > 0, \varphi_1(3) = 3n-17 > 0, \varphi_1(4) = -8 < 0, \varphi_1(n) = -8 < 0$, and $\varphi_1(n+1) = (n+1)(n-3)-8 > 0$, then it follows that $0 < q_n(F_n^1) < 0.3$ and $n < q_1(F_n^1) < n+1$. Thus,

$$SQ(F_n^1) = q_1(F_n^1) - q_n(F_n^1) > n-0.3.$$

Then we have $SQ(F_n^1) > n-0.3 > n-1$.

Claim 2 If $n \ge 9$, then $SQ(F_n^2) > n-0.35 > n-1$.

Indeed by Lemma 2.5, we have

$$\Phi(Q(F_n^2), x) = (x-1)^{n-4}(x-3)\varphi_2(x)$$

where $\varphi_2(x) = x^3-(n+3)x^2+3nx-8$. Since $n \ge 9$, $\varphi_2(0) = -8 < 0$, $\varphi_2(0.35) = 0.9275n - 8.32463 > 0$, $\varphi_2(1) = 2n-10 > 0$, $\varphi_2(3) = -8 < 0$, $\varphi_2(n) = -8 < 0$ and $\varphi_2(n+1) = n^2-n-10 > 0$, then it follows that $0 < q_n(F_n^2) < 0.35$ and $n < q_1(F_n^2) < n+1$. Thus,

$$SQ(F_n^2) = q_1(F_n^2) - q_n(F_n^2) > n-0.35.$$

Then we have $SQ(F_n^2) > n-0.35 > n-1$.

Claim 3 If $n \ge 9$, then $SQ(H_n^5) < n-0.35 < \min\{SQ(F_n^1), SQ(F_n^2)\}$.

By Lemma 2.5, we have

$$\Phi(Q(H_n^5),x)=(x-1)^{n-6}(x-2)\varphi_3(x),$$

where $\varphi_3(x)=x^5-(n+6)x^4+(7n+4)x^3-(11n+2)x^2+(4n+16)x-8$. Since $n\geq 9$, $\varphi_3(0)=-8<0, \varphi_3(0.4)=0.2624n-1.80736>0, \varphi_3(0.6)=-0.1776n+1.04416<0$, $\varphi_3(1.6)=0.3584n+0.02816>0, \varphi_3(4)=32n-232>0$, $\varphi_3(5)=-5n-103<0$, $\varphi_3(n-1)=-9n^2+38n-37<0$, $\varphi_3(n-0.35)=\frac{13}{20}n^4-\frac{273}{50}n^3+\frac{13621}{4000}n^2+\frac{84557}{5000}n-\frac{45157727}{3200000}>0$ and $\varphi_3(n-0.3)=\frac{7}{10}n^4-\frac{287}{50}n^3+\frac{436}{125}n^2+\frac{82907}{5000}n-\frac{1313903}{100000}>0$, and by Claims 1 and 2, then $SQ(H_n^5)=q_1(H_n^5)-q_n(H_n^5)<q_1(H_n^5)<n-0.35<\min\{SQ(F_n^1),SQ(F_n^2)\}$.

Claim 4 If $n\geq 9$, then $SQ(H_n^8)<n-0.35<\min\{SQ(F_n^1),SQ(F_n^2)\}$.

Similarly, for the graph H_n^8, by Lemma 2.5, we have that

$$\Phi(Q(H_n^8),x)=(x-1)^{n-6}\varphi_4(x),$$

where $\varphi_4(x)=x^6-(n+8)x^5+(9n+18)x^4-(27n+12)x^3+(30n+29)x^2-(9n+64)x+24$. Since $n\geq 9$, $\varphi_4(0)=24>0, \varphi_4(0.4)=-0.30784n+2.65498<0$, $\varphi_4(0.5)=0.15625n-1.35938>0$, $\varphi_4(1)=2n-12>0, \varphi_4(2)=-2n+12<0$, $\varphi_4(3)=12>0, \varphi_4(4)=-4n-24<0, \varphi_4(n-1)=-9n^3+77n^2-200n+156<0$, $\varphi_4(n-0.35)=\frac{13}{20}n^5-\frac{559}{80}n^4+\frac{14029}{800}n^3+\frac{54221}{3200}n^2-\frac{53955731}{640000}n+\frac{3249981969}{64000000}>0$ and $\varphi_4(n-0.3)=\frac{7}{10}n^5-\frac{147}{20}n^4+\frac{1809}{100}n^3+\frac{16499}{1000}n^2-\frac{322873}{4000}n-\frac{46299969}{1000000}>0$, and by Claims 1 and 2, then $SQ(H_n^8))=q_1(H_n^8)-q_n(H_n^8)<q_1(H_n^8)<n-0.35<\min\{SQ(F_n^1),SQ(F_n^2)\}$.

