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Categorification of Spin Modules of Enveloping Algebra of Lie Algebra of Type D_4

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Abstract In this paper we give a categorification of the n-th tensor products of the spin modules of enveloping algebra of lie algebra of type D_4 via some subcategories and projective functors of the BGG category of the general linear Lie algebra gl_n over the complex field $\mathbb{C}$.

Keywords categorification; spin modules; BGG category; projective functors.

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1. Introduction

The terminology of categorification was introduced by Crane [6] and the idea originates from the earlier joint work with Frenkel [5]. The general idea is that, replacing a simpler object by something more complicated, one gets a bonus in the form of some extra structure which may be used to study the original object.

One of sources for categorification models is the Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$ associated to a fixed triangular decomposition of a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$. This category appears as a non-semisimple extension of the semisimple category of finite-dimensional $\mathfrak{g}$-modules, which not only contains a lot of new objects, but also has several nice properties [3]. It was found that category $\mathcal{O}$ has a number of different kinds of symmetries and connections to combinatorics and geometry. Most importantly, the category $\mathcal{O}$ leads to a variety of naturally defined functorial actions on this category, which are very useful and play an important role in various categorifications. Khovanov, Mazorchuk and Stroppel [11] presented several examples about categorifications of various representations of the symmetric group S_n via projective functors acting on certain subcategories of $\mathcal{O}$. Moreover, they categorified integral Specht modules over S_n and its Hecke algebra via some translation functors [12]. Mazorchuk and Stroppel [14] constructed a subcategory of $\mathcal{O}$ on which the actions of translation functors categorify cell modules and induced cell modules for Hecke algebras of finite Weyl groups. Soon after, they gave a categorification of Wedderburn's basis for $\mathbb{C}[S_n]$ (see [15]). Bernstein, Frenkel and Khovanov [1] studied a categorification of the n-th tensor products of the fundamental

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representation of $U(\mathrm{sl}_2)$ via certain singular blocks and projective functors of $\mathcal{O}(\mathrm{gl}_n)$. Sussan [17] generalized the case of sl_2 in [1] to that of sl_k and studied sl_k-link invariants. Xu and Yang [18] gave a categorification of the n-th tensor products of the spin representation of enveloping algebra of simple Lie algebra of type B_3 by using certain subcategories of $\mathcal{O}$ and projective functors of $\mathcal{O}(\mathrm{gl}_n)$. In this paper we give a categorification of the n-th tensor products of the spin modules of enveloping algebra of Lie algebra of type D_4.

The paper is organized as follows. In Section 2 we give some notations and results used in the sequal. In Section 3 we give a categorification of the n-th tensor products of the spin representation $V_{+\mathrm{sp}}$ of $U(\mathrm{so}(8,\mathbb{C}))$, and give a remark about the categorification of $(V_{-\mathrm{sp}})^{\otimes n}$.

2. Preliminaries

In this section we give some known notations and results. All the vector spaces and algebras are considered over the complex field $\mathbb{C}$. Denote by $K(\mathcal{C})$ the Grothendieck group of an abelian category $\mathcal{C}$. For an object $M \in \mathcal{C}$, denote by $[M]$ the equivalent class in $K(\mathcal{C})$, and denote by $[F]$ the abelian group homomorphism induced by the exact functor F. Let $\mathfrak{g}$ be a finite dimension reductive Lie algebra over $\mathbb{C}$, and $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be the triangular decomposition associated to a fixed Cartan subalgebra $\mathfrak{h}$. Let us denote the universal enveloping algebra of $\mathfrak{g}$ by $U(\mathfrak{g})$, the center of $U(\mathfrak{g})$ by $Z(\mathfrak{g})$ and the set of central characters by Θ. W denotes the Weyl group of $\mathfrak{g}$. Let ρ be the half sum of all the positive roots, and $M(\lambda)$ the Verma module with the highest weight $\lambda \in \mathfrak{h}^*$. We recall a shifted action of W defined as follows.

Definition 2.1 ([10]) *For any $w \in W$ and $\lambda \in \mathfrak{h}^*$, the dot action of W on $\mathfrak{h}^*$ is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$.*

Let θ_λ be the central character corresponding to $\lambda \in \mathfrak{h}^*$ and Λ^+_{dot} be the set of dominant weight with respect to the dot action defined above. Then we have the following conclusion.

Theorem 2.2 ([10]) *There exists a bijective map between Λ^+_{dot} and Θ defined by sending λ to θ_λ.*

Next we introduce some results about general lineal Lie algebra gl_n. Let $\mathrm{gl}_n = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a triangular decomposition of gl_n, and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be a standard orthogonal basis in $\mathbb{R}^n$. We identify $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n$ with $\mathfrak{h}^*$ so that $\Phi^+ = \{\varepsilon_i - \varepsilon_j | 1 \le i < j \le n\}$ is the set of positive roots and $\Pi = \{\beta_i = \varepsilon_i - \varepsilon_{i+1} | 1 \le i \le n-1\}$ is the set of simple roots. Then by definition

$$\rho = \frac{n-1}{2}\varepsilon_1 + \frac{n-3}{2}\varepsilon_2 + \cdots + \frac{1-n}{2}\varepsilon_n.$$

The Weyl group of gl_n is isomorphic to the symmetric group S_n, and the generator s_i of S_n acts on $\mathfrak{h}^*$ by permuting ε_i and ε_{i+1}. Let L_n be the n-dimensional fundamental representation of gl_n, and $u_1, u_2, \ldots, u_n$ be the weight vectors with weights $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, respectively. Then L_n^* has weights $-\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_n$. Denote by $\mathrm{Sym}^2 L_n$ the symmetric square of L_n generated by $\{u_i \otimes u_i | 1 \le i \le n\}$ and $\{u_i \otimes u_j + u_j \otimes u_i | 1 \le i < j \le n\}$. Denote by $\mathrm{Alt}^2 L_n$ the alternative square of L_n generated by $\{u_i \otimes u_j - u_j \otimes u_i | 1 \le i < j \le n\}$. It is easy to obtain $\{2\varepsilon_i | 1 \le i \le n\}$

and $\{\varepsilon_i+\varepsilon_j|1\le i<j\le n\}$ are weights of Sym^2L_n, $\{\varepsilon_i+\varepsilon_j|1\le i<j\le n\}$ are weights of Alt^2L_n. Denote by $\mathrm{Sym}^2L_n^*$ and $\mathrm{Alt}^2L_n^*$ the symmetric square and alternative square of L_n^*, respectively. It is also easy to obtain $\{-2\varepsilon_i|1\le i\le n\}$ and $\{-(\varepsilon_i+\varepsilon_j)|1\le i<j\le n\}$ are weights of $\mathrm{Sym}^2L_n^*$, $\{-(\varepsilon_i+\varepsilon_j)|1\le i<j\le n\}$ are weights of $\mathrm{Alt}^2L_n^*$.

Definition 2.3 ([10]) *The BGG category $\mathcal{O}$ is defined to be the full subcategory of Mod-$U(\mathfrak{g})$ whose objects are the modules satisfying: M is a finite generated, $\mathfrak{h}$-semisimple $U(\mathfrak{g})$-module, and M is locally $\mathfrak{n}^+$-finite, that is, for every $v\in M$, $U(\mathfrak{n}^+)v$ is of finite dimension.*

For $M\in\mathcal{O}$, define a $U(\mathfrak{g})$-submodule of M for each fixed $\theta\in\Theta$ by

$$M_\theta:=\{v\in M|(z-\theta(z))^n v=0 \text{ for some } n>0 \text{ depending on } z\}.$$

Denote by $\mathcal{O}_\theta$ the full subcategory of $\mathcal{O}$ whose objects are the modules M_θ.

Proposition 2.4 ([10]) *Category $\mathcal{O}$ is the direct sum of the subcategory $\mathcal{O}_\theta$ as θ ranges over the central characters of the form θ_λ.*

Denote by proj_θ the functor from $\mathcal{O}$ to $\mathcal{O}$ defined by $\mathrm{proj}_\theta(M)=M_\theta$ for $M=\bigoplus_{\theta=\theta_\lambda}M_\theta$. For any $V\in$ Mod-$U(\mathfrak{g})$, we denote $V\otimes$- by F_V for convenience.

Definition 2.5 ([10]) *A functor $F:\mathcal{O}\longrightarrow\mathcal{O}$ is called projective if it is isomorphic to direct summand of some F_V, where V is finite dimensional.*

Here are some basic general properties of projective functors [2].

Proposition 2.6 *(1) Projective functors are exact.*

(2) Any direct sum and composition of projective functors is a projective functor.

(3) The functor proj_θ is a projective functor.

(4) Let F,G be projective functors. If $[F]=[G]$, then $F\simeq G$.

For a fixed central character $\theta\in\Theta$, the set $\{[M(\lambda)]|\theta=\theta_\lambda\}=\{[M(\mu)]|\mu\in W\cdot\lambda\}$ forms a $\mathbb{Z}$-basis of $K(\mathcal{O}_\theta)$. Hence the set $\{[M(\lambda)]|$ there exists $\theta\in\Theta$ such that $\theta=\theta_\lambda\}$ forms a $\mathbb{Z}$-basis of $K(\mathcal{O})$. Moreover, the following conclusion holds.

Proposition 2.7 ([1]) *Let V be a finite dimension $U(\mathfrak{g})$-module, $v_i\in V$ be the weight vectors corresponding to weights μ_i, where $1\le i\le n$. Then $[V\bigotimes M(\lambda)]=\sum_{i=1}^n[M(\lambda+\mu_i)]$.*

Definition 2.8 ([9]) *The universal enveloping algebra $U(\mathrm{so}(8,\mathbb{C}))$ is the associative algebra over $\mathbb{C}$ with unity generated by e_i,f_i,h_i $(1\le i\le 4)$ subject to the following defining relations*

(1) $h_ih_j=h_jh_i$, $e_if_j-f_je_i=\delta_{ij}h_i$, $1\le i,\,j\le 4$;

(2) $h_ie_j-e_jh_i=a_{ij}e_j$, $h_if_j-f_jh_i=-a_{ij}f_j$, $1\le i,\,j\le 4$;

(3) $\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}e_i^{1-a_{ij}-k}e_je_i^k=0$ for $i\ne j$;

(4) $\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}f_i^{1-a_{ij}-k}f_jf_i^k=0$ for $i\ne j$,

where a_{ij} $(1\le i,j\le 4)$ are the entries of the Cartan matrix A of $\mathrm{so}(8,\mathbb{C})$.

Definition 2.9 *The spin modules $V_{\pm\mathrm{sp}}$ are 8-dimensional vector spaces*

$$V_{+\mathrm{sp}} = \sum_{\substack{\xi_i=\pm,\\ \xi_1\xi_2\xi_3\xi_4=+}} (\xi_1,\xi_2,\xi_3,\xi_4), \quad V_{-\mathrm{sp}} = \sum_{\substack{\xi_i=\pm,\\ \xi_1\xi_2\xi_3\xi_4=-}} (\xi_1,\xi_2,\xi_3,\xi_4)$$

with the $U(\mathrm{so}(8,\mathbb{C}))$-action given by

$$e_i(\xi_1,\xi_2,\xi_3,\xi_4) = \begin{cases} (+,-,\xi_3,\xi_4), & \text{if } (\xi_1,\xi_2)=(-,+);\\ (\xi_1,+,-,\xi_4), & \text{if } (\xi_2,\xi_3)=(-,+);\\ (\xi_1,\xi_2,+,+,), & \text{if } (\xi_3,\xi_4)=(-,-);\\ 0, & \text{otherwise.} \end{cases}$$

$$f_i(\xi_1,\xi_2,\xi_3,\xi_4) = \begin{cases} (-,+,\xi_3,\xi_4), & \text{if } (\xi_1,\xi_2)=(+,-);\\ (\xi_1,-,+,\xi_4), & \text{if } (\xi_2,\xi_3)=(+,-);\\ (\xi_1,\xi_2,-,-,), & \text{if } (\xi_3,\xi_4)=(+,+);\\ 0, & \text{otherwise.} \end{cases}$$

$$h_i(\xi_1,\xi_2,\xi_3,\xi_4) = (e_if_i - f_ie_i)(\xi_1,\xi_2,\xi_3,\xi_4).$$

For $V_{+\mathrm{sp}}$, we set

$$v_1=(-,-,-,-), v_2=(-,-,+,+), v_3=(-,+,-,+), v_4=(-,+,+,-)$$

$$v_5=(+,-,-,+), v_6=(+,-,+,-), v_7=(+,+,-,-), v_8=(+,+,+,+)$$

for convenience. For $V_{-\mathrm{sp}}$, we can do this similarly.

3. Categorification of the spin modules $V_{\pm\mathrm{sp}}$

In this section we give a categorification of the n-th tensor products $V_{+\mathrm{sp}}^{\otimes n}$ of the spin module $V_{+\mathrm{sp}}$ for $U(\mathrm{so}(8,\mathbb{C}))$. The categorification of $V_{-\mathrm{sp}}^{\otimes n}$ can be obtained similarly.

Let

$$\mathbf{A} = \{\mathbf{a}=(a_1,a_2,\ldots,a_n) \mid 1\le a_i\le 8 \text{ for } 1\le i\le n\},$$

$$\mathbf{D} = \{\mathbf{d}=(d_1,d_2,\ldots,d_8) \mid d_i\in\mathbb{Z}^{\ge 0} \text{ for any } 1\le i\le 8 \text{ and } \sum_{i=1}^{8} d_i = n\}.$$

For any $\mathbf{d}=(d_1,d_2,\ldots,d_8)$, $\mathbf{d'}=(d'_1,d'_2,\ldots,d'_8)\in\mathbf{D}$, we define

$$\mathbf{d}\sim\mathbf{d'} \Leftrightarrow \begin{cases} -d_3-d_4+d_5+d_6 = -d'_3-d'_4+d'_5+d'_6\\ -d_2+d_3-d_6+d_7 = -d'_2+d'_3-d'_6+d'_7\\ -d_3+d_4-d_5+d_6 = -d'_3+d'_4-d'_5+d'_6\\ -d_1+d_2-d_7+d_8 = -d'_1+d'_2-d'_7+d'_8 \end{cases}$$

It is easy to know that $\sim$ is an equivalence relation on $\mathbf{D}$. We denote by $[\mathbf{d}]$ and $\widetilde{\mathbf{D}}$ the equivalent class of $\mathbf{d}$ and the set of all the equivalent classes, respectively. For $\mathbf{d}\in\mathbf{D}$ and a sequence $\mathbf{a}\in\mathbf{A}$. Set

$$\mathbf{d}_i=(d_1,\ldots,d_{i-1},d_i-1,d_{i+1},\ldots,d_8),\ \mathbf{d}^i=(d_1,\ldots,d_{i-1},d_i+1,d_{i+1},\ldots,d_8)$$

$$\mathbf{d_a} = (d_1^{\mathbf{a}}, d_2^{\mathbf{a}}, \ldots, d_8^{\mathbf{a}}),\ \text{where } d_k^{\mathbf{a}} = \sharp\{a_m | a_m = k, 1 \le m \le n\} \text{ for } 1 \le k \le 8.$$

$$B_{[\mathbf{d}]} := \{v_{a_1} \otimes v_{a_2} \otimes \cdots \otimes v_{a_n} |\ \mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{A} \text{ and } \mathbf{d_a} \in [\mathbf{d}]\}.$$

Denote by ${}^{\mathbb{Z}}(V_{+\mathrm{sp}}^{\otimes n})_{[\mathbf{d}]}$ the $\mathbb{Z}$-module spanned by $B_{[\mathbf{d}]}$. Set

$${}^{\mathbb{Z}}(V_{+\mathrm{sp}}^{\otimes n}) = \bigoplus_{[\mathbf{d}] \in \widetilde{\mathbf{D}}} {}^{\mathbb{Z}}(V_{+\mathrm{sp}}^{\otimes n})_{[\mathbf{d}]}, \quad (V_{+\mathrm{sp}}^{\otimes n})_{[\mathbf{d}]} = \mathbb{C} \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}(V_{+\mathrm{sp}}^{\otimes n})_{[\mathbf{d}]}, \quad V_{+\mathrm{sp}}^{\otimes n} = \mathbb{C} \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}(V_{+\mathrm{sp}}^{\otimes n}).$$

It is easy to prove $V_{+\mathrm{sp}}^{\otimes n} = \bigoplus_{[\mathbf{d}] \in \widetilde{\mathbf{D}}} (V_{+\mathrm{sp}}^{\otimes n})_{[\mathbf{d}]}$. For any $\mathbf{a} \in \mathbf{A}$, set $M(a_1, \ldots, a_n) := M(a_1\varepsilon_1 + \cdots + a_n\varepsilon_n - \rho)$. For each $\mathbf{d} \in \mathbf{D}$, set $\lambda_{\mathbf{d}} = \sum_{i=0}^{7} (8-i) \sum_{j=1}^{d_{8-i}} \varepsilon_{d_8 + d_7 + \cdots + d_{9-i} + j}$. Denote by $\theta_{\mathbf{d}} = \eta(\lambda_{\mathbf{d}} - \rho)$ in Θ. Set $\mathcal{O}_{\mathbf{d}} := \mathcal{O}_{\theta_{\mathbf{d}}}$, $\mathcal{O}_{[\mathbf{d}]} := \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} \mathcal{O}_{\mathbf{d}'}$, $\mathcal{O}^n := \bigoplus_{[\mathbf{d}] \in \widetilde{\mathbf{D}}} \mathcal{O}_{[\mathbf{d}]}$. It is easy to see that there exists $w \in W$ satisfying $a_1\varepsilon_1 + \cdots + a_n\varepsilon_n - \rho = w \cdot \lambda_{\mathbf{d}}$. Hence we have the following theorem.

Theorem 3.1 *There exists an isomorphism of abelian groups $\gamma_n : K(\mathcal{O}^n) \to^{\mathbb{Z}} (V_{+\mathrm{sp}}^{\otimes n})$ defined by*

$$\gamma_n([M(a_1, a_2, \ldots, a_n)]) = v_{a_1} \otimes v_{a_2} \otimes \cdots \otimes v_{a_n},$$

for any $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{A}$. Moreover, the restriction of γ_n on $K(\mathcal{O}_{[\mathbf{d}]})$ is an abelian group isomorphism between $K(\mathcal{O}_{[\mathbf{d}]})$ and ${}^{\mathbb{Z}}(V_{+\mathrm{sp}}^{\otimes n})_{[\mathbf{d}]}$.

For $\mathbf{d} = (d_1, d_2, \ldots, d_8) \in \mathbf{D}$, we define

$$c_1(\mathbf{d}) = d_6 + d_5 - d_4 - d_3;\ c_2(\mathbf{d}) = d_7 - d_6 + d_3 - d_2;$$

$$c_3(\mathbf{d}) = d_6 - d_5 + d_4 - d_3;\ c_4(\mathbf{d}) = d_8 - d_7 + d_2 - d_1,$$

and for $1 \le i \le 4$, denote the sign function of $c_i(\mathbf{d})$ by $\mathrm{sgn}(c_i(\mathbf{d}))$. Then set

$$\mathcal{H}_i^{\mathrm{sgn}(c_i(\mathbf{d}))}([\mathbf{d}]) := (\mathrm{Id}_{\mathcal{O}_{[\mathbf{d}]}})^{\oplus c_i(\mathbf{d})\mathrm{sgn}(c_i(\mathbf{d}))} : \mathcal{O}_{[\mathbf{d}]} \longrightarrow \mathcal{O}_{[\mathbf{d}]},$$

where $\mathrm{Id}_{\mathcal{O}_{[\mathbf{d}]}}$ is the identity functor on $\mathcal{O}_{[\mathbf{d}]}$. It is easy to check that the definition of $\mathcal{H}_i^{\mathrm{sgn}(c_i(\mathbf{d}))}([\mathbf{d}])$ is independent of the choice of the representative $\mathbf{d}$ in $[\mathbf{d}]$.

For any $\mathbf{d} = (d_1, d_2, \ldots, d_8) \in \mathbf{D}$, define

$$\mathcal{E}_{+1}^{+3}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_3^5}} \circ F_{\mathrm{sym}^2 L_n} : \mathcal{O}_{\mathbf{d}} \longrightarrow \mathcal{O}_{\mathbf{d}_3^5};$$

$$\mathcal{E}_{+1}^{+4}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_4^6}} \circ F_{\mathrm{sym}^2 L_n} : \mathcal{O}_{\mathbf{d}} \longrightarrow \mathcal{O}_{\mathbf{d}_4^6};$$

$$\mathcal{E}_{+1}^{-3}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_3^5}} \circ F_{\mathrm{alt}^2 L_n} : \mathcal{O}_{\mathbf{d}} \longrightarrow \mathcal{O}_{\mathbf{d}_3^5};$$

$$\mathcal{E}_{+1}^{-4}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_4^6}} \circ F_{\mathrm{alt}^2 L_n} : \mathcal{O}_{\mathbf{d}} \longrightarrow \mathcal{O}_{\mathbf{d}_4^6}.$$

By equivalent relations we set

$$[\overleftarrow{\mathbf{d}_{+1}}] := [\mathbf{d}_4^6] = [\mathbf{d}_3^5],\ [\overleftarrow{\mathbf{d}_{+2}}] := [\mathbf{d}_2^3] = [\mathbf{d}_6^7],$$

$$[\overleftarrow{\mathbf{d}_{+3}}] := [\mathbf{d}_3^4] = [\mathbf{d}_5^6],\ [\overleftarrow{\mathbf{d}_{+4}}] := [\mathbf{d}_1^2] = [\mathbf{d}_7^8],$$

$$[\overrightarrow{\mathbf{d}_{+1}}] := [\mathbf{d}_5^3] = [\mathbf{d}_6^4],\ [\overrightarrow{\mathbf{d}_{+2}}] := [\mathbf{d}_3^2] = [\mathbf{d}_7^6],$$

$$[\overrightarrow{\mathbf{d}_{+3}}] := [\mathbf{d}_4^3] = [\mathbf{d}_6^5],\ [\overrightarrow{\mathbf{d}_{+4}}] := [\mathbf{d}_2^1] = [\mathbf{d}_8^7].$$

Define

$$\mathcal{E}_{+1}^{+}([\mathbf{d}]) := \bigoplus_{\mathbf{d}'\in[\mathbf{d}]} (\mathcal{E}_{+1}^{+3}(\mathbf{d}') \oplus \mathcal{E}_{+1}^{+4}(\mathbf{d}')) : \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{[\overleftarrow{\mathbf{d}_{+1}}]},$$

$$\mathcal{E}_{+1}^{-}([\mathbf{d}]) := \bigoplus_{\mathbf{d}'\in[\mathbf{d}]} (\mathcal{E}_{+1}^{-3}(\mathbf{d}') \oplus \mathcal{E}_{+1}^{-4}(\mathbf{d}')) : \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{[\overleftarrow{\mathbf{d}_{+1}}]}.$$

$$\mathcal{E}_{+2}^{2}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_2^3}} \circ F_{L_n} : \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}_2^3};$$

$$\mathcal{E}_{+2}^{6}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_6^7}} \circ F_{L_n} : \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}_6^7},$$

$$\mathcal{E}_{+2}([\mathbf{d}]) := \bigoplus_{\mathbf{d}'\in[\mathbf{d}]} (\mathcal{E}_{+2}^{2}(\mathbf{d}') \oplus \mathcal{E}_{+2}^{6}(\mathbf{d}')) : \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{[\overleftarrow{\mathbf{d}_{+2}}]}.$$

$$\mathcal{E}_{+3}^{3}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_3^4}} \circ F_{L_n} : \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}_3^4};$$

$$\mathcal{E}_{+3}^{5}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_5^6}} \circ F_{L_n} : \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}_5^6},$$

$$\mathcal{E}_{+3}([\mathbf{d}]) := \bigoplus_{\mathbf{d}'\in[\mathbf{d}]} (\mathcal{E}_{+3}^{3}(\mathbf{d}') \oplus \mathcal{E}_{+3}^{5}(\mathbf{d}')) : \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{[\overleftarrow{\mathbf{d}_{+3}}]}.$$

$$\mathcal{E}_{+4}^{1}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_1^2}} \circ F_{L_n} : \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}_1^2};$$

$$\mathcal{E}_{+4}^{7}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_7^8}} \circ F_{L_n} : \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}_7^8},$$

$$\mathcal{E}_{+4}([\mathbf{d}]) := \bigoplus_{\mathbf{d}'\in[\mathbf{d}]} (\mathcal{E}_{+4}^{1}(\mathbf{d}') \oplus \mathcal{E}_{+4}^{7}(\mathbf{d}')) : \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{[\overleftarrow{\mathbf{d}_{+4}}]}.$$

$$\mathcal{F}_{+1}^{+5}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_5^3}} \circ F_{\mathrm{sym}^2 L_n^*} : \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}_5^3};$$

$$\mathcal{F}_{+1}^{-5}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_5^3}} \circ F_{\mathrm{alt}^2 L_n^*} : \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}_5^3};$$

$$\mathcal{F}_{+1}^{+6}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_6^4}} \circ F_{\mathrm{sym}^2 L_n^*} : \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}_6^4};$$

$$\mathcal{F}_{+1}^{-6}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_6^4}} \circ F_{\mathrm{alt}^2 L_n^*} : \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}_6^4}.$$

$$\mathcal{F}_{+1}^{+}([\mathbf{d}]) := \bigoplus_{\mathbf{d}'\in[\mathbf{d}]} (\mathcal{F}_{+1}^{+5}(\mathbf{d}') \oplus \mathcal{F}_{+1}^{+6}(\mathbf{d}')) : \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{[\overrightarrow{\mathbf{d}_{+1}}]},$$

$$\mathcal{F}_{+1}^{-}([\mathbf{d}]) := \bigoplus_{\mathbf{d}'\in[\mathbf{d}]} (\mathcal{F}_{+1}^{-5}(\mathbf{d}') \oplus \mathcal{F}_{+1}^{-6}(\mathbf{d}')) : \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{[\overrightarrow{\mathbf{d}_{+1}}]}.$$

$$\mathcal{F}_{+2}^{3}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_3^2}} \circ F_{L_n^*} : \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}_3^2};$$

$$\mathcal{F}_{+2}^{7}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_7^6}} \circ F_{L_n^*} : \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}_7^6},$$

$$\mathcal{F}_{+2}([\mathbf{d}]) := \bigoplus_{\mathbf{d}'\in[\mathbf{d}]} (\mathcal{F}_{+2}^{3}(\mathbf{d}') \oplus \mathcal{F}_{+2}^{7}(\mathbf{d}')) : \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{[\overrightarrow{\mathbf{d}_{+2}}]}.$$

$$\mathcal{F}_{+3}^{4}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_4^3}} \circ F_{L_n^*} : \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}_4^3};$$

$$\mathcal{F}_{+3}^{6}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_6^5}} \circ F_{L_n^*} : \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}_6^5},$$

$$\mathcal{F}_{+3}([\mathbf{d}]) := \bigoplus_{\mathbf{d}'\in[\mathbf{d}]} (\mathcal{F}_{+3}^{4}(\mathbf{d}') \oplus \mathcal{F}_{+3}^{6}(\mathbf{d}')) : \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{[\rightarrow \mathbf{d}_{+3}]}.$$

$$\mathcal{F}_{+4}^{2}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}_2^1}} \circ F_{L_n^*} : \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}_2^1};$$

$$\mathcal{F}^8_{+4}(\mathbf{d}) := \mathrm{proj}_{\theta_{\mathbf{d}^7_8}} \circ F_{L^*_n} : \mathcal{O}_{\mathbf{d}} \to \mathcal{O}_{\mathbf{d}^7_8},$$
$$\mathcal{F}_{+4}([\mathbf{d}]) := \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\mathcal{F}^2_{+4}(\mathbf{d}') \oplus \mathcal{F}^8_{+4}(\mathbf{d}')) : \mathcal{O}_{[\mathbf{d}]} \to \mathcal{O}_{\overrightarrow{[\mathbf{d}_{+4}]}}.$$

It is easy to know that the functors defined above are projective functors by Definition 2.5 and Proposition 2.6. Therefore, they can induce abelian group homomorphisms of the corresponding Grothendieck groups. Next we give some formulas which will be useful.

Proposition 3.2 The actions of abelian group homomorphisms $[\mathcal{E}^{\pm j}_{+i}(\mathbf{d})], [\mathcal{F}^{\pm j}_{+i}(\mathbf{d})]$ are described as follows.

$$[\mathcal{E}^{+3}_{+1}(\mathbf{d})]([M(a_1, a_2, \ldots, a_n)]) = \sum_{\substack{m=1,\\ a_m=3}}^{n} [M(a_1, a_2, \ldots, a_{m-1}, a_m + 2, a_{m+1}, \ldots, a_n)] +$$
$$\sum_{\substack{1 \le i < j \le n,\\ (a_i, a_j) = (3,4) \text{or} (4,3)}} [M(a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_{j-1}, a_j + 1, a_{j+1}, \ldots, a_n)];$$

$$[\mathcal{E}^{+4}_{+1}(\mathbf{d})]([M(a_1, a_2, \ldots, a_n)]) = \sum_{\substack{m=1,\\ a_m=4}}^{n} [M(a_1, a_2, \ldots, a_{m-1}, a_m + 2, a_{m+1}, \ldots, a_n)] +$$
$$\sum_{\substack{1 \le i < j \le n,\\ (a_i, a_j) = (4,5) \text{or} (5,4)}} [M(a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_{j-1}, a_j + 1, a_{j+1}, \ldots, a_n)];$$

$$[\mathcal{E}^{-3}_{+1}(\mathbf{d})]([M(a_1, a_2, \ldots, a_n)])$$
$$= \sum_{\substack{1 \le i < j \le n,\\ (a_i, a_j) = (3,4) \text{or} (4,3)}} [M(a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_{j-1}, a_j + 1, a_{j+1}, \ldots, a_n)];$$

$$[\mathcal{E}^{-4}_{+1}(\mathbf{d})]([M(a_1, a_2, \ldots, a_n)])$$
$$= \sum_{\substack{1 \le i < j \le n,\\ (a_i, a_j) = (4,5) \text{or} (5,4)}} [M(a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_{j-1}, a_j + 1, a_{j+1}, \ldots, a_n)];$$

$$[\mathcal{E}^{2}_{+2}(\mathbf{d})]([M(a_1, a_2, \ldots, a_n)]) = \sum_{\substack{m=1,\\ a_m=2}}^{n} [M(a_1, a_2, \ldots, a_{m-1}, a_m + 1, a_{m+1}, \ldots, a_n)];$$

$$[\mathcal{E}^{6}_{+2}(\mathbf{d})]([M(a_1, a_2, \ldots, a_n)]) = \sum_{\substack{m=1,\\ a_m=6}}^{n} [M(a_1, a_2, \ldots, a_{m-1}, a_m + 1, a_{m+1}, \ldots, a_n)];$$

$$[\mathcal{E}^{3}_{+3}(\mathbf{d})]([M(a_1, a_2, \ldots, a_n)]) = \sum_{\substack{m=1,\\ a_m=3}}^{n} [M(a_1, a_2, \ldots, a_{m-1}, a_m + 1, a_{m+1}, \ldots, a_n)];$$

$$[\mathcal{E}^{5}_{+3}(\mathbf{d})]([M(a_1, a_2, \ldots, a_n)]) = \sum_{\substack{m=1,\\ a_m=5}}^{n} [M(a_1, a_2, \ldots, a_{m-1}, a_m + 1, a_{m+1}, \ldots, a_n)];$$

$$[\mathcal{E}^{1}_{+4}(\mathbf{d})]([M(a_1, a_2, \ldots, a_n)]) = \sum_{\substack{m=1,\\ a_m=1}}^{n} [M(a_1, a_2, \ldots, a_{m-1}, a_m + 1, a_{m+1}, \ldots, a_n)];$$

$$[\mathcal{E}_{+4}^{7}(\mathbf{d})]([M(a_1,a_2,\ldots,a_n)])=\sum_{\substack{m=1,\\ a_m=7}}^{n}[M(a_1,a_2,\ldots,a_{m-1},a_m+1,a_{m+1},\ldots,a_n)];$$

$$[\mathcal{F}_{+1}^{+5}(\mathbf{d})]([M(a_1,a_2,\ldots,a_n)])=\sum_{\substack{m=1,\\ a_m=5}}^{n}[M(a_1,a_2,\ldots,a_{m-1},a_m-2,a_{m+1},\ldots,a_n)]+$$

$$\sum_{\substack{1\le i<j\le n,\\ (a_i,a_j)=(5,4)\text{or}(4,5)}}[M(a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_{j-1},a_j-1,a_{j+1},\ldots,a_n)];$$

$$[\mathcal{F}_{+1}^{-5}(\mathbf{d})]([M(a_1,a_2,\ldots,a_n)])$$
$$=\sum_{\substack{1\le i<j\le n,\\ (a_i,a_j)=(5,4)\text{or}(4,5)}}[M(a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_{j-1},a_j-1,a_{j+1},\ldots,a_n)];$$

$$[\mathcal{F}_{+1}^{+6}(\mathbf{d})]([M(a_1,a_2,\ldots,a_n)])=\sum_{\substack{m=1,\\ a_m=6}}^{n}[M(a_1,a_2,\ldots,a_{m-1},a_m-2,a_{m+1},\ldots,a_n)]+$$

$$\sum_{\substack{1\le i<j\le n,\\ (a_i,a_j)=(5,6)\text{or}(6,5)}}[M(a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_{j-1},a_j-1,a_{j+1},\ldots,a_n)];$$

$$[\mathcal{F}_{+1}^{-6}(\mathbf{d})]([M(a_1,a_2,\ldots,a_n)])$$
$$=\sum_{\substack{1\le i<j\le n,\\ (a_i,a_j)=(6,5)\text{or}(5,6)}}[M(a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_{j-1},a_j-1,a_{j+1},\ldots,a_n)];$$

$$[\mathcal{F}_{+2}^{3}(\mathbf{d})]([M(a_1,a_2,\ldots,a_n)])=\sum_{\substack{m=1,\\ a_m=3}}^{n}[M(a_1,a_2,\ldots,a_{m-1},a_m-1,a_{m+1},\ldots,a_n)];$$

$$[\mathcal{F}_{+2}^{7}(\mathbf{d})]([M(a_1,a_2,\ldots,a_n)])=\sum_{\substack{m=1,\\ a_m=7}}^{n}[M(a_1,a_2,\ldots,a_{m-1},a_m-1,a_{m+1},\ldots,a_n)];$$

$$[\mathcal{F}_{+3}^{4}(\mathbf{d})]([M(a_1,a_2,\ldots,a_n)])=\sum_{\substack{m=1,\\ a_m=4}}^{n}[M(a_1,a_2,\ldots,a_{m-1},a_m-1,a_{m+1},\ldots,a_n)];$$

$$[\mathcal{F}_{+3}^{6}(\mathbf{d})]([M(a_1,a_2,\ldots,a_n)])=\sum_{\substack{m=1,\\ a_m=6}}^{n}[M(a_1,a_2,\ldots,a_{m-1},a_m-1,a_{m+1},\ldots,a_n)];$$

$$[\mathcal{F}_{+4}^{2}(\mathbf{d})]([M(a_1,a_2,\ldots,a_n)])=\sum_{\substack{m=1,\\ a_m=2}}^{n}[M(a_1,a_2,\ldots,a_{m-1},a_m-1,a_{m+1},\ldots,a_n)];$$

$$[\mathcal{F}_{+4}^{8}(\mathbf{d})]([M(a_1,a_2,\ldots,a_n)])=\sum_{\substack{m=1,\\ a_m=8}}^{n}[M(a_1,a_2,\ldots,a_{m-1},a_m-1,a_{m+1},\ldots,a_n)].$$

Proof As examples, we only prove the first one, the second one and the fifth one. The rest formulas can be shown similarly.

$$[\mathcal{E}_{+1}^{+3}(\mathbf{d})]([M(a_1,a_2,\ldots,a_n)])=[\mathcal{E}_{+1}^{+3}(\mathbf{d})(M(a_1,a_2,\ldots,a_n))]\quad\text{(by Proposition 2.7)}$$

$$= \sum_{\substack{m=1,\\ a_m=3}}^{n} [M(a_1, a_2, \ldots, a_{m-1}, a_m+2, a_{m+1}, \ldots, a_n)] +$$

$$\sum_{\substack{1\le i<j\le n,\\ (a_i,a_j)=(3,4)\text{or}(4,3)}} M(a_1, \ldots, a_{i-1}, a_i+1, a_{i+1}, \ldots, a_{j-1}, a_j+1, a_{j+1}, \ldots, a_n)].$$

$$[\mathcal{E}^{-3}_{+1}(\mathbf{d})]([M(a_1, a_2, \ldots, a_n)]) = [\mathcal{E}^{-3}_{+1}(\mathbf{d})([M(a_1, a_2, \ldots, a_n))] \quad \text{(by Proposition 2.7)}$$

$$= [\mathrm{proj}_{\theta_{\mathbf{d}^5_3}}(\sum_{1\le i<j\le n} M(a_1\varepsilon_1 + \cdots + a_n\varepsilon_n - \rho + \varepsilon_i + \varepsilon_j))]$$

$$= \sum_{\substack{1\le i<j\le n,\\ (a_i,a_j)=(3,4)\text{or}(4,3)}} [a_1, \ldots, a_{i-1}, a_i+1, \ldots, a_{j-1}, a_j+1, \ldots, a_n].$$

$$[\mathcal{E}^{2}_{+2}(\mathbf{d})]([M(a_1, a_2, \ldots, a_n)]) = [\mathrm{proj}_{\theta_{\mathbf{d}^3_2}}(\sum_{i=1}^{n} M(a_1\varepsilon_1 + \cdots + a_n\varepsilon_n - \rho + \varepsilon_i))]$$

$$= \sum_{\substack{m=1,\\ a_m=2}}^{n} [M(a_1, a_2, \ldots, a_{m-1}, a_m+1, a_{m+1}, \ldots, a_n)].$$

The proof is completed. □

Now we give the first main theorem of this section which categorifies the actions of h_i, e_i, f_i on ${}^{\mathbb{Z}}(V^{\otimes n}_{+\mathrm{sp}})_{[\mathbf{d}]}$.

Theorem 3.3 *(1) For any $[d] \in \widetilde{\mathbf{D}}$, the actions of h_i $(1 \le i \le 4)$ on ${}^{\mathbb{Z}}(V^{\otimes n}_{+\mathrm{sp}})_{[\mathbf{d}]}$ can be categorified by the exact functors $\mathcal{H}^{\mathrm{sgn}(c_i(\mathbf{d}))}_i([\mathbf{d}])$. That is, $\mathrm{sgn}(c_i(\mathbf{d}))h_i \circ \gamma_n = \gamma_n \circ [\mathcal{H}^{\mathrm{sgn}(c_i(\mathbf{d}))}_i([\mathbf{d}])]$.*

(2) (i) For any $[d] \in \widetilde{\mathbf{D}}$, the actions of e_1 on ${}^{\mathbb{Z}}(V^{\otimes n}_{+\mathrm{sp}})_{[\mathbf{d}]}$ can be categorified by the exact functors $\mathcal{E}^{+}_{+1}([\mathbf{d}]), \mathcal{E}^{-}_{+1}([\mathbf{d}])$. That is, $\gamma_n \circ ([\mathcal{E}^{+}_{+1}([\mathbf{d}])] - [\mathcal{E}^{-}_{+1}([\mathbf{d}])]) = \gamma_n \circ e_1$.

(ii) For any $[d] \in \widetilde{\mathbf{D}}$, the actions of e_i $(2 \le i \le 4)$ on ${}^{\mathbb{Z}}(V^{\otimes n}_{+\mathrm{sp}})_{[\mathbf{d}]}$ can be categorified by the exact functors $\mathcal{E}_{+i}([\mathbf{d}])$. That is, $e_i \circ \gamma_n = \gamma_n \circ [\mathcal{E}_{+i}([\mathbf{d}])]$.

(3) (i) For any $[d] \in \widetilde{\mathbf{D}}$, the actions of f_1 on ${}^{\mathbb{Z}}(V^{\otimes n}_{+\mathrm{sp}})_{[\mathbf{d}]}$ can be categorified by the exact functors $\mathcal{F}^{+}_{+1}([\mathbf{d}]), \mathcal{F}^{-}_{+1}([\mathbf{d}])$. That is, $f_1 \circ \gamma_n = \gamma_n \circ ([\mathcal{F}^{+}_{+1}([\mathbf{d}])] - [\mathcal{F}^{-}_{+1}([\mathbf{d}])])$.

(ii) For any $[d] \in \widetilde{\mathbf{D}}$, the actions of f_i $(2 \le i \le 4)$ on ${}^{\mathbb{Z}}(V^{\otimes n}_{+\mathrm{sp}})_{[\mathbf{d}]}$ can be categorified by the exact functors $\mathcal{F}_{+i}([\mathbf{d}])$. That is, $f_i \circ \gamma_n = \gamma_n \circ [\mathcal{F}_{+i}([\mathbf{d}])]$.

Proof (1) For any $M(a_1, a_2, \ldots, a_n) \in B_{[\mathbf{d}]} = \bigcup_{\mathbf{d}' \in [\mathbf{d}]} B_{[\mathbf{d}']}$. It is enough to prove that for any $1 \le i \le 4$

$$\mathrm{sgn}(c_i(\mathbf{d}))h_i \circ \gamma_n([M(a_1, a_2, \ldots, a_n)]) = \gamma_n \circ \mathcal{H}^{\mathrm{sgn}(c_i(\mathbf{d}))}_i([\mathbf{d}])([M(a_1, a_2, \ldots, a_n)]).$$

On the one hand,

$$\begin{aligned}\mathrm{sgn}(c_i(\mathbf{d}))h_i \circ \gamma_n([M(a_1, a_2, \ldots, a_n)]) &= \mathrm{sgn}(c_i(\mathbf{d}))h_i(v_{a_1} \otimes v_{a_2} \otimes \cdots \otimes v_{a_n})\\ &= \mathrm{sgn}(c_i(\mathbf{d}))(\sum_{k=1}^{n} v_{a_1} \otimes \cdots \otimes h_i \cdot v_{a_k} \otimes \cdots \otimes v_{a_n}).\end{aligned}$$

On the other hand,

$$\gamma_n \circ \mathcal{H}^{\mathrm{sgn}(c_i(\mathbf{d}))}_i([\mathbf{d}])([M(a_1, a_2, \ldots, a_n)]) = \gamma_n c_i(\mathbf{d})\mathrm{sgn}(c_i(\mathbf{d}))([M(a_1, a_2, \ldots, a_n)])$$

$$=\mathrm{sgn}(c_i(\mathbf{d}))c_i(\mathbf{d})(v_{a_1}\otimes v_{a_2}\otimes\cdots\otimes v_{a_n})$$
$$=\mathrm{sgn}(c_i(\mathbf{d}))(\sum_{k=1}^{n} v_{a_1}\otimes\cdots\otimes h_i\cdot v_{a_k}\otimes\cdots\otimes v_{a_n}).$$

Hence $\mathrm{sgn}(c_i(\mathbf{d}))h_i\circ\gamma_n=\gamma_n\circ[\mathcal{H}_i^{\mathrm{sgn}(c_i(\mathbf{d}))}([\mathbf{d}])]$.

(2) (i) For any $M(a_1,a_2,\dots,a_n)\in B_{[\mathbf{d}]}=\bigcup_{\mathbf{d}'\in[\mathbf{d}]}B_{[\mathbf{d}']}$. It is enough to prove that

$$e_1\circ\gamma_n([M(a_1,a_2,\dots,a_n)])=\gamma_n\circ([\mathcal{E}_{+1}^{+}([\mathbf{d}])]-[\mathcal{E}_{+1}^{-}([\mathbf{d}])])([M(a_1,a_2,\dots,a_n)]).$$

On the one hand,

$$\begin{aligned}e_1\circ\gamma_n([M(a_1,a_2,\dots,a_n)])&=\sum_{k=1}^{n}v_{a_1}\otimes v_{a_2}\otimes\cdots\otimes e_1\cdot v_{a_k}\otimes\cdots\otimes v_{a_n}\\&=\sum_{\substack{k=1,\\a_k=3}}^{n}v_{a_1}\otimes v_{a_2}\otimes\cdots\otimes v_{a_k+2}\otimes\cdots\otimes v_{a_n}+\\&\quad\sum_{\substack{k=1,\\a_k=4}}^{n}v_{a_1}\otimes v_{a_2}\otimes\cdots\otimes v_{a_k+2}\otimes\cdots\otimes v_{a_n}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\gamma_n\circ([\mathcal{E}_{+1}^{+}([\mathbf{d}])]-[\mathcal{E}_{+1}^{-}([\mathbf{d}])])([M(a_1,a_2,\dots,a_n)])\\&=\gamma_n(\sum_{\substack{k=1,\\a_k=3}}^{n}[M(a_1,\dots,a_k+2,\dots,a_n)]+\sum_{\substack{k=1,\\a_k=4}}^{n}[M(a_1,\dots,a_k+2,\dots,a_n)])\\&=\sum_{\substack{k=1,\\a_k=3}}^{n}v_{a_1}\otimes v_{a_2}\otimes\cdots\otimes v_{a_k+2}\otimes\cdots\otimes v_{a_n}+\sum_{\substack{k=1,\\a_k=4}}^{n}v_{a_1}\otimes v_{a_2}\otimes\cdots\otimes v_{a_k+2}\otimes\cdots\otimes v_{a_n}\end{aligned}$$

Hence $\gamma_n\circ([\mathcal{E}_{+1}^{+}([\mathbf{d}])]-[\mathcal{E}_{+1}^{-}([\mathbf{d}])])=\gamma_n\circ e_1$.

(ii) For any $M(a_1,a_2,\dots,a_n)\in B_{[\mathbf{d}]}=\bigcup_{\mathbf{d}'\in[\mathbf{d}]}B_{[\mathbf{d}']}$. It is enough to prove that for $2\le i\le 4$

$$e_i\circ\gamma_n([M(a_1,a_2,\dots,a_n)])=\gamma_n\circ[\mathcal{E}_{+i}([\mathbf{d}])]([M(a_1,a_2,\dots,a_n)]).$$

We take $i=2$ for example here. The cases for $i=3,4$ can be proved similarly. On the one hand,

$$\begin{aligned}&e_2\circ\gamma_n([M(a_1,a_2,\dots,a_n)])=\sum_{k=1}^{n}v_{a_1}\otimes\cdots\otimes e_2v_{a_k}\otimes\cdots\otimes v_{a_n}\\&=\sum_{\substack{k=1,\\a_k=2}}^{n}v_{a_1}\otimes v_{a_2}\otimes\cdots\otimes v_{a_k+1}\otimes\cdots\otimes v_{a_n}+\sum_{\substack{k=1,\\a_k=6}}^{n}v_{a_1}\otimes v_{a_2}\otimes\cdots\otimes v_{a_k+1}\otimes\cdots\otimes v_{a_n}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\gamma_n\circ[\mathcal{E}_{+2}([\mathbf{d}])]([M(a_1,a_2,\dots,a_n)])\\&=\gamma_n(\sum_{\substack{k=1,\\a_k=2}}^{n}[M(a_1,\dots,a_k+1,\dots,a_n)]+\sum_{\substack{k=1,\\a_k=6}}^{n}[M(a_1,\dots,a_k+1,\dots,a_n)])\end{aligned}$$

$$= \sum_{\substack{k=1,\\ a_k=2}}^{n} v_{a_1}\otimes v_{a_2}\otimes\cdots\otimes v_{a_k+1}\otimes\cdots\otimes v_{a_n} + \sum_{\substack{k=1,\\ a_k=6}}^{n} v_{a_1}\otimes v_{a_2}\otimes\cdots\otimes v_{a_k+1}\otimes\cdots\otimes v_{a_n}.$$

(3) Similar to (2). $\square$

Set

$$\mathcal{E}_{+1}^{+} := \bigoplus_{[\mathbf{d}]\in\widetilde{\mathbf{D}}} \mathcal{E}_{+1}^{+}([\mathbf{d}]) : \mathcal{O}^n \longrightarrow \mathcal{O}^n,\ \mathcal{E}_{+1}^{-} := \bigoplus_{[\mathbf{d}]\in\widetilde{\mathbf{D}}} \mathcal{E}_{+1}^{-}([\mathbf{d}]) : \mathcal{O}^n \longrightarrow \mathcal{O}^n;$$

$$\mathcal{F}_{+1}^{+} := \bigoplus_{[\mathbf{d}]\in\widetilde{\mathbf{D}}} \mathcal{F}_{+1}^{+}([\mathbf{d}]) : \mathcal{O}^n \longrightarrow \mathcal{O}^n,\ \mathcal{F}_{+1}^{-} := \bigoplus_{[\mathbf{d}]\in\widetilde{\mathbf{D}}} \mathcal{F}_{+1}^{-}([\mathbf{d}]) : \mathcal{O}^n \longrightarrow \mathcal{O}^n;$$

$$\mathcal{E}_{+j} := \bigoplus_{[\mathbf{d}]\in\widetilde{\mathbf{D}}} \mathcal{E}_{+j}([\mathbf{d}]) : \mathcal{O}^n \longrightarrow \mathcal{O}^n,\ \mathcal{F}_{+j} := \bigoplus_{[\mathbf{d}]\in\widetilde{\mathbf{D}}} \mathcal{F}_{+j}([\mathbf{d}]) : \mathcal{O}^n \longrightarrow \mathcal{O}^n,$$

$$\mathcal{H}_{i}^{+} := \bigoplus_{\substack{[\mathbf{d}]\in\widetilde{\mathbf{D}},\\ c_i(\mathbf{d})\ge 0}} \mathcal{H}_{i}^{\mathrm{sgn}(c_i(\mathbf{d}))}([\mathbf{d}]) : \mathcal{O}^n \longrightarrow \mathcal{O}^n,\ \mathcal{H}_{i}^{-} := \bigoplus_{\substack{[\mathbf{d}]\in\widetilde{\mathbf{D}},\\ c_i(\mathbf{d})< 0}} \mathcal{H}_{i}^{\mathrm{sgn}(c_i(\mathbf{d}))}([\mathbf{d}]) : \mathcal{O}^n \longrightarrow \mathcal{O}^n,$$

where $2\le j\le 4$, $1\le i\le 4$. By the Proposition 2.6, we know $\mathcal{E}_{+1}^{+}$, $\mathcal{E}_{+1}^{-}$, $\mathcal{F}_{+1}^{+}$, $\mathcal{F}_{+1}^{-}$, $\mathcal{E}_{+j}$, $\mathcal{F}_{+j}$ $(2\le j\le 4)$ and $\mathcal{H}_i^{+}$, $\mathcal{H}_i^{-}$ $(1\le i\le 4)$ are exact functors. Now we give the second main theorem.

Theorem 3.4 *(1) The action of h_i $(1\le i\le 4)$ on ${}^{\mathbb{Z}}(V_{+\mathrm{sp}}^{\otimes n})$ can be categorified by functors $\mathcal{H}_i^{+},\mathcal{H}_i^{-}$. That is, $h_i\circ\gamma_n=\gamma_n\circ([\mathcal{H}_i^{+}]-[\mathcal{H}_i^{-}])$.*

(2) (i) The action of e_1 on ${}^{\mathbb{Z}}(V_{+\mathrm{sp}}^{\otimes n})$ can be categorified by functors $\mathcal{E}_{+1}^{+},\mathcal{E}_{+1}^{-}$. That is, $e_1\circ\gamma_n=\gamma_n\circ([\mathcal{E}_{+1}^{+}]-[\mathcal{E}_{+1}^{-}])$.

(ii) The action of e_i $(2\le i\le 4)$ on ${}^{\mathbb{Z}}(V_{+\mathrm{sp}}^{\otimes n})$ can be categorified by functors $\mathcal{E}_{+i}$. That is, $e_i\circ\gamma_n=\gamma_n\circ[\mathcal{E}_{+i}]$.

(3) (i) The action of f_1 on ${}^{\mathbb{Z}}(V_{+\mathrm{sp}}^{\otimes n})$ can be categorified by functors $\mathcal{F}_{+1}^{+},\mathcal{F}_{+1}^{-}$. That is, $f_1\circ\gamma_n=\gamma_n\circ([\mathcal{F}_{+1}^{+}]-[\mathcal{F}_{+1}^{-}])$.

(ii) The action of f_i $(2\le i\le 4)$ on ${}^{\mathbb{Z}}(V_{+\mathrm{sp}}^{\otimes n})$ can be categorified by functors $\mathcal{F}_{+i}$. That is, $f_i\circ\gamma_n=\gamma_n\circ[\mathcal{F}_{+i}]$.

Proof It is easy to check all the diagrams above are commutative by Theorem 3.3. $\square$

In the following Theorem 3.5, we will give the categorification of the defining relations of $U(\mathrm{so}(8,\mathbb{C}))$ on $V_{+\mathrm{sp}}^{\otimes n}$.

Theorem 3.5 *The defining relations of $U(\mathrm{so}(8,\mathbb{C}))$ on $V_{+\mathrm{sp}}^{\otimes n}$ can be categorified as follows.*

(1) $\mathcal{H}_i^{+}\mathcal{H}_j^{+}\oplus\mathcal{H}_j^{+}\mathcal{H}_i^{-}\oplus\mathcal{H}_j^{-}\mathcal{H}_i^{+}\oplus\mathcal{H}_i^{-}\mathcal{H}_j^{-}\simeq\mathcal{H}_j^{+}\mathcal{H}_i^{+}\oplus\mathcal{H}_i^{+}\mathcal{H}_j^{-}\oplus\mathcal{H}_i^{-}\mathcal{H}_j^{+}\oplus\mathcal{H}_j^{-}\mathcal{H}_i^{-}$ for $1\le i,j\le 4$.

(2) (i) $\mathcal{E}_{+1}^{+}\mathcal{F}_{+1}^{+}\oplus\mathcal{E}_{+1}^{-}\mathcal{F}_{+1}^{-}\oplus\mathcal{F}_{+1}^{+}\mathcal{E}_{+1}^{-}\oplus\mathcal{F}_{+1}^{-}\mathcal{E}_{+1}^{+}\oplus\mathcal{H}_1^{-}\simeq\mathcal{E}_{+1}^{+}\mathcal{F}_{+1}^{-}\oplus\mathcal{E}_{+1}^{-}\mathcal{F}_{+1}^{+}\oplus\mathcal{F}_{+1}^{+}\mathcal{E}_{+1}^{+}\mathcal{F}_{+1}^{-}\mathcal{E}_{+1}^{-}\oplus\mathcal{H}_1^{+}$;

(ii) $\mathcal{E}_{+1}^{+}\mathcal{F}_{+i}\oplus\mathcal{F}_{+i}\mathcal{E}_{+1}^{-}\simeq\mathcal{E}_{+1}^{-}\mathcal{F}_{+i}\oplus\mathcal{F}_{+i}\mathcal{E}_{+1}^{+}$, where $i=2,3,4$;

(iii) $\mathcal{E}_{+i}\mathcal{F}_{+1}^{+}\oplus\mathcal{F}_{+1}^{-}\mathcal{E}_{+i}\simeq\mathcal{E}_{+i}\mathcal{F}_{+1}^{-}\oplus\mathcal{F}_{+1}^{+}\mathcal{E}_{+i}$, where $i=2,3,4$;

(iv) $\mathcal{E}_{+i}\mathcal{F}_{+j}\oplus\delta_{ij}\mathcal{H}_i^{-}\simeq\mathcal{F}_{+j}\mathcal{E}_{+i}\oplus\delta_{ij}\mathcal{H}_i^{+}$, where $i,j=2,3,4$.

(3) (i) $\mathcal{H}_1^{+}\mathcal{E}_{+1}^{+}\oplus\mathcal{H}_1^{-}\mathcal{E}_{+1}^{-}\oplus\mathcal{E}_{+1}^{+}\mathcal{H}_1^{-}\oplus\mathcal{E}_{+1}^{-}\mathcal{H}_1^{+}\oplus(\mathcal{E}_{+1}^{-})^{\oplus 2}\simeq\mathcal{H}_1^{+}\mathcal{E}_{+1}^{-}\oplus\mathcal{H}_1^{-}\mathcal{E}_{+1}^{+}\oplus\mathcal{E}_{+1}^{+}\mathcal{H}_1^{+}\oplus\mathcal{E}_{+1}^{-}\mathcal{H}_1^{-}\oplus(\mathcal{E}_{+1}^{+})^{\oplus 2}$;

(ii) $\mathcal{H}_2^+\mathcal{E}_{+1}^+ \oplus \mathcal{H}_2^-\mathcal{E}_{+1}^- \oplus \mathcal{E}_{+1}^+\mathcal{H}_2^- \oplus \mathcal{E}_{+1}^-\mathcal{H}_2^+ \oplus \mathcal{E}_{+1}^+ \simeq \mathcal{H}_2^+\mathcal{E}_{+1}^- \oplus \mathcal{H}_2^-\mathcal{E}_{+1}^+ \oplus \mathcal{E}_{+1}^+\mathcal{H}_2^+ \oplus \mathcal{E}_{+1}^-\mathcal{H}_2^- \oplus \mathcal{E}_{+1}^-$;

(iii) $\mathcal{H}_i^+\mathcal{E}_{+1}^+ \oplus \mathcal{H}_i^-\mathcal{E}_{+1}^- \oplus \mathcal{E}_{+1}^+\mathcal{H}_i^- \oplus \mathcal{E}_{+1}^-\mathcal{H}_i^+ \simeq \mathcal{H}_i^+\mathcal{E}_{+1}^- \oplus \mathcal{H}_i^-\mathcal{E}_{+1}^+ \oplus \mathcal{E}_{+1}^+\mathcal{H}_i^+ \oplus \mathcal{E}_{+1}^-\mathcal{H}_i^-$, $i = 3, 4$;

(iv) $\mathcal{H}_i^+\mathcal{E}_{+j} \oplus \mathcal{E}_{+j}\mathcal{H}_i^- \oplus \mathcal{E}_{+j} \simeq \mathcal{H}_i^-\mathcal{E}_{+j} \oplus \mathcal{E}_{+j}\mathcal{H}_i^+$, *where* $(i,j) \in \{(1,2),(2,3),(2,4),(3,2),(4,2)\}$;

(v) $\mathcal{H}_i^+\mathcal{E}_{+j} \oplus \mathcal{E}_{+j}\mathcal{H}_i^- \simeq \mathcal{H}_i^-\mathcal{E}_{+j} \oplus \mathcal{E}_{+j}\mathcal{H}_i^+$, *where* $(i,j) \in \{(1,3),(1,4),(3,4),(4,3)\}$;

(vi) $\mathcal{H}_i^+\mathcal{E}_{+i} \oplus \mathcal{E}_{+i}\mathcal{H}_i^- \simeq \mathcal{H}_i^-\mathcal{E}_{+i} \oplus \mathcal{E}_{+i}\mathcal{H}_i^+ \oplus (\mathcal{E}_{+i})^{\oplus 2}$, *where* $i = 2, 3, 4$.

(4) *(i)* $\mathcal{H}_1^+\mathcal{F}_{+1}^+ \oplus \mathcal{H}_1^-\mathcal{F}_{+1}^- \oplus \mathcal{F}_{+1}^+\mathcal{H}_1^- \oplus \mathcal{F}_{+1}^-\mathcal{H}_1^+ \oplus (\mathcal{F}_{+1}^+)^{\oplus 2} \simeq \mathcal{H}_1^+\mathcal{F}_{+1}^- \oplus \mathcal{H}_1^-\mathcal{F}_{+1}^+ \oplus \mathcal{F}_{+1}^+\mathcal{H}_1^+ \oplus \mathcal{F}_{+1}^-\mathcal{H}_1^- \oplus (\mathcal{F}_{+1}^-)^{\oplus 2}$;

(ii) $\mathcal{H}_2^+\mathcal{F}_{+1}^+ \oplus \mathcal{H}_2^-\mathcal{F}_{+1}^- \oplus \mathcal{F}_{+1}^+\mathcal{H}_2^- \oplus \mathcal{F}_{+1}^-\mathcal{H}_2^+ \oplus \mathcal{F}_{+1}^- \simeq \mathcal{H}_2^+\mathcal{F}_{+1}^- \oplus \mathcal{H}_2^-\mathcal{F}_{+1}^+ \oplus \mathcal{F}_{+1}^+\mathcal{H}_2^+ \oplus \mathcal{F}_{+1}^-\mathcal{H}_2^- \oplus \mathcal{F}_{+1}^+$;

(iii) $\mathcal{H}_i^+\mathcal{F}_{+1}^+ \oplus \mathcal{H}_i^-\mathcal{F}_{+1}^- \oplus \mathcal{F}_{+1}^+\mathcal{H}_i^- \oplus \mathcal{F}_{+1}^-\mathcal{H}_i^+ \simeq \mathcal{H}_i^+\mathcal{F}_{+1}^- \oplus \mathcal{H}_i^-\mathcal{F}_{+1}^+ \oplus \mathcal{F}_{+1}^+\mathcal{H}_i^+ \oplus \mathcal{F}_{+1}^-\mathcal{H}_i^-$, *where* $i = 3, 4$;

(iv) $\mathcal{H}_i^+\mathcal{F}_{+i} \oplus \mathcal{F}_{+i}\mathcal{H}_i^- \oplus (\mathcal{F}_{+i})^{\oplus 2} \simeq \mathcal{H}_i^-\mathcal{F}_{+i} \oplus \mathcal{F}_{+i}\mathcal{H}_i^+$, *where* $i = 2, 3, 4$;

(v) $\mathcal{H}_i^+\mathcal{F}_{+j} \oplus \mathcal{F}_{+j}\mathcal{H}_i^- \simeq \mathcal{H}_i^-\mathcal{F}_{+j} \oplus \mathcal{F}_{+j}\mathcal{H}_i^+ \oplus \mathcal{F}_{+j}$, *where* $(i,j) \in \{(1,2),(2,3),(2,4),(3,2),(4,2)\}$;

(vi) $\mathcal{H}_i^+\mathcal{F}_{+j} \oplus \mathcal{F}_{+j}\mathcal{H}_i^- \simeq \mathcal{H}_i^-\mathcal{F}_{+j} \oplus \mathcal{F}_{+j}\mathcal{H}_i^+$, *where* $(i,j) \in \{(1,3),(1,4),(3,4),(4,3)\}$;

(5) *(i)* $\mathcal{E}_{+1}^+\mathcal{E}_{+1}^+\mathcal{E}_{+2} \oplus \mathcal{E}_{+1}^-\mathcal{E}_{+1}^-\mathcal{E}_{+2} \oplus \mathcal{E}_{+1}^+\mathcal{E}_{+2}\mathcal{E}_{+1}^- \oplus \mathcal{E}_{+1}^-\mathcal{E}_{+2}\mathcal{E}_{+1}^- \oplus \mathcal{E}_{+2}\mathcal{E}_{+1}^+\mathcal{E}_{+1}^+ \oplus \mathcal{E}_{+2}\mathcal{E}_{+1}^-\mathcal{E}_{+1}^- \simeq \mathcal{E}_{+1}^+\mathcal{E}_{+1}^-\mathcal{E}_{+2} \oplus \mathcal{E}_{+1}^-\mathcal{E}_{+1}^+\mathcal{E}_{+2} \oplus \mathcal{E}_{+1}^+\mathcal{E}_{+2}\mathcal{E}_{+1}^+ \oplus \mathcal{E}_{+1}^-\mathcal{E}_{+2}\mathcal{E}_{+1}^- \oplus \mathcal{E}_{+2}\mathcal{E}_{+1}^+\mathcal{E}_{+1}^- \oplus \mathcal{E}_{+2}\mathcal{E}_{+1}^-\mathcal{E}_{+1}^+$;

(ii) $\mathcal{E}_{+1}^+\mathcal{E}_{+i} \oplus \mathcal{E}_{+i}\mathcal{E}_{+1}^- \simeq \mathcal{E}_{+1}^-\mathcal{E}_{+i} \oplus \mathcal{E}_{+i}\mathcal{E}_{+1}^+$, *where* $i = 3, 4$;

(iii) $\mathcal{E}_{+i}\mathcal{E}_{+1}^+ \oplus \mathcal{E}_{+1}^-\mathcal{E}_{+i} \simeq \mathcal{E}_{+i}\mathcal{E}_{+1}^- \oplus \mathcal{E}_{+1}^+\mathcal{E}_{+i}$, *where* $i = 3, 4$;

(iv) $\mathcal{E}_{+2}\mathcal{E}_{+2}\mathcal{E}_{+1}^+ \oplus (\mathcal{E}_{+2}\mathcal{E}_{+1}^-\mathcal{E}_{+2})^{\oplus 2} \oplus \mathcal{E}_{+1}^+\mathcal{E}_{+2}\mathcal{E}_{+2} \simeq \mathcal{E}_{+2}\mathcal{E}_{+2}\mathcal{E}_{+1}^- \oplus (\mathcal{E}_{+2}\mathcal{E}_{+1}^+\mathcal{E}_{+2})^{\oplus 2} \oplