Journal of Mathematical Research with Applications
May, 2014, Vol. 34, No. 3, pp. 349–361
DOI:10.3770/j.issn:2095-2651.2014.03.012
Http://jmre.dlut.edu.cn

Well-Posedness for a New Two-Component Integrable System

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Abstract In this paper, we consider a new two-component integrable system with cubic nonlinearity, which can be deduced by a curve flow and it is integrable with its Lax pair, bi-Hamiltonian structure, and infinitely many conservation laws. We mainly establish the local well-posedness of this system in a range of the Besov spaces $B^s_{p,r}$ with $s > \max\{2+\frac{1}{p}, \frac{5}{2}\}$.

Keywords Besov space; two-component integrable system; local well-posedness.

MR(2010) Subject Classification 35A01; 35A02; 35E15: 35L05

1. Introduction

The present paper focuses on the Cauchy problem of the new two-component integrable system with cubic nonlinearity:

$$\begin{cases} m_t = bu_x + \frac{1}{2}[m(uv-u_xv_x)]_x - \frac{1}{2}m(uv_x-u_xv), \quad t>0,\ x\in R \\ n_t = bv_x + \frac{1}{2}[n(uv-u_xv_x)]_x + \frac{1}{2}n(uv_x-u_xv), \quad t>0,\ x\in R \\ m=u-u_{xx}, n=v-v_{xx}, \quad t>0,\ x\in R, \\ u(x,0)=u\_0(x), v(x,0)=v\_0(x) \end{cases} \tag{1}$$

where b is an arbitrary constant. This system was proposed by Qu, Song and Yao in [1]. Note that system (1) can be considered as the two-component case of Eq. (40) in [1]. And Qu et al. [1] showed that Eq. (40) can be deduced by a curve flow $\gamma(x,t)$. Then system (1) can be induced by a curve flow $\gamma(x,t)$, where $\overrightarrow{m}$ denotes the curvature vector.

Note that when we take $v=2$, the system (1) becomes the famous Camassa-Holm equation:

$$m_t + bu_x + 2mu_x + m_xu = 0, \tag{2}$$

which has been derived independently by Fokas and Fuchssteiner [2] by the method of recursion operators, and by Camassa and Holm in [3]. Like the celebrated KdV equation, Camassa-Holm equation describes the unidirectional propagation of shallow water waves over a flat bottom. Here $u(t,x)$ stands for the fluid velocity at time t in the spatial x direction (or equivalently the height of the free surface of water above a float bottom), b is a constant related to the critical

Received May 16, 2013; Accepted September 11, 2013
Supported by National Natural Science Foundation of China (Grant No. 11371384).
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shallow water wave speed. It turns out that it is also a model for the propagation of nonlinear waves in cylindrical hyperplastic rods [4] with $u(x,t)$ representing the radial stretch relative to a pre-stressed state. Since the work of Camassa and Holm, various studies on this equation have been developed. For instance, it has been found that Eq. (2) conforms with many conservation laws, is completely integrable [5] and admits bi-Hamiltonian structures [6], Lax representation, multi-soliton solutions and algebra-geometric solutions [7]. Meanwhile, Eq. (2) possesses smooth solitary wave solutions if $b > 0$ (see [8]) or peakons if $b = 0$ (see [3]). It is also regarded as a model of the geodesic flow for the H^1 right invariant metric on the Bott-Virasoro group if $b > 0$ and on the diffeomorphism group if $b = 0$ (see [9]). The local well-posedness, global existence, blow-up structures and the well-posedness of global weak solutions of Eq. (2) have been given in [10]. The sharpest results for the global existence and blow-up solutions were found in Bressan and Constantin [11].

For $v = 2u$, system (1) is exactly the cubic CH equation:

$$m_t + (u^2 - u_x^2)m_x + 2u_x m^2 + bu_x = 0, \tag{3}$$

which was derived independently by Fuchssteiner [12], Olver and Rosenau [13] and Fokas [14] by applying the general method of tri-Hamiltonian duality to the bi-Hamiltonian representation of the modified Korteweg-de Vries equation. Later, it was derived by Qiao [15] from the two-dimensional Euler equations, where the variables $u(t,x)$ and $m(t,x)$ represent, respectively, the velocity of the fluid and its potential density. In [15, 16], it was shown that equation (3) admits a Lax pair, and hence can be solved by the method of inverse scattering. In [17], the local well-posedness and blow-up scenario to its Cauchy problem were considered. In [18], it was shown that the singularities of the solutions can occur only in the form of wave-breaking, and a new wave-breaking mechanism was described for solutions with certain initial profiles.

If we choose $v = k_1 u + k_2$, system (1) became the generalized CH equation:

$$\begin{cases} m_t + bu_x + \frac{1}{2}k_1[m(u^2 - u_x^2)]_x + \frac{1}{2}k_2(2mu_x + m_x u) = 0, & t > 0, x \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{4}$$

where b, k_1 and k_2 are arbitrary constants, which was proposed by Qiao, Xia and Li [19]. In [19], Qiao et al showed that the above equation was completely integrable, and derived its Lax pair, bi-Hamiltonian structure, peakons, weak kinks, kink-peakon interactional and smooth soliton solutions. Very recently, we have established the well-posedness, blow-up criteria and the lower bound of the maximal time of existence for Eq. (4) in our another paper.

Since the work of CH equation, the study on the two-component CH system has also been remarkably developed [20–22]. In [23], Gui and Liu established the local well-posedness for the two-component Camassa-Holm system. Inspired by the references cited above, the goal of the present paper is to establish the local well-posedness for the strong solutions to the Cauchy problem (1). The proof of the local well-posedness is inspired by the argument of approximate solutions by Danchin [24] in the study of the local well-posedness to the Camassa-Holm equation.

However, one problematic issue is that we here deal with two-component system with a higher order nonlinearity in the Besov spaces, making the proof of several required nonlinear estimates somewhat delicate. These difficulties are nevertheless overcome by carefully estimating each iterative approximation of solutions to (1).

2. Basic facts and notation

For convenience, we need to recall some basic facts on the transport equations and Moser-type estimates.

Definition 2.1 ([25]) *(Besov space) Let $s \in \mathbb{R}, 1 \leq p, r \leq \infty$. The inhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ ($B^s_{p,r}$ for short) is defined by*

$$B^s_{p,r} \doteq \{f \in \mathcal{S}'(\mathbb{R}^d); \|f\|_{B^s_{p,r}} < \infty\},$$

where

$$\|f\|_{B^s_{p,r}} \doteq \begin{cases} (\sum_{q\in Z} \ 2^{qsr}\|\triangle_q f\|^r_{L^p})^{\frac{1}{r}}, \ \text{for } r < \infty, \\ \sup_{q\in Z} 2^{qs}\|\triangle_q f\|_{L_p}, \ \text{for } r = \infty. \end{cases}$$

If $s = \infty, B^\infty_{p,r} \doteq \bigcap_{s\in\mathbb{R}} B^s_{p,r}$.

