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Expansion Formulas for Orthogonal Projectors onto Ranges of Row Block Matrices

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Abstract We give in this note some expansion formulas for the orthogonal projectors onto the range of the row block matrix $[\,A,\ B\,]$, and use the expansion formulas to examine relations among the orthogonal projectors onto the ranges of A, B and $[\,A,\ B\,]$. In particular, we present some identifying conditions for a pair of orthogonal projectors of the same size to commute.

Keywords Moore-Penrose inverse; orthogonal projector; equality; inequality; rank formula.

MR(2010) Subject Classification 15A03; 15A09; 15A27

1. Introduction

Throughout this note, $\mathbb{C}^{m\times n}$ denotes the set of all $m\times n$ complex matrices; A^*, $r(A)$ and $\mathscr{R}(A)$ stand for the conjugate transpose, rank, range (column space) of a matrix $A\in\mathbb{C}^{m\times n}$, respectively; $[\,A,\ B\,]$ denotes a row block matrix consisting of A and B. We write $A>0$ ($A\geqslant 0$) if A is Hermitian positive definite (positive semi-definite). Two Hermitian matrices A and B of the same size are said to satisfy the inequality $A>B$ ($A\geqslant B$) in the Löwner partial ordering if $A-B$ is positive definite (positive semi-definite). The inertia of Hermitian matrix A is defined to be the triplet $\mathrm{In}(A)=\{\,i_+(A),\ i_-(A),\ i_0(A)\,\}$, where $i_+(A)$, $i_-(A)$ and $i_0(A)$ are the numbers of the positive, negative and zero eigenvalues of A counted with multiplicities, respectively, and $s(A)=i_+(A)-i_-(A)$.

The Moore-Penrose inverse of $A\in\mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique matrix $X\in\mathbb{C}^{n\times m}$ satisfying the four matrix equations

$$\text{(i)}\ \ AXA=A,\quad \text{(ii)}\ \ XAX=X,\quad \text{(iii)}\ \ (AX)^*=AX,\quad \text{(iv)}\ \ (XA)^*=XA.$$

A matrix X is called a generalized inverse of A, denoted by $X=A^-$, if it satisfies (i); the collection of all generalized inverses of A is denoted by $\{A^-\}$.

A matrix $A\in\mathbb{C}^{m\times m}$ is called an orthogonal projector if it is both idempotent and Hermitian, i.e., $A^2=A=A^*$. A matrix $X\in\mathbb{C}^{m\times m}$ is called the orthogonal projector onto the range $\mathscr{R}(A)$ of $A\in\mathbb{C}^{m\times n}$, denoted by $X=P_A$, if it satisfies

$$\mathscr{R}(X)=\mathscr{R}(A)\ \ \text{and}\ \ X^2=X=X^*.$$

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It can be seen from the definition of the Moore-Penrose inverse that the product $AA^{\dagger}$ is the orthogonal projector onto $\mathscr{R}(A)$, i.e., $P_A = AA^{\dagger}$; while $P_A^{\perp} = I_m - AA^{\dagger}$ is the orthogonal projector onto the null space of A^*.

Orthogonal projectors are fundamental objects of study in matrix theory, which play important roles in the study of matrix factorizations of Hermitian matrices and matrix computations. Various expressions or equalities consisting of orthogonal projectors may occur in matrix theory and applications. In particular, much attention was paid to orthogonal projectors onto the ranges of row block matrices and their submatrices [1, 3]. The purpose of this note is to revisit the orthogonal projectors onto the ranges of the row block matrix $[\,A,\ B\,]$ and its two submatrices A and B. We shall give some new expansion formulas for the orthogonal projectors onto the ranges of the row block matrix $[\,A,\ B\,]$ and use the expansion formulas to examine the relations among the orthogonal projectors onto the ranges of A, B and $[\,A,\ B\,]$. In particular, we give some identifying conditions for a pair of orthogonal projectors of the same size to commute.

The following are some known results on ranks of matrices, which will be used in the latter part of this paper.