Claim 5 If $n\geq 9$, $B\in\mathcal{B}_n$ and $\Delta(B)=n-2$, then $SQ(B)<n-0.35<\min\{SQ(F_n^1), SQ(F_n^2)\}$.

Indeed if $n\geq 9$, $B\in\mathcal{B}_n$ and $\Delta(B)=n-2$, then B is one of the bicyclic graphs H_n^1,H_n^2,H_n^3,H_n^4, H_n^5,H_n^6,H_n^7,H_n^8 and H_n^9. By Theorem 2.4, Claims 1, 2, 3 and 4, we have $SQ(B)=q_1(B)-q_n(B)<q_1(B)<n-0.35<\min\{SQ(F_n^1),SQ(F_n^2)\}$.

Claim 6 If $n\geq 9$, then $SQ(F_n^2)<SQ(F_n^1)$.

Indeed by Lemma 2.5, we have $\Phi(Q(F_n^1),x)=(x-1)^{n-4}(x-2)\varphi_1(x)$ and $\Phi(Q(F_n^2),x)=(x-1)^{n-4}(x-3)\varphi_2(x)$, where $\varphi_1(x)=x^3-(n+4)x^2+4nx-8$ and $\varphi_2(x)=x^3-(n+3)x^2+3nx-8$.

By Theorem 2.4, we have $q(F_n^1)>q(F_n^2)$. Next we will prove $q_n(F_n^1)<q_n(F_n^2)$ as follows.

According to the proof of Theorem 2.4, suppose

$$F(x)=\Phi(Q(F_n^2),x)-\Phi(Q(F_n^1),x)=(x-1)^{n-4}\psi(x),$$

where $\psi(x)=x^2-nx+8$.

When $0<x<\frac{n-\sqrt{n^2-32}}{2}$, $n\geq 9$, $\psi(x)>0$. For $\varphi_1(x)=x^3-(n+4)x^2+4nx-8$ and $\varphi_2(x)=x^3-(n+3)x^2+3nx-8$, $\varphi_1(0)=-8<0$ and $\varphi_1(\frac{n-\sqrt{n^2-32}}{2})=4\sqrt{n^2-32}-4n+24>0$, $\varphi_2(0)=-8<0$ and $\varphi_2(\frac{n-\sqrt{n^2-32}}{2})=4\sqrt{n^2-32}-4n+16>0$, then $0<q_n(F_n^1)<\frac{n-\sqrt{n^2-32}}{2}$ and $0<q_n(F_n^2)<\frac{n-\sqrt{n^2-32}}{2}$.

Case 1 When n is odd, $0<x<\frac{n-\sqrt{n^2-32}}{2}\leq 1$ and $n\geq 9$, then $(x-1)^{n-4}<0$ and $\psi(x)>0$, then $F(x)<0$. We can obtain $q_n(F_n^1)<q_n(F_n^2)$.

Case 2 When n is even, $0<x<\frac{n-\sqrt{n^2-32}}{2}\leq 1$ and $n\geq 9$, then $(x-1)^{n-4}>0$ and $\psi(x)>0$, then $F(x)>0$. We also obtain $q_n(F_n^1)<q_n(F_n^2)$.

By combining with the above arguments, $SQ(F_n^1)=q_1(F_n^1)-q_n(F_n^1)>q_1(F_n^2)-q_n(F_n^2)=SQ(F_n^2)$.

By Lemma 3.2, and Claims 1–6, Theorem 3.1 is proved. $\square$

Acknowledgements The authors are grateful to the referees for helpful comments that improved the paper.

References

[1] Yanhong BAO, Yingying TAN, Yizheng FAN. *The Laplacian spread of unicyclic graphs.* Appl. Math. Lett., 2009, **22**(7): 1011–1015.

[2] D. M. CARDOSO, D. CVETKOVIĆ, P. ROWLINSON, et al. *A sharp lower bound for the least eigenvalue of the signless Laplacian of a non-bipartite graph.* Linear Algebra Appl., 2008, **429**(11-12): 2770–2780.

[3] D. M. CVETKOVIĆ, M. DOOB, H. SACHS. *Spectra of Graphs-Theory and Application.* V. E. B. Deutscher Verlag der Wissenschaften, Berlin, 1980.

[4] An CHANG. *On the largest eigenvalue of a tree with perfect matchings.* Discrete Math., 2003, **269**(1-3): 45–63.

[5] D. CVETKOVIĆ, P. ROWLINSON, S. K. SIMIĆ. *Signless Laplacians of finite graphs.* Linear Algebra Appl., 2007, **423**(1): 155–171.

[6] D. CVETKOVIĆ, S. K. SIMIĆ. *Towards a spectral theory of graphs based on the signless Laplacian (I).* Publ. Inst. Math. (Beograd) (N.S.), 2009, **85**(99): 19–33.

[7] D. CVETKOVIĆ, S. K. SIMIĆ. *Towards a spectral theory of graphs based on the signless Laplacian (II).* Linear Algebra Appl., 2010, **432**(9): 2257–2272.

[8] D. CVETKOVIĆ, S. K. SIMIĆ. *Towards a spectral theory of graphs based on the signless Laplacian (III).* Appl. Anal. Discrete Math., 2010, **4**(1): 156–166.

[9] Yanqing CHEN, Ligong WANG. *The Laplacian spread of tricyclic graphs.* Electron. J. Combin., 2009, **16**(1): 1–18.

[10] K. C. DAS. *The Laplacian spectrum of a graph.* Comput. Math. Appl., 2004, **48**(5-6): 715–724.

[11] Yizheng FAN, Shuangdong LI, Yingying TAN. *The Laplacian spread of bicyclic graphs.* J. Math. Res. Exposition, 2010, **30**(1): 17–28.

[12] Yizheng FAN, Yi WANG, Yubin GAO. *Minimizing the least eigenvalues of unicyclic graphs with application to spectral spread.* Linear Algebra Appl., 2008, **429**(2-3): 577–588.

[13] Yizheng FAN, Jing XU, Yi WANG, et al. *The Laplacian spread of a tree.* Discrete Math. Theor. Comput. Sci., 2008, **10**(1): 79–86.

[14] D. A. GREGORY, D. HERSHKOWITZ, S. J. KIRKLAND. *The spread of the spectrum of a graph.* Linear Algebra Appl., 2001, **332/334**: 23–35.

[15] Jianxi LI, W. C. SHIU, W. H. CHAN. *Some results on the Laplacian eigenvalues of unicyclic graphs.* Linear Algebra Appl., 2009, **430**(8-9): 2080–2093.

[16] Xueliang LI, Jianbin ZHANG, Bo ZHOU. *The spread of unicyclic graphs with given size of maximum matchings.* J. Math. Chem., 2007, **42**(4): 775–788.

[17] Bolian LIU, Muhuo LIU. *On the spread of the spectrum of a graph.* Discrete Math., 2009, **309**(9): 2727–2732.

[18] Muhuo LIU, Bolian LIU. *The signless Laplacian spread.* Linear Algebra Appl., 2010, **432**(2-3): 505–514.

[19] Muhuo LIU, Bolian LIU. *A computer-aided method to calculate characteristic polynomials of graphs on n vertices.* Numer. Math. J. Chinese Univ., 2012, **34**(3): 260–266. (in Chinese)

[20] Muhuo LIU. *On the Laplacian spread of trees and unicyclic graphs.* Kragujevac J. Math., 2010, **34**: 151–159.

[21] Muhuo LIU. *Some results on the signless Laplacian spectra of unicyclic graphs.* International Scholarly Research Network Algebra, 2011, 1–10, doi:10.5402/2011/282940.

[22] R. MERRIS. *Laplacian matrices of graphs: a survey.* Linear Algebra Appl., 1994, **197/198**: 143–176.

[23] M. PETROVIĆ, B. BOROVIĆANIN, T. ALEKSIĆ. *Bicyclic graphs for which the least eigenvalue is minimum.* Linear Algebra Appl., 2009, **430**(4): 1328–1335.

[24] M. PETROVIĆ. *On graphs whose spectral spread does not exceed* 4. Publ. Inst. Math. (Beograd) (N.S.), 1983, **34**(48): 169–174.

[25] Yongliang PAN. *Sharp upper bounds for the Laplacian graph eigenvalues.* Linear Algebra Appl., 2002, **355**(1-3): 287–295.

[26] Zhifu YOU, Bolian LIU. *The minimum Laplacian spread of unicyclic graphs.* Linear Algebra Appl., 2010, **432**(2-3): 499–504.