Definition 2.2 *For $T > 0, s \in \mathbb{R}$ and $1 \leq p \leq +\infty$, we set*

$$E^s_{p,r}(T) \doteq C\left([0,T]; B^s_{p,r}\right) \cap C^1\left([0,T]; B^{s-1}_{p,r}\right) \ \text{if } r < +\infty,$$

$$E^s_{p,\infty}(T) \doteq L^\infty\left([0,T]; B^s_{p,\infty}\right) \cap \mathrm{Lip}\left([0,T]; B^{s-1}_{p,\infty}\right)$$

and $E^s_{p,r} \doteq \cap_{T>0} E^s_{p,r}(T)$.

Proposition 2.3 ([25, 26]) *The following properties hold.*

I) Density: if $p, r < \infty$, then $\mathcal{S}(\mathbb{R}^d)$ is dense in $B^s_{p,r}(\mathbb{R}^d)$.

II) Sobolev embedding: if $p_1 \leq p_2$, and $r_1 \leq r_2$, then $B^s_{p_1,r_1} \hookrightarrow B^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_1,r_2}$. If $s_1 < s_2, 1 \leq p \leq +\infty$ and $1 \leq r_1, r_2 \leq +\infty$, then the embedding $B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1}$ is locally compact.

III) Algebraic properties: for $s > 0, B^s_{p,r} \cap L^\infty$ is an algebra. Moreover, $B^s_{p,r}$ is an algebra $\Leftrightarrow B^s_{p,r} \hookrightarrow L^\infty \Leftrightarrow s > \frac{d}{p}$ or ($s \geq \frac{d}{p}$ and $r = 1$).

IV) Complex interpolation: if $u \in B^s_{p,r} \cap B^{\tilde{s}}_{p,r}$ and $\theta \in [0,1], 1 \leq p, r \leq \infty$, then $u \in B^{\theta s+(1-\theta)\tilde{s}}_{p,r}$, and $\|u\|_{B^{\theta s+(1-\theta)\tilde{s}}_{p,r}} \leq \|u\|^\theta_{B^s_{p,r}}\|u\|^{1-\theta}_{B^{\tilde{s}}_{p,r}}$.

Lemma 2.1 ([25, 26]) *Suppose that $(p, r) \in [1,\infty]^2$, and $s > -\frac{d}{p}$. Let v be a vector field such that ∇v belongs to $L^1([0,T]; B^{s-1}_{p,r})$ if $s > 1 + \frac{d}{p}$ or to $L^1([0,T]; B^{\frac{d}{p}}_{p,r} \cap L^\infty)$ otherwise. Suppose also that $f_0 \in B^s_{p,r}, F \in L^1([0,T]; B^s_{p,r})$ and that $f \in L^\infty([0,T]; B^s_{p,r}) \cap C([0,T]; \mathcal{S}')$ solves the d-dimensional linear transport equations*

$$\begin{cases} \partial_t f + v \cdot \nabla f = F, \\ f|_{t=0} = f_0. \end{cases} \tag{5}$$

Then there exists a constant C depending only on s, p and d such that the following statements hold:

I) If $r = 1$ or $s \neq 1 + \frac{d}{p}$, then

$$\|f\|_{B^s_{p,r}} \leq \|f_0\|_{B^s_{p,r}} + \int_0^t \|F(\tau)\|_{B^s_{p,r}} \mathrm{d}\tau + C \int_0^t V'(\tau)\|f(\tau)\|_{B^s_{p,r}} \mathrm{d}\tau,$$

or

$$\|f\|_{B^s_{p,r}} \leq e^{CV(t)}\Big(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B^s_{p,r}} \mathrm{d}\tau\Big),$$

where $V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{\frac{d}{p}}_{p,r} \cap L^\infty} \mathrm{d}\tau$ if $s < 1 + \frac{d}{p}$ and $V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{s-1}_{p,r}} \mathrm{d}\tau$ else.

II) If $s \leq 1 + \frac{d}{p}$ and, in addition, $\nabla f_0 \in L^\infty$, $\nabla f \in L^\infty([0,T]\times\mathbb{R}^d)$ and $\nabla F \in L^1([0,T]; L^\infty)$, then

$$\|f\|_{B^s_{p,r}} + \|\nabla f(t)\|_{L^\infty} \leq e^{CV(t)}\Big(\|f_0\|_{B^s_{p,r}} + \|\nabla f_0\|_{L^\infty} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B^s_{p,r}} + \|\nabla F(\tau)\|_{L^\infty} \mathrm{d}\tau\Big)$$

with $V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{\frac{d}{p}}_{p,r} \cap L^\infty} \mathrm{d}\tau$.

III) If $f = v$, then for all $s > 0$, the estimate (6) holds with $V(t) = \int_0^t \|\partial_x u(\tau)\|_{L^\infty} \mathrm{d}\tau$.

IV) If $r < +\infty$, then $f \in C([0,T]; B^s_{p,r})$. If $r = +\infty$, then $f \in C([0,T]; B^{s'}_{p,1})$ for all $s' < s$.

Lemma 2.2 ([26]) *Let $(p, p_1, r) \in [1, +\infty]^3$. Assume that $s > -d\min\{\frac{1}{p}, \frac{1}{p'}\}$ with $p' := (1-\frac{1}{p})^{-1}$. Let $f_0 \in B^s_{p,r}$ and $F \in L^1([0,T]; B^s_{p,r})$. Let v be a time dependent vector field such that $v \in L^\rho([0,T]; B^{-M}_{\infty,\infty})$ for some $\rho > 1, M > 0$, and $\nabla v \in L^1([0,T]; B^{\frac{d}{p_1}}_{p_1,\infty} \cap L^\infty)$ if $s < 1 + \frac{d}{p_1}$, and $\nabla v \in L^1([0,T]; B^{s-1}_{p_1,r})$ if $s > 1 + \frac{d}{p_1}$ or $s = 1 + \frac{d}{p_1}$ and $r = 1$. Then the transport equations (5) has a unique solution $f \in L^\infty([0,T]; B^s_{p,r}) \cap (\cap_{s'<s} C([0,T]; B^{s'}_{p,1}))$ and the inequalities in Lemma 2.1 hold true. If, moreover, $r < \infty$, then we have $f \in C([0,T]; B^s_{p,r})$.*

Lemma 2.3 ([13]) *(1-D Moser-type estimates) Assume that $1 \leq p, r \leq +\infty$. Then the following estimates hold:*

(I) For $s > 0$, $\|fg\|_{B^s_{p,r}} \leq C(\|f\|_{B^s_{p,r}}\|g\|_{L^\infty} + \|g\|_{B^s_{p,r}}\|f\|_{L^\infty})$;

(II) For $s_1 \leq \frac{1}{p}, s_2 > \frac{1}{p}$ ($s_2 \geq \frac{1}{p}$ if $r = 1$) and $s_1 + s_2 > 0$,

$$\|fg\|_{B^{s_1}_{p,r}} \leq C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}},$$

where the constant C is independent of f and g.

3. Local well-posedness

In this section, we will prove the local well-posedness of the Cauchy problem (1). For convenience, applying the operator $(1-\partial_x^2)^{-1}$ to both sides of the system (1), we have

$$\begin{cases} u_t =(1-\partial_x^2)^{-1}(bu_x)+\partial_x(1-\partial_x^2)^{-1}\big[(u^2+uu_x+\frac{1}{2}u_x^2)v\big]+ \\ \qquad (1-\partial_x^2)^{-1}(\frac{1}{3}u_x^2v-u^2v)+(uv-\frac{1}{3}u_xv_x)u_x, \\ v_t =(1-\partial_x^2)^{-1}(bv_x)+\partial_x(1-\partial_x^2)^{-1}\big[(v^2+vv_x+\frac{1}{2}v_x^2)u\big]+ \\ \qquad (1-\partial_x^2)^{-1}(\frac{1}{3}uv_x^2-uv^2)+(uv-\frac{1}{3}u_xv_x)v_x, \end{cases} \tag{7}$$

which enables us to define the solutions of the Cauchy problem (1).

Theorem 3.1 *Suppose that $1\le p,r\le+\infty$, $s>\max\{2+\frac{1}{p},\frac{5}{2}\}$ and $(u_0,v_0)\in B^s_{p,r}\times B^s_{p,r}$. Then there exists a time $T>0$ such that the initial-value problem (1.1) has a unique solution $(u,v)\in E^s_{p,r}(T)\times E^s_{p,r}(T)$, and the map $(u_0,v_0)\mapsto(u,v)$ is continuous from a neighborhood of $(u_0,v_0)\in B^s_{p,r}\times B^s_{p,r}$ into*

$$C([0,T];B^{s'}_{p,r})\cap C^1([0,T];B^{s'-1}_{p,r})\times C([0,T];B^{s'}_{p,r})\cap C^1([0,T];B^{s'-1}_{p,r})$$

for every $s'<s$ when $r=+\infty$ and $s'=s$ whereas $r<+\infty$.

To prove this theorem, we need to prove the following Proposition and Lemma first. Uniqueness and continuity with respect to the initial data are an immediate consequence of the following result.

Proposition 3.2 *Let $1\le p,r\le+\infty$ and $s>\max\{2+\frac{1}{p},\frac{5}{2}\}$. Let $(u^{(1)},v^{(1)}),(u^{(2)},v^{(2)})$ be two given solutions of the initial-value problem (1) with the initial data $(u_0^{(1)},v_0^{(1)}),(u_0^{(2)},v_0^{(2)})\in B^s_{p,r}\times B^s_{p,r}$ satisfying $u^{(1)},v^{(1)},u^{(2)},v^{(2)}\in L^\infty([0,T];B^s_{p,r})\cap C([0,T];\mathcal{S}')$. Then for every $t\in[0,T]$:*

$$\begin{aligned} &\|(u^{(1)}-u^{(2)})(t)\|_{B^{s-1}_{p,r}}+\|(v^{(1)}-v^{(2)})(t)\|_{B^{s-1}_{p,r}} \\ &\le(\|(u_0^{(1)}-u_0^{(2)})\|_{B^{s-1}_{p,r}}+\|(v_0^{(1)}-v_0^{(2)})\|_{B^{s-1}_{p,r}})\cdot\exp\Big\{C\int_0^t\big(\|u^{(1)}(\tau)\|^2_{B^s_{p,r}}+\|u^{(2)}(\tau)\|^2_{B^s_{p,r}}+ \\ &\qquad \|v^{(1)}(\tau)\|^2_{B^s_{p,r}}+\|v^{(2)}(\tau)\|^2_{B^s_{p,r}}+|b|\big)\Big\}, \end{aligned} \tag{8}$$

where $C>0$ is a generic constant only depending on p,r,s.

Proof Denote $u^{(12)}:=u^{(2)}-u^{(1)}$ and $v^{(12)}:=v^{(2)}-v^{(1)}$. It is obvious that

$$u^{(12)},v^{(12)}\in L^\infty\left([0,T];B^s_{p,r}\right)\cap C\left([0,T];\mathcal{S}'\right),$$

which along with the equivalent formulation (3.1) of (1.1) implies that

$$(u^{(12)},v^{(12)})\in C\left([0,T];B^{s-1}_{p,r}\right)$$

and $(u^{(12)}, v^{(12)})$ solves the transport equation

$$\begin{cases} \partial_t u^{(12)} + [\frac{1}{3}(u_x^{(1)} + u_x^{(2)})v_x^{(1)} - u^{(1)}v^{(2)}]\partial_x u^{(12)} = f(u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}), \\ \partial_t v^{(12)} + [\frac{1}{3}(v_x^{(1)} + v_x^{(2)})u_x^{(1)} - v^{(1)}u^{(2)}]\partial_x v^{(12)} = g(u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}), \\ u^{(12)}|_{t=0} = u_0^{(12)} := u_0^{(2)} - u_0^{(1)}, \\ v^{(12)}|_{t=0} = v_0^{(12)} := v_0^{(2)} - v_0^{(1)} \end{cases} \tag{9}$$

with

$$\begin{aligned} f(u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}) =& \partial_x(1-\partial_x^2)^{-1}\Big[bu^{(12)} + (u^{(1)} + u^{(2)} + u_x^{(2)})v^{(2)}u^{(12)} + v^{(12)} \times \\ & (\frac{1}{2}(u_x^{(2)})^2 + (u^{(1)})^2 + u^{(1)}u_x^{(1)}) + (u^{(1)}v^{(2)} + \frac{1}{2}(u_x^{(1)} + u_x^{(2)})v^{(1)})\partial_x u^{(12)}\Big] + \\ & (1-\partial_x^2)^{-1}\Big[\frac{1}{3}(u_x^{(1)} + u_x^{(2)})v^{(1)}\partial_x u^{(12)} + (\frac{1}{3}(u_x^{(2)})^2 - (u^{(1)})^2)v^{(12)} - \\ & (u^{(1)} + u^{(2)})v^{(2)}u^{(12)}\Big] + u_x^{(2)}u^{(12)}v^{(2)} + u^{(1)}u_x^{(1)}v^{(12)} - \frac{1}{3}(u_x^{(2)})^2 v_x^{(12)}, \\ g(u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}) =& \partial_x(1-\partial_x^2)^{-1}\Big[bv^{(12)} + (v^{(1)} + v^{(2)} + v_x^{(2)})u^{(2)}v^{(12)} + u^{(12)} \times \\ & (\frac{1}{2}(v_x^{(2)})^2 + (v^{(1)})^2 + v^{(1)}v_x^{(1)}) + (v^{(1)}u^{(2)} + \frac{1}{2}(v_x^{(1)} + v_x^{(2)})u^{(1)})\partial_x v^{(12)}\Big] + \\ & (1-\partial_x^2)^{-1}\Big[\frac{1}{3}(v_x^{(1)} + v_x^{(2)})u^{(1)}\partial_x v^{(12)} + (\frac{1}{3}(v_x^{(2)})^2 - (v^{(1)})^2)u^{(12)} - \\ & (v^{(1)} + v^{(2)})u^{(2)}v^{(12)}\Big] + v_x^{(2)}v^{(12)}u^{(2)} + v^{(1)}v_x^{(1)}u^{(12)} - \frac{1}{3}(v_x^{(2)})^2 u_x^{(12)}. \end{aligned}$$

Thanks to the transport theory in Lemma 2.1, we get

$$\begin{aligned} \|u^{(12)}(t)\|_{B_{p,r}^{s-1}} \le& \|u_0^{(12)}\|_{B_{p,r}^{s-1}} + \int_0^t \|f(u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)})\|_{B_{p,r}^{s-1}} d\tau + \\ & C\int_0^t \|\frac{1}{3}\big(u_x^{(1)} + u_x^{(2)}\big)v_x^{(1)} - u^{(1)}v^{(2)}\|_{B_{p,r}^{s-1}} \cdot \|u^{(12)}(\tau)\|_{B_{p,r}^{s-1}} d\tau. \end{aligned} \tag{10}$$

$$\begin{aligned} \|v^{(12)}(t)\|_{B_{p,r}^{s-1}} \le& \|v_0^{(12)}\|_{B_{p,r}^{s-1}} + \int_0^t \|g(u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)})\|_{B_{p,r}^{s-1}} d\tau + \\ & C\int_0^t \|\frac{1}{3}\big(v_x^{(1)} + v_x^{(2)}\big)u_x^{(1)} - v^{(1)}u^{(2)}\|_{B_{p,r}^{s-1}} \cdot \|v^{(12)}(\tau)\|_{B_{p,r}^{s-1}} d\tau. \end{aligned} \tag{11}$$

Applying the product law in the Besov spaces, we have

$$\|\frac{1}{3}\big(u_x^{(1)} + u_x^{(2)}\big)v_x^{(1)} - u^{(1)}v^{(2)}\|_{B_{p,r}^{s-1}} \le C\big(\|u^{(1)}\|_{B_{p,r}^s}^2 + \|u^{(2)}\|_{B_{p,r}^s}^2 + \|v^{(1)}\|_{B_{p,r}^s}^2\big). \tag{12}$$

$$\begin{aligned} &\|u_x^{(2)}u^{(12)}v^{(2)} + u^{(1)}u_x^{(1)}v^{(12)} - \frac{1}{3}(u_x^{(2)})^2 v_x^{(12)}\|_{B_{p,r}^{s-1}} \\ &\quad \le C\big(\|u^{(2)}\|_{B_{p,r}^{s-1}}^2 + \|v^{(2)}\|_{B_{p,r}^{s-1}}^2\big)\|u^{(12)}\|_{B_{p,r}^{s-1}} + C\|u^{(2)}\|_{B_{p,r}^s}^2 \|v^{(12)}\|_{B_{p,r}^{s-1}}, \end{aligned}$$

$$\begin{aligned} &\Big\|(1-\partial_x^2)^{-1}[\frac{1}{3}(u_x^{(1)} + u_x^{(2)})v^{(1)}\partial_x u^{(12)} + (\frac{1}{3}(u_x^{(2)})^2 - (u^{(1)})^2)v^{(12)} - (u^{(1)} + u^{(2)})v^{(2)}u^{(12)}]\Big\|_{B_{p,r}^{s-1}} \\ &\le C\Big[\big(\|u^{(1)}\|_{B_{p,r}^{s-1}}^2 + \|u^{(2)}\|_{B_{p,r}^{s-1}}^2 + \|v^{(1)}\|_{B_{p,r}^{s-1}}^2\big)\|u^{(12)}\|_{B_{p,r}^{s-1}} + (\|u^{(1)}\|_{B_{p,r}^{s-1}}^2 + \|u^{(2)}\|_{B_{p,r}^{s-1}}^2)\|v^{(12)}\|_{B_{p,r}^{s-1}}\Big], \end{aligned}$$

and

$$\Big\|\partial_x(1-\partial_x^2)^{-1}[bu^{(12)}+(u^{(1)}+u^{(2)}+u_x^{(2)})v^{(2)}u^{(12)}+v^{(12)}\cdot(\frac{1}{2}(u_x^{(2)})^2+(u^{(1)})^2+u^{(1)}u_x^{(1)})+ \\ (u^{(1)}v^{(2)}+\frac{1}{2}(u_x^{(1)}+u_x^{(2)})v^{(1)})\partial_x u^{(12)}]\Big\|_{B^{s-1}_{p,r}} \\ \leq C\big(\|u^{(1)}\|^2_{B^{s-1}_{p,r}}+\|u^{(2)}\|^2_{B^{s-1}_{p,r}}+\|v^{(1)}\|^2_{B^{s-1}_{p,r}}+\|v^{(2)}\|^2_{B^{s-1}_{p,r}}+|b|\big)\|u^{(12)}\|_{B^{s-1}_{p,r}}+ \\ C(\|u^{(1)}\|^2_{B^{s-1}_{p,r}}+\|u^{(2)}\|^2_{B^{s-1}_{p,r}})\|v^{(12)}\|_{B^{s-1}_{p,r}},$$

which leads to

$$\|f(u^{(1)},u^{(2)},v^{(1)},v^{(2)})\|_{B^{s-1}_{p,r}}\leq C\Big[\big(\|u^{(1)}\|^2_{B^{s-1}_{p,r}}+\|u^{(2)}\|^2_{B^{s}_{p,r}}+\|v^{(1)}\|^2_{B^{s-1}_{p,r}}+\|v^{(2)}\|^2_{B^{s}_{p,r}}+ \\ |b|\big)\|u^{(12)}\|_{B^{s-1}_{p,r}}+(\|u^{(1)}\|^2_{B^{s}_{p,r}}+\|u^{(2)}\|^2_{B^{s}_{p,r}})\|v^{(12)}\|_{B^{s-1}_{p,r}}\Big]. \tag{13}$$

Similarly, we have

$$\|g(u^{(1)},u^{(2)},v^{(1)},v^{(2)})\|_{B^{s-1}_{p,r}}\leq C\Big[\big(\|u^{(1)}\|^2_{B^{s-1}_{p,r}}+\|u^{(2)}\|^2_{B^{s}_{p,r}}+\|v^{(1)}\|^2_{B^{s-1}_{p,r}}+\|v^{(2)}\|^2_{B^{s}_{p,r}}+ \\ |b|\big)\|v^{(12)}\|_{B^{s-1}_{p,r}}+(\|v^{(1)}\|^2_{B^{s}_{p,r}}+\|v^{(2)}\|^2_{B^{s}_{p,r}})\|u^{(12)}\|_{B^{s-1}_{p,r}}\Big]. \tag{14}$$

Then, inserting Eq. (12–14) into (10) and (11) implies that

$$\|u^{(12)}(t)\|_{B^{s-1}_{p,r}}+\|v^{(12)}(t)\|_{B^{s-1}_{p,r}}\leq(\|u^{(12)}(0)\|_{B^{s-1}_{p,r}}+\|v^{(12)}(0)\|_{B^{s-1}_{p,r}})+ \\ C\int_0^t\big(\|u^{(1)}(\tau)\|^2_{B^{s}_{p,r}}+\|u^{(2)}(\tau)\|^2_{B^{s}_{p,r}}+\|v^{(1)}(\tau)\|^2_{B^{s}_{p,r}}+ \\ \|v^{(2)}(\tau)\|^2_{B^{s}_{p,r}}+|b|\big)\cdot(\|u^{(12)}(t)\|_{B^{s-1}_{p,r}}+\|v^{(12)}(t)\|_{B^{s-1}_{p,r}})\mathrm{d}\tau.$$

Applying Gronwall's inequality, we get (8).

Next, we use the classical Friedrichs regularization method to construct the approximate solutions to the Cauchy problem (1).

Lemma 3.3 *Let $1\leq p,r\leq+\infty, s>\max\{2+\frac{1}{p},\frac{5}{2}\},(u_0,v_0)\in B^s_{p,r}\times B^s_{p,r}$. Assume that $u^{(0)}=v^{(0)}=0$. There exists a sequence of smooth functions $\big(u^{(n)},v^{(n)}\big)_{n\in\mathbb{N}}\in C(\mathbb{R}^+;B^\infty_{p,r})^2$ solving the following linear transport equations by induction:*

$$\begin{cases}\partial_t m^{(n+1)}+\frac{1}{2}\big(u_x^{(n)}v_x^{(n)}-u^{(n)}v^{(n)}\big)\partial_x m^{(n+1)}=P(u^{(n)},v^{(n)}),\ t>0,x\in\mathbb{R},\\ \partial_t n^{(n+1)}+\frac{1}{2}\big(u_x^{(n)}v_x^{(n)}-u^{(n)}v^{(n)}\big)\partial_x n^{(n+1)}=Q(u^{(n)},v^{(n)}),\ t>0,x\in\mathbb{R},\\ u^{(n+1)}|_{t=0}=u_0^{(n+1)}(x)=S_{n+1}u_0,\quad x\in\mathbb{R},\\ v^{(n+1)}|_{t=0}=v_0^{(n+1)}(x)=S_{n+1}v_0,\quad x\in\mathbb{R},\end{cases} \tag{15}$$

where

$$\begin{cases}P(u^{(n)},v^{(n)})=bu_x^{(n)}+\frac{1}{2}\big(2u_x^{(n)}v^{(n)}-(u_x^{(n)}v_x^{(n)})_x\big)m^{(n)},\\ Q(u^{(n)},v^{(n)})=bv_x^{(n)}+\frac{1}{2}\big(2v_x^{(n)}u^{(n)}-(u_x^{(n)}v_x^{(n)})_x\big)n^{(n)}.\end{cases} \tag{16}$$

Moreover, there exists a $T>0$ such that the solutions satisfy the following properties:

(I) $\big(u^{(n)},v^{(n)}\big)_{n\in\mathbb{N}}$ *is uniformly bounded in* $E^s_{p,r}(T)\times E^s_{p,r}(T)$.

(II) $\big(u^{(n)},v^{(n)}\big)_{n\in\mathbb{N}}$ *is a Cauchy sequence in* $C([0,T];B^{\infty}_{p,r})^2$.

Proof As all data $(S_{n+1}u_0,S_{n+1}v_0)$ belongs to $B^\infty_{p,r}\times B^\infty_{p,r}$, Lemma 2.2 enables us to show by induction that for all $n\in\mathbb{N}$, the equation (15) has a global solution which belongs to $C(\mathbb{R};B^\infty_{p,r})^2$. Applying Lemma 2.1 to (15), we get for all $n\in\mathbb{N}$:

$$\begin{aligned}&\|m^{(n+1)}(t)\|_{B^{s-2}_{p,r}}+\|n^{(n+1)}(t)\|_{B^{s-2}_{p,r}}\\&\quad\le\exp\Big(C\int_0^t\frac12\|u_x^{(n)}v_x^{(n)}-u^{(n)}v^{(n)}\|_{B^{s-1}_{p,r}}\mathrm{d}\tau\Big)\cdot\big(\|S_{n+1}u_0\|_{B^s_{p,r}}+\|S_{n+1}v_0\|_{B^s_{p,r}}\big)+\\&\qquad C\int_0^t\exp\Big(C\int_\tau^t\|\frac12\|u_x^{(n)}v_x^{(n)}-u^{(n)}v^{(n)}\|_{B^{s-1}_{p,r}}\mathrm{d}\tau'\Big)\cdot\\&\qquad(\|P(u^{(n)},v^{(n)})\|_{B^{s-2}_{p,r}}+\|Q(u^{(n)},v^{(n)})\|_{B^{s-2}_{p,r}})\mathrm{d}\tau.\end{aligned}\tag{17}$$

Thanks to the product law in Besov spaces, one has

$$\|u_x^{(n)}v_x^{(n)}-u^{(n)}v^{(n)}\|_{B^{s-1}_{p,r}}\le C(\|u^{(n)}\|_{B^s_{p,r}}+\|v^{(n)}\|_{B^s_{p,r}})^2,$$

$$\begin{aligned}&\|P(u^{(n)},v^{(n)})\|_{B^{s-2}_{p,r}}+\|Q(u^{(n)},v^{(n)})\|_{B^{s-2}_{p,r}}\\&\quad\le C\Big[(\|u^{(n)}\|_{B^s_{p,r}}+\|v^{(n)}\|_{B^s_{p,r}})+(\|u^{(n)}\|_{B^s_{p,r}}+\|v^{(n)}\|_{B^s_{p,r}})^2+(\|u^{(n)}\|_{B^s_{p,r}}+\|v^{(n)}\|_{B^s_{p,r}})^4\Big],\end{aligned}$$

which combined with (17) leads to

$$\begin{aligned}&\|u^{(n+1)}(t)\|_{B^s_{p,r}}+\|v^{(n+1)}(t)\|_{B^s_{p,r}}\\&\quad\le\exp\Big(C\int_0^t(\|u^{(n)}(\tau)\|_{B^s_{p,r}}+\|v^{(n)}(\tau)\|_{B^s_{p,r}})^2\mathrm{d}\tau\Big)\cdot(\|u_0\|_{B^s_{p,r}}+\|v_0\|_{B^s_{p,r}})+\\&\qquad C\int_0^t\exp\Big(C\int_\tau^t(\|u^{(n)}(\tau')\|_{B^s_{p,r}}+\|v^{(n)}(\tau')\|_{B^s_{p,r}})^2\mathrm{d}\tau'\Big)\cdot\Big[(\|u^{(n)}\|_{B^s_{p,r}}+\|v^{(n)}\|_{B^s_{p,r}})+\\&\qquad(\|u^{(n)}\|_{B^s_{p,r}}+\|v^{(n)}\|_{B^s_{p,r}})^2+(\|u^{(n)}\|_{B^s_{p,r}}+\|v^{(n)}\|_{B^s_{p,r}})^4\Big]\mathrm{d}\tau.\end{aligned}\tag{18}$$

Let us choose a $T>0$ such that

$$T\le\min\{\frac{3-2M_0}{4C(1+6M_0)},\frac{1}{16CM_0^2}\},$$

where $M_0=\|u_0\|_{B^s_{p,r}}+\|v_0\|_{B^s_{p,r}}$ and suppose by induction that for all $t\in[0,T]$

$$\|u^{(n)}(t)\|_{B^s_{p,r}}+\|v^{(n)}(t)\|_{B^s_{p,r}}\le\frac{2M_0}{(1-16CM_0^2t)^{\frac12}}.\tag{19}$$

Then we obtain from (19) that for any $0\le\tau\le t$

$$C\int_\tau^t(\|u^{(n)}(\tau')\|_{B^s_{p,r}}+\|v^{(n)}(\tau')\|_{B^s_{p,r}})^2\mathrm{d}\tau'=\frac14[\ln(1-16CM_0^2\tau)-\ln(1-16CM_0^2t)].$$

And inserting the above inequality and (19) into (18) leads to

$$\begin{aligned}&\|u^{(n+1)}(t)\|_{B^s_{p,r}}+\|v^{(n+1)}(t)\|_{B^s_{p,r}}\\&\quad\le(1-16CM_0^2t)^{-\frac14}\Big[M_0+C\int_0^t(2M_0(1-16CM_0^2t)^{-\frac14}+4M_0^2(1-16CM_0^2t)^{-\frac34}+\end{aligned}$$

$$16M_0^4(1-16CM_0^2t)^{-\frac{7}{4}})\mathrm{d}\tau\Big]$$
$$\leq \frac{8C(1+6M_0)M_0t+4M_0^2}{3(1-16CM_0^2t)^{\frac{1}{2}}},$$

which implies

$$\|u^{(n+1)}(t)\|_{B^s_{p,r}}+\|v^{(n+1)}(t)\|_{B^s_{p,r}}\leq \frac{2M_0}{(1-16CM_0^2t)^{\frac{1}{2}}}.$$

Therefore, $\big(u^{(n)},v^{(n)}\big)_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,T];B^s_{p,r})^2$.

Since $B^s_{p,r}$ is an algebra, one can deduce from the Moser-type estimates (see Lemma 2.3 (II)) and equation (15) that

$$(\partial_t u^{(n+1)},\partial_t v^{(n+1)})\in C([0,T];B^{s-1}_{p,r})^2$$

uniformly bounded, which yields that the sequence $\big(u^{(n)},v^{(n)}\big)_{n\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)\times E^s_{p,r}(T)$.

Now, let us prove that $\big(u^{(n)},v^{(n)}\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})^2$. By (15), we have that, for all $n,k\in\mathbb{N}$,

$$\begin{cases}\big[\partial_t+\frac{1}{2}(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})\partial_x\big](m^{(n+k+1)}-m^{(n+1)})=F(u^{(n+k)},u^{(n)},v^{(n+k)},v^{(n)}),\\ \big[\partial_t+\frac{1}{2}(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})\partial_x\big](n^{(n+k+1)}-n^{(n+1)})=G(u^{(n+k)},u^{(n)},v^{(n+k)},v^{(n)}),\end{cases}\tag{20}$$

where

$$\begin{aligned}&F(u^{(n+k)},u^{(n)},v^{(n+k)},v^{(n)})\\ &=b(u_x^{(n+k)}-u_x^{(n)})+\frac{1}{2}\big[(u_x^{(n+k)}v_x^{(n+k)}-u_x^{(n)}v_x^{(n)})-(u^{(n+k)}v^{(n+k)}-u^{(n)}v^{(n)})\big]\partial_x m^{(n+1)}+\\ &\quad\frac{1}{2}[2u_x^{(n)}v^{(n)}-(u_x^{(n)}v_x^{(n)})_x](m^{(n+k)}-m^{(n)})+\frac{1}{2}\Big[2\big[(u_x^{(n+k)}-u_x^{(n)})v^{(n+k)}+(v^{(n+k)}-v^{(n)})u_x^{(n)}\big]-\\ &\quad\big[(u_x^{(n+k)}-u_x^{(n)})v_x^{(n+k)}+(v_x^{(n+k)}-v_x^{(n)})u_x^{(n)}\big]_x\Big]m^{(n+k)},\\ &G(u^{(n+k)},u^{(n)},v^{(n+k)},v^{(n)})\\ &=b(v_x^{(n+k)}-v_x^{(n)})+\frac{1}{2}\big[(u_x^{(n+k)}v_x^{(n+k)}-u_x^{(n)}v_x^{(n)})-(u^{(n+k)}v^{(n+k)}-u^{(n)}v^{(n)})\big]\partial_x n^{(n+1)}+\\ &\quad\frac{1}{2}[2v_x^{(n)}u^{(n)}-(u_x^{(n)}v_x^{(n)})_x](n^{(n+k)}-n^{(n)})+\frac{1}{2}\Big[2\big[(v_x^{(n+k)}-v_x^{(n)})u^{(n+k)}+(u^{(n+k)}-u^{(n)})v_x^{(n)}\big]-\\ &\quad\big[(v_x^{(n+k)}-v_x^{(n)})u_x^{(n+k)}+(u_x^{(n+k)}-u_x^{(n)})v_x^{(n)}\big]_x\Big]n^{(n+k)}.\end{aligned}$$

Note that the equation (20) is equivalent to the following one:

$$\begin{cases}(1-\partial_x^2)[\big[\partial_t+\frac{1}{2}(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})\partial_x\big](u^{(n+k+1)}-u^{(n+1)})]=\overline{F}^{(n,k)}\\ (1-\partial_x^2)[\big[\partial_t+\frac{1}{2}(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})\partial_x\big](v^{(n+k+1)}-v^{(n+1)})]=\overline{G}^{(n,k)}\end{cases}\tag{21}$$

with

$$\overline{F}^{(n,k)}=(1-\partial_x^2)[\frac{1}{2}(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})\partial_x(u^{(n+k+1)}-u^{(n+1)})]-$$

$$\frac{1}{2}(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})\partial_x(1-\partial_x^2)(u^{(n+k+1)}-u^{(n+1)})+F,$$
$$\overline{G}^{(n,k)}=(1-\partial_x^2)[\frac{1}{2}(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})\partial_x(v^{(n+k+1)}-v^{(n+1)})]-$$
$$\frac{1}{2}(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})\partial_x(1-\partial_x^2)(v^{(n+k+1)}-v^{(n+1)})+G.$$

Applying the operator $(1-\partial_x^2)^{-1}$ to (21), one obtains that

$$\begin{cases}\big[\partial_t+\frac{1}{2}(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})\partial_x\big](u^{(n+k+1)}-u^{(n+1)})=(1-\partial_x^2)^{-1}\overline{F}^{(n,k)}\\ \big[\partial_t+\frac{1}{2}(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})\partial_x\big](v^{(n+k+1)}-v^{(n+1)})=(1-\partial_x^2)^{-1}\overline{G}^{(n,k)}.\end{cases}\tag{22}$$

Thanks to Lemma 2.1, for every $t\in[0,T]$, one gets

$$\begin{aligned}&\exp\Big(-C\int_0^t\|\frac{1}{2}(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})\|_{B^{s-1}_{p,r}}\mathrm{d}\tau\Big)\cdot\|(u^{(n+k+1)}-u^{(n+1)})(t)\|_{B^{s-1}_{p,r}}\\&\le C\int_0^t\exp\Big(-C\int_0^\tau\|\frac{1}{2}(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})(\tau')\|_{B^{s-1}_{p,r}}\mathrm{d}\tau'\Big)\cdot\|\overline{F}^{(n,k)}\|_{B^{s-3}_{p,r}}\mathrm{d}\tau+\\&\|u_0^{(n+k+1)}-u_0^{(n+1)}\|_{B^{s-1}_{p,r}},\\&\exp\Big(-C\int_0^t\|\frac{1}{2}(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})\|_{B^{s-1}_{p,r}}\mathrm{d}\tau\Big)\cdot\|(v^{(n+k+1)}-v^{(n+1)})(t)\|_{B^{s-1}_{p,r}}\\&\le C\int_0^t\exp\Big(-C\int_0^\tau\|\frac{1}{2}(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})(\tau')\|_{B^{s-1}_{p,r}}\mathrm{d}\tau'\Big)\cdot\|\overline{G}^{(n,k)}\|_{B^{s-3}_{p,r}}\mathrm{d}\tau+\\&\|v_0^{(n+k+1)}-v_0^{(n+1)}\|_{B^{s-1}_{p,r}}.\end{aligned}\tag{23}$$

Since $s>\max\{2+\frac{1}{p},\frac{5}{2}\}$, by the product law in Besov spaces, one has

$$\begin{aligned}&\|(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})\partial_x(u^{(n+k+1)}-u^{(n+1)})\|_{B^{s-1}_{p,r}}\\&\quad\le C\|u^{(n+k+1)}-u^{(n+1)}\|_{B^{s-1}_{p,r}}(\|u^{(n+k)}\|^2_{B^s_{p,r}}+\|v^{(n+k)}\|^2_{B^s_{p,r}}),\\&\|(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})\partial_x(1-\partial_x^2)(u^{(n+k+1)}-u^{(n+1)})\|_{B^{s-3}_{p,r}}\\&\quad\le C\|u^{(n+k+1)}-u^{(n+1)}\|_{B^{s-1}_{p,r}}(\|u^{(n+k)}\|^2_{B^s_{p,r}}+\|v^{(n+k)}\|^2_{B^s_{p,r}}),\end{aligned}$$

and

$$\begin{aligned}\|F\|_{B^{s-3}_{p,r}}\le&C\|u^{(n+k)}-u^{(n)}\|_{B^{s-1}_{p,r}}\big(\|u^{(n)}\|^2_{B^s_{p,r}}+\|u^{(n+1)}\|^2_{B^s_{p,r}}+\|u^{(n+k)}\|^2_{B^s_{p,r}}+\\&\|v^{(n+k)}\|^2_{B^s_{p,r}}+\|v^{(n)}\|^2_{B^s_{p,r}}+|b|\big)+C\|v^{(n+k)}-v^{(n)}\|_{B^{s-1}_{p,r}}(\|u^{(n)}\|^2_{B^s_{p,r}}+\\&\|u^{(n+1)}\|^2_{B^s_{p,r}}+\|u^{(n+k)}\|^2_{B^s_{p,r}}),\\\|G\|_{B^{s-3}_{p,r}}\le&C\|v^{(n+k)}-v^{(n)}\|_{B^{s-1}_{p,r}}\big(\|v^{(n)}\|^2_{B^s_{p,r}}+\|v^{(n+1)}\|^2_{B^s_{p,r}}+\|v^{(n+k)}\|^2_{B^s_{p,r}}+\\&\|u^{(n+k)}\|^2_{B^s_{p,r}}+\|u^{(n)}\|^2_{B^s_{p,r}}+|b|\big)+C\|u^{(n+k)}-u^{(n)}\|_{B^{s-1}_{p,r}}(\|v^{(n)}\|^2_{B^s_{p,r}}+\\&\|v^{(n+1)}\|^2_{B^s_{p,r}}+\|v^{(n+k)}\|^2_{B^s_{p,r}}).\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}&\|\overline{F}^{(n,k)}\|_{B^{s-3}_{p,r}}+\|\overline{G}^{(n,k)}\|_{B^{s-3}_{p,r}}\\&\quad\le C\big[(\|u^{(n+k)}-u^{(n)}\|_{B^{s-1}_{p,r}}+\|v^{(n+k)}-v^{(n)}\|_{B^{s-1}_{p,r}})\big(\|u^{(n)}\|^2_{B^s_{p,r}}+\|u^{(n+1)}\|^2_{B^s_{p,r}}+\|u^{(n+k)}\|^2_{B^s_{p,r}}+\end{aligned}$$

$$\|v^{(n+k)}\|^2_{B^s_{p,r}}+\|v^{(n+1)}\|^2_{B^s_{p,r}}+\|v^{(n)}\|^2_{B^s_{p,r}}+|b|\big)+(\|u^{(n+k+1)}-u^{(n+1)}\|_{B^{s-1}_{p,r}}+ \|v^{(n+k+1)}-v^{(n+1)}\|_{B^{s-1}_{p,r}})\cdot(\|u^{(n+k)}\|^2_{B^s_{p,r}}+\|v^{(n+k)}\|^2_{B^s_{p,r}})\big].$$

Thus, we have that

$$\begin{aligned}
&\exp\Big(-C\int_0^t\|\frac{1}{2}(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})\|_{B^{s-1}_{p,r}}\mathrm{d}\tau\Big)\cdot\big(\|(u^{(n+k+1)}-u^{(n+1)})(t)\|_{B^{s-1}_{p,r}}+\\
&\quad\|(v^{(n+k+1)}-v^{(n+1)})(t)\|_{B^{s-1}_{p,r}}\big)\\
&\le\big(\|u_0^{(n+k+1)}-u_0^{(n+1)}\|_{B^{s-1}_{p,r}}+\|v_0^{(n+k+1)}-v_0^{(n+1)}\|_{B^{s-1}_{p,r}}\big)+\\
&\quad C\int_0^t\exp\Big(-C\int_0^\tau\|\frac{1}{2}(u_x^{(n+k)}v_x^{(n+k)}-u^{(n+k)}v^{(n+k)})(\tau')\|_{B^{s-1}_{p,r}}\mathrm{d}\tau'\Big)\cdot\big[(\|u^{(n+k)}-u^{(n)}\|_{B^{s-1}_{p,r}}+\\
&\quad\|v^{(n+k)}-v^{(n)}\|_{B^{s-1}_{p,r}})\big(\|u^{(n)}\|^2_{B^s_{p,r}}+\|u^{(n+1)}\|^2_{B^s_{p,r}}+\|u^{(n+k)}\|^2_{B^s_{p,r}}+\|v^{(n+k)}\|^2_{B^s_{p,r}}+\\
&\quad\|v^{(n+1)}\|^2_{B^s_{p,r}}+\|v^{(n)}\|^2_{B^s_{p,r}}+|b|\big)+(\|u^{(n+k+1)}-u^{(n+1)}\|_{B^{s-1}_{p,r}}+\|v^{(n+k+1)}-v^{(n+1)}\|_{B^{s-1}_{p,r}})\times\\
&\quad(\|u^{(n+k)}\|^2_{B^s_{p,r}}+\|v^{(n+k)}\|^2_{B^s_{p,r}})\big]\mathrm{d}\tau. \qquad (24)
\end{aligned}$$

Since $(u^{(n)},v^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)\times E^s_{p,r}(T)$ and

$$u_0^{(n+k+1)}-u_0^{(n+1)}=S_{n+k+1}u_0-S_{n+1}u_0=\sum_{q=n+1}^{n+k}\Delta_q u_0,$$

$$v_0^{(n+k+1)}-v_0^{(n+1)}=S_{n+k+1}v_0-S_{n+1}v_0=\sum_{q=n+1}^{n+k}\Delta_q v_0,$$

there exists a constant $C(T)$ independent of n and k such that for all $t\in[0,T]$

$$\begin{aligned}
&\|(u^{(n+k+1)}-u^{(n+1)})(t)\|_{B^{s-1}_{p,r}}+\|(v^{(n+k+1)}-v^{(n+1)})(t)\|_{B^{s-1}_{p,r}}\\
&\le C(T)\Big(2^{-n}+\int_0^t\|(u^{(n+k)}-u^{(n)})(\tau)\|_{B^{s-1}_{p,r}}+\|(v^{(n+k)}-v^{(n)})(\tau)\|_{B^{s-1}_{p,r}}\mathrm{d}\tau\Big).
\end{aligned}$$

And by induction with regard to index n, one can deduce that

$$\begin{aligned}
&\|u^{(n+k+1)}-u^{(n+1)}\|_{L^\infty_T(B^{s-1}_{p,r})}+\|v^{(n+k+1)}-v^{(n+1)}\|_{L^\infty_T(B^{s-1}_{p,r})}\\
&\le\frac{(tC(T))^{n+1}}{(n+1)!}(\|u^{(k)}\|_{L^\infty_T(B^s_{p,r})}+\|v^{(k)}\|_{L^\infty_T(B^s_{p,r})})+C(T)\sum_{i=0}^n 2^{-(n-i)}\frac{(tC(T))^i}{i!}.
\end{aligned}$$

Similarly, $\|u^{(k)}\|_{L^\infty_T(B^s_{p,r})},\|v^{(k)}\|_{L^\infty_T(B^s_{p,r})}$ can be bounded independent of k. Hence, we conclude that there exists some new constant $C'(T)$ independent of n and k such that

$$\|(u^{(n+k+1)}-u^{(n+1)})(t)\|_{B^{s-1}_{p,r}}+\|v^{(n+k+1)}-v^{(n+1)}\|_{L^\infty_T(B^{s-1}_{p,r})}\le 2^{-n}C'(T).$$

Therefore, $(u^{(n)},v^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})^2$.

Proof of Theorem 3.1 According to Lemma 3.3, we have that $(u^{(n)},v^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})\times C([0,T];B^{s-1}_{p,r})$, so it converges to some function $(u,v)\in C([0,T];B^{s-1}_{p,r})\times C([0,T];B^{s-1}_{p,r})$. The only thing that we need to do is to check that (u,v) belongs to $E^s_{p,r}(T)\times E^s_{p,r}(T)$ and solves the Cauchy problem (1). Since $(u^{(n)},v^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded in

$L^\infty([0,T];B^s_{p,r})\times L^\infty([0,T];B^s_{p,r})$ according to Lemma 3.3, the Fatou property for the Besov spaces guarantees that (u,v) also belongs to $L^\infty([0,T];B^s_{p,r})\times L^\infty([0,T];B^s_{p,r})$.

On the other hand, as $(u^{(n)},v^{(n)})_{n\in\mathbb{N}}$ converges to (u,v) in $C([0,T];B^{s-1}_{p,r})\times C([0,T];B^{s-1}_{p,r})$, an interpolation argument ensures that the convergence holds in $C([0,T];B^{s'}_{p,r})\times C([0,T];B^{s-1}_{p,r})$, for any $s'<s$. It is then easy to cross the limit in the equation (15) and to conclude that (u,v) is indeed a solution to the Cauchy problem (1). Since (u,v) belongs to $L^\infty([0,T];B^s_{p,r})\times L^\infty([0,T];B^s_{p,r})$, the right-hand side of the equation

$$\partial_t m^{(n+1)}+\frac{1}{2}\big(u^{(n)}_x v^{(n)}_x-u^{(n)}v^{(n)}\big)\partial_x m^{(n+1)}=P(u^{(n)},v^{(n)})$$

belongs to $L^\infty([0,T];B^{s-2}_{p,r})\times L^\infty([0,T];B^{s-2}_{p,r})$. Especially, for the case $r<\infty$, Lemma 2.2 implies that $(u,v)\in C([0,T];B^{s'}_{p,r})\times C([0,T];B^{s'}_{p,r})$ for any $s'<s$. Finally, using the equation again, we see that $(\partial_t u,\partial_t v)\in C([0,T];B^{s-1}_{p,r})\times C([0,T];B^{s-1}_{p,r})$ if $r<\infty$, and in $L^\infty([0,T];B^{s-1}_{p,r})\times L^\infty([0,T];B^{s-1}_{p,r})$ otherwise. Moreover, a standard use of a sequence of viscosity approximate solutions $(u_\epsilon,v_\epsilon)_{\epsilon>0}$ for the Cauchy problem (1) which converges uniformly in

$$C([0,T];B^s_{p,r})\cap C^1([0,T];B^{s-1}_{p,r})\times C([0,T];B^s_{p,r})\cap C^1([0,T];B^{s-1}_{p,r})$$

leads to the continuity of the solution (u,v) in $E^s_{p,r}(T)\times E^s_{p,r}(T)$. The proof is completed. □

Acknowledgements We thank the referees for their time and comments.

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