Lemma 1.1 ([2]) *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times k}$. Then, the following rank formulas hold*

$$r[\,A,\ B\,] = r(A) + r[\,(\,I_m - AA^{\dagger}\,)B\,] = r(B) + r[\,(\,I_m - BB^{\dagger}\,)A\,], \tag{1.1}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r[\,(\,I_m - BB^{\dagger}\,)A(\,I_n - C^{\dagger}C\,)\,], \tag{1.2}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r(\,D - CA^{\dagger}B\,), \quad \text{if } \mathscr{R}(B) \subseteq \mathscr{R}(A) \text{ and } \mathscr{R}(C^*) \subseteq \mathscr{R}(A^*). \tag{1.3}$$

Lemma 1.2 *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times m}$. Then, the following rank formula holds*

$$r\begin{bmatrix} AA^{\dagger} & B \\ C & 0 \end{bmatrix} = r\begin{bmatrix} CB & 0 & C \\ 0 & 0 & A^* \\ B & A & 0 \end{bmatrix} - r(A). \tag{1.4}$$

Proof Applying (1.3) and $AA^{\dagger} = A(A^*A)^{\dagger}A^*$ to the left-hand side of (1.4), and simplifying by elementary matrix operations, we obtain

$$\begin{aligned} r\begin{bmatrix} AA^{\dagger} & B \\ C & 0 \end{bmatrix} &= r\begin{bmatrix} -A^*A & A^* & 0 \\ A & 0 & B \\ 0 & C & 0 \end{bmatrix} - r(A) = r\begin{bmatrix} 0 & A^* & A^*B \\ A & 0 & B \\ 0 & C & 0 \end{bmatrix} - r(A) \\ &= r\begin{bmatrix} 0 & A^* & 0 \\ A & 0 & B \\ 0 & C & -CB \end{bmatrix} - r(A) = r\begin{bmatrix} CB & 0 & C \\ 0 & 0 & A^* \\ B & A & 0 \end{bmatrix} - r(A), \end{aligned}$$

establishing (1.4). $\square$

Lemma 1.3 ([11]) *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times k}$. Then, the following rank formulas hold*

$$r(\, P_A - P_B\,) = 2r[\, A,\, B\,] - r(A) - r(B), \tag{1.5}$$

$$r(\, P_AP_B - P_BP_A\,) = 2r[\, A,\, B\,] - 2r(A) - 2r(B) + 2r(A^*B). \tag{1.6}$$

Lemma 1.4 ([9]) *Let $P,\, P_1,\, P_2 \in \mathbb{C}^{m\times m}$ be three orthogonal projectors and assume that*

$$\mathscr{R}(P_1) \subseteq \mathscr{R}(P) \ \text{ and } \ \mathscr{R}(P_2) \subseteq \mathscr{R}(P). \tag{1.7}$$

Then, the following hold.

(a) $P - P_1 - P_2$ satisfies the following equalities

$$i_+(\, P - P_1 - P_2\,) = r(P) - r(P_1) - r(P_2) + r(P_1P_2), \tag{1.8}$$

$$i_-(\, P - P_1 - P_2\,) = r(P_1P_2), \tag{1.9}$$

$$r(\, P - P_1 - P_2\,) = r(P) - r(P_1) - r(P_2) + 2r(P_1P_2), \tag{1.10}$$

$$s(\, P - P_1 - P_2\,) = r(P) - r(P_1) - r(P_2). \tag{1.11}$$

Hence, the following hold.

(i) $P - P_1 - P_2$ is nonsingular if and only if $r(P) = r(P_1) + r(P_2) - 2r(P_1P_2) + m$.

(ii) $P - P_1 - P_2 \geqslant 0$ if and only if $P_1P_2 = 0$.

(iii) $P - P_1 - P_2 \leqslant 0$ if and only if $r(P) = r(P_1) + r(P_2) - r(P_1P_2)$.

(iv) $P = P_1 + P_2$ if and only if $P_1P_2 = 0$ and $r(P) = r(P_1) + r(P_2)$.

(v) The signature of $P - P_1 - P_2$ is zero if and only if $r(P) = r(P_1) + r(P_2)$.

(b) $2P - P_1 - P_2$ satisfies the following equalities

$$i_+(\, 2P - P_1 - P_2\,) = r(\, 2P - P_1 - P_2\,) = r(P) - r(P_1) - r(P_2) + r[\, P_1,\, P_2\,]. \tag{1.12}$$

Hence, $2P = P_1 + P_2$ if and only if $r(P) = r(P_1) + r(P_2) - r[\, P_1,\, P_2\,]$.

Lemma 1.5 ([5, 10]) *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$ be given. Then the minimal rank of $A - BXC$ with respect to $X \in \mathbb{C}^{k\times l}$ is given by the following closed-form formula*

$$\min_{X\in\mathbb{C}^{k\times l}} r(\, A - BXC\,) = r[\, A,\, B\,] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{1.13}$$

The matrix X satisfying (1.13) was also given in [5, 10].

2. Main results

We first show a group of results on the Moore-Penrose inverse of product of two orthogonal projectors.

Lemma 2.1 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times k}$. Then,*

(a) The following expansion formulas hold:

$$(P_AP_B)^{\dagger} = P_BP_A - P_B(P_B^{\perp}P_A^{\perp})^{\dagger}P_A, \tag{2.1}$$

$$(P_AP_B)(P_AP_B)^{\dagger} = P_AP_BP_A - P_AP_B(P_B^{\perp}P_A^{\perp})^{\dagger}P_A, \tag{2.2}$$

$$(P_AP_B)^{\dagger}(P_AP_B) = P_BP_AP_B - P_B(P_B^{\perp}P_A^{\perp})^{\dagger}P_AP_B. \tag{2.3}$$

(b) The following inequalities hold:

$$(P_AP_B)(P_AP_B)^{\dagger} \geqslant P_AP_BP_A, \tag{2.4}$$

$$(P_AP_B)^{\dagger}(P_AP_B) \geqslant P_BP_AP_B. \tag{2.5}$$

(c) The following rank expansion formulas hold:

$$r[\,(P_AP_B)^{\dagger} - P_BP_A\,] = r[\,A,\, B\,] - r(A) - r(B) + r(A^*B), \tag{2.6}$$

$$r[\,(P_AP_B)(P_AP_B)^{\dagger} - P_AP_BP_A\,] = r[\,A,\, B\,] - r(A) - r(B) + r(A^*B), \tag{2.7}$$

$$r[\,(P_AP_B)^{\dagger}(P_AP_B) - P_BP_AP_B\,] = r[\,A,\, B\,] - r(A) - r(B) + r(A^*B). \tag{2.8}$$

Proof Eq. (2.1) was shown in [7, 8]. Pre- and post-multiplying P_AP_B yield (2.2) and (2.3), respectively. Recall that $P_AP_BP_A \geqslant 0$ and $I_m - P_AP_BP_A \geqslant 0$. Then, we have

$$(P_AP_B)(P_AP_B)^{\dagger} - P_AP_BP_A = (P_AP_B)(P_AP_B)^{\dagger}(\,I_m - P_AP_BP_A\,)(P_AP_B)(P_AP_B)^{\dagger} \geqslant 0, \tag{2.9}$$

$$(P_AP_B)^{\dagger}(P_AP_B) - P_BP_AP_B = (P_AP_B)^{\dagger}(P_AP_B)(\,I_m - P_BP_AP_B\,)(P_AP_B)^{\dagger}(P_AP_B) \geqslant 0, \tag{2.10}$$

establishing (2.4) and (2.5). Eq. (2.6) was shown in [8], while (2.7) and (2.8) follow from (2.6). □

Let $M = [\,A,\, B\,]$, $A_1 = P_B^{\perp}P_A$ and $B_1 = P_A^{\perp}P_B$. Then,

$$M\begin{bmatrix} I_n & -A^{\dagger}P_B \\ 0 & B^{\dagger} \end{bmatrix} = [\,A,\, B_1\,], \quad M\begin{bmatrix} A^{\dagger} & 0 \\ -B^{\dagger}P_A & I_k \end{bmatrix} = [\,A_1,\, B\,].$$

Also note from (1.1) that $r(M) = r(A) + r(B_1) = r(A_1) + r(B)$. In consequence,

$$\mathscr{R}(M) = \mathscr{R}[\,A,\, B_1\,] = \mathscr{R}[\,P_A,\, P_{B_1}\,] = \mathscr{R}(\,P_A + P_{B_1}\,), \tag{2.11}$$

$$\mathscr{R}(M) = \mathscr{R}[\,A_1,\, B\,] = \mathscr{R}[\,P_{A_1},\, P_B\,] = \mathscr{R}(\,P_{A_1} + P_B\,). \tag{2.12}$$

Also note that both $P_AP_{B_1} = P_BP_{A_1} = 0$. So that

$$(\,P_A + P_{B_1}\,)^2 = P_A + P_{B_1}, \quad (\,P_B + P_{A_1}\,)^2 = P_B + P_{A_1}. \tag{2.13}$$

Thus, both $P_A + P_{B_1}$ and $P_{A_1} + P_B$ are orthogonal projectors, and P_M can be decomposed as

$$P_M = P_A + P_{B_1} = P_A + B_1B_1^{\dagger}, \tag{2.14}$$

$$P_M = P_{A_1} + P_B = P_B + A_1A_1^{\dagger}, \tag{2.15}$$

which were due to Rao and Yanai [4], see also [3]. Two further expansion formulas derived from (2.13) and (2.14) for P_M are given below.

Theorem 2.2 *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, and define $M = [\,A,\, B\,]$, $A_1 = P_B^{\perp}P_A$ and $B_1 = P_A^{\perp}P_B$. Then, the following hold.*

(a) P_M can be decomposed as

$$P_M = P_A + B_1B_1^* - B_1A_1^{\dagger}P_A^{\perp}, \tag{2.16}$$

$$P_M = P_B + A_1A_1^* - A_1B_1^{\dagger}P_B^{\perp}. \tag{2.17}$$

(b) P_M *satisfies the following inequalities*

$$P_M \geqslant P_A + B_1B_1^* = P_A + P_B - P_AP_B - P_BP_A + P_AP_BP_A, \tag{2.18}$$

$$P_M \geqslant P_B + A_1A_1^* = P_A + P_B - P_AP_B - P_BP_A + P_BP_AP_B. \tag{2.19}$$

(c) P_M *satisfies the following rank and inertia equalities*

$$r(\, P_M - P_A - B_1B_1^* \,) = r(M) - r(A) - r(B) + r(A^*B), \tag{2.20}$$

$$r(\, P_M - P_B - A_1A_1^* \,) = r(M) - r(A) - r(B) + r(A^*B), \tag{2.21}$$

$$r(\, P_M - P_A - P_B + P_AP_B \,) = r(M) - r(A) - r(B) + r(A^*B), \tag{2.22}$$

$$\begin{aligned} i_-(\, P_M - P_{A_1} - P_{B_1} \,) &= i_-(\, P_A - P_{A_1} \,) = i_-(\, P_B - P_{B_1} \,) \\ &= r(M) - r(A) - r(B) + r(A^*B). \end{aligned} \tag{2.23}$$

(d) *The following formulas for minimum matrix rank optimization hold*

$$\begin{aligned} \min_{X \in \mathbb{C}^{n\times m}} r\left(M - M \begin{bmatrix} X \\ B^\dagger \end{bmatrix} M \right) &= \min_{Y \in \mathbb{C}^{k\times m}} r\left(M - M \begin{bmatrix} A^\dagger \\ Y \end{bmatrix} M \right) \\ &= r(M) - r(A) - r(B) + r(A^*B). \end{aligned} \tag{2.24}$$

(e) *The following statements are equivalent:*

(i) $P_AP_B = P_BP_A$.

(ii) $(P_AP_B)^\dagger = P_BP_A$.

(iii) $(P_AP_B)(P_AP_B)^\dagger = P_AP_BP_A$.

(iv) $(P_AP_B)^\dagger(P_AP_B) = P_BP_AP_B$.

(v) $P_M = P_A + B_1B_1^*$.

(vi) $P_M = P_B + A_1A_1^*$.

(vii) $P_M \geqslant P_{A_1} + P_{B_1}$.

(viii) $P_A \geqslant P_{A_1}$.

(ix) $P_B \geqslant P_{B_1}$.

(x) $P_M = P_A + P_B - P_AP_B$.

(xi) *There exists an* $X \in \mathbb{C}^{n\times m}$ *such that* $\begin{bmatrix} X \\ B^\dagger \end{bmatrix} \in \{[A,\, B\,]^-\}$.

(xii) *There exists a* $Y \in \mathbb{C}^{k\times m}$ *such that* $\begin{bmatrix} A^\dagger \\ Y \end{bmatrix} \in \{[A,\, B\,]^-\}$.

(xiii) $r(M) = r(A) + r(B) - r(A^*B)$.

Proof Applying (2.1)–(2.5) to $A_1 = P_B^\perp P_A$ and $B_1 = P_A^\perp P_B$ gives

$$\begin{aligned} A_1^\dagger &= P_AP_B^\perp - P_A(P_A^\perp P_B)^\dagger P_B^\perp = A_1^* - P_AB_1^\dagger P_B^\perp, \\ B_1^\dagger &= P_BP_A^\perp - P_B(P_B^\perp P_A)^\dagger P_A^\perp = B_1^* - P_BA_1^\dagger P_A^\perp, \\ A_1A_1^\dagger &= A_1A_1^* - A_1B_1^\dagger P_B^\perp, \\ B_1B_1^\dagger &= B_1B_1^* - B_1A_1^\dagger P_A^\perp, \\ A_1A_1^\dagger &\geqslant A_1A_1^*, \end{aligned}$$

$$B_1 B_1^{\dagger} \geqslant B_1 B_1^{*}.$$

Substituting them into (2.14) and (2.15) gives

$$\begin{aligned} P_M &= P_A + P_{B_1} = P_A + B_1B_1^* - B_1A_1^{\dagger}P_A^{\perp}, \\ P_M &= P_B + P_{A_1} = P_B + A_1A_1^* - A_1B_1^{\dagger}P_B^{\perp}, \\ P_M &= P_A + P_{B_1} \geqslant P_A + B_1B_1^*, \\ P_M &= P_B + P_{A_1} \geqslant P_B + A_1A_1^*, \end{aligned}$$

as required for (2.16)–(2.19). It is also easy to verify that

$$P_M - P_A - B_1B_1^* = P_M - P_A - P_A^{\perp}P_BP_A^{\perp} = P_A^{\perp}(\, P_M - P_B\,)P_A^{\perp}, \tag{2.25}$$

$$P_M - P_A - P_B + P_AP_B = P_A^{\perp}(\, P_M - P_B\,), \tag{2.26}$$

where $P_M - P_B \geqslant 0$. Hence, applying (1.1) to (2.25) and simplifying, we obtain

$$\begin{aligned} r(\, P_M - P_A - B_1B_1^*\,) &= r(\, P_M - P_A - P_B + P_AP_B\,) \\ &= r[\, P_A^{\perp}(\, P_M - P_B\,)] = r[\, P_A,\ P_B^{\perp}P_A\,] - r(P_A) \\ &= r[\, P_BP_A,\ P_B^{\perp}P_A\,] - r(P_A) = r(P_BP_A) + r(P_B^{\perp}P_A) - r(P_A) \\ &= r(M) - r(A) - r(B) + r(A^*B), \end{aligned} \tag{2.27}$$

as required for (2.20) and (2.22). Eq. (2.21) can be shown similarly. Eq. (2.22) was also shown in [6]. Eq. (2.23) was shown in [9].

Applying (1.13) and simplifying by (1.1) and elementary matrix operations, we obtain

$$\begin{aligned} \min_Y r\left(M - M \begin{bmatrix} A^{\dagger} \\ Y \end{bmatrix} M \right) &= \min_Y r\big([\, 0,\ P_A^{\perp}B\,] - BYM\big) \\ &= r[\, P_A^{\perp}B,\ B\,] + r\begin{bmatrix} 0 & P_A^{\perp}B \\ A & B \end{bmatrix} - r\begin{bmatrix} 0 & P_A^{\perp}B & B \\ A & B & 0 \end{bmatrix} \\ &= r[\, P_A^{\perp}B,\ P_AB\,] + r\begin{bmatrix} 0 & 0 \\ A & B \end{bmatrix} - r\begin{bmatrix} 0 & 0 & B \\ A & B & 0 \end{bmatrix} \\ &= r(P_A^{\perp}B) + r(P_AB) - r(B) \\ &= r[\, A,\ B\,] + r(A^*B) - r(A) - r(B), \end{aligned}$$

establishing the second equality in (2.24). The first equality in (2.24) can be shown similarly. Setting the right-hand sides of (1.6), (2.6)–(2.8), (2.20)–(2.24) equal to zero leads to the equivalences in (e). □

It is of interest to consider extensions of the previous results to some general row block matrices. A special case for the orthogonal projectors onto the ranges of a row block matrix $[\, A,\ B,\ C\,]$ is formulated below.

Theorem 2.3 *Let* $N = [\, A,\ B,\ C\,]$, *where* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{m\times l}$. *Then,* *(a) The following rank equality holds*

$$
\begin{aligned}
&r(\, P_N - P_A - P_B - P_C + P_AP_B + P_AP_C + P_BP_C - P_AP_BP_C \,) \\
&= r\begin{bmatrix} A^*C & A^*B \\ B^*C & 0 \end{bmatrix} + r[\, A,\, B,\, C\,] - r(A) - r(B) - r(C).
\end{aligned} \tag{2.28}
$$

(b) The following statements are equivalent:

(i) $P_N = P_A + P_B + P_C - P_AP_B - P_AP_C - P_BP_C + P_AP_BP_C$.

(ii) $(\, I_m - P_A\,)(\, P_N - P_B\,)(\, I_m - P_C\,) = 0$.

(iii) $(\, P_N - P_A\,)(\, P_N - P_B\,)(\, P_N - P_C\,) = 0$.

(iv) $r\begin{bmatrix} A^*C & A^*B \\ B^*C & 0 \end{bmatrix} = r(A) + r(B) + r(C) - r(N)$.

(c) (see [4]) If $P_AP_B = P_BP_A$, $P_AP_C = P_CP_A$ *and* $P_BP_C = P_CP_B$, *then (i) of (b) holds.*

Proof Note that $P_NP_A = P_A$, $P_NP_B = P_B$ and $P_NP_C = P_C$ hold. Then it is easy to verify

$$
\begin{aligned}
&P_N - P_A - P_B - P_C + P_AP_B + P_AP_C + P_BP_C - P_AP_BP_C \\
&= (\, I_m - P_A\,)(\, P_N - P_B\,)(\, I_m - P_C\,).
\end{aligned} \tag{2.29}
$$

It follows from (2.12) that P_N can be decomposed as

$$
P_N = P_B + (P_B^{\perp}[A, C])(P_B^{\perp}[A, C])^{\dagger}. \tag{2.30}
$$

Applying (1.2) to (2.29) and simplifying by (1.1), (1.4), (2.30) and elementary matrix operations, we obtain

$$
\begin{aligned}
&r[\,(\, I_m - P_A\,)(\, P_N - P_B\,)(\, I_m - P_C\,)\,] \\
&= r\begin{bmatrix} P_N - P_B & P_A \\ P_C & 0 \end{bmatrix} - r(P_A) - r(P_C) \\
&= r\begin{bmatrix} (P_B^{\perp}[A,\, C])(P_B^{\perp}[A,\, C])^{\dagger} & P_A \\ P_C & 0 \end{bmatrix} - r(A) - r(C) \\
&= r\begin{bmatrix} P_CP_A & 0 & P_C \\ 0 & 0 & (P_B^{\perp}[A,\, C])^* \\ P_A & P_B^{\perp}[A,\, C] & 0 \end{bmatrix} - r(P_B^{\perp}[A,\, C]) - r(A) - r(C) \\
&= r\begin{bmatrix} P_CP_A & 0 & P_CP_B \\ 0 & 0 & (P_B^{\perp}[A,\, C])^* \\ P_BP_A & P_B^{\perp}[A,\, C] & 0 \end{bmatrix} - r(P_B^{\perp}[A,\, C]) - r(A) - r(C) \\
&= r\begin{bmatrix} P_CP_A & P_CP_B \\ P_BP_A & 0 \end{bmatrix} + r(P_B^{\perp}[A,\, C]) - r(A) - r(C) \\
&= r\begin{bmatrix} A^*C & A^*B \\ B^*C & 0 \end{bmatrix} + r[\, A,\, B,\, C\,] - r(A) - r(B) - r(C),
\end{aligned}
$$

establishing (2.28). Setting both hands of (2.28) equal to zero leads to the equivalence of (i), (ii) and (iv) in (b). It is also easy to verify that

$$\begin{aligned} & P_N - P_A - P_B - P_C + P_A P_B + P_A P_C + P_B P_C - P_A P_B P_C \\ & = (P_N - P_A)(P_N - P_B)(P_N - P_C). \end{aligned}$$

Setting both hands of (2.28) equal to zero leads to the equivalence of (i) and (iii) in (b).

Under $P_A P_B = P_B P_A$, $P_A P_C = P_C P_A$, and $P_B P_C = P_C P_B$, both $P_M = P_A + P_B - P_A P_B$ and $P_M P_C = P_C P_M$ hold by Theorem 2.2(e), where $M = [\, A,\, B\,]$. In this case, $P_N = P_M + P_C - P_M P_C$ by Theorem 2.2(d). Substituting $P_M = P_A + P_B - P_A P_B$ into $P_N = P_M + P_C - P_M P_C$ yields (i) of (b). □

Many matrix expressions consisting of the orthogonal projectors onto the range of $N = [\, A,\, B,\, C\,]$, A, B and C can be constructed, for instance,

$$P_{[\, A,\, B,\, C\,]} - P_A - P_B - P_C, \quad P_{[\, A,\, B,\, C\,]} - P_{A_1} - P_{B_1} - P_{C_1},$$

where $A_1 = P^{\perp}_{[\, B,\, C\,]} P_A$, $B_1 = P^{\perp}_{[\, A,\, C\,]} P_B$ and $C_1 = P^{\perp}_{[\, A,\, B\,]} P_C$. Thus, it is an attractive topic to extend the previous results to the orthogonal projectors onto the range of a general row block matrix $[\, A_1, \ldots, A_k\,]$. In particular, it can also be derived from Theorem 2.2(e) that if $P_{A_i} P_{A_j} = P_{A_j} P_{A_i}$ for $i,\ j = 1, \ldots, s$, then

$$\begin{aligned} P_N = & P_{A_1} + \cdots + P_{A_s} - P_{A_1} P_{A_2} - \cdots - P_{A_{s-1}} P_{A_s} + \\ & P_{A_1} P_{A_2} P_{A_3} + \cdots + P_{A_{s-2}} P_{A_{s-1}} P_{A_s} - \cdots + (-1)^{s-1} P_{A_1} \cdots P_{A_s}. \end{aligned}$$

This result was first shown in Rao and Yanai [4].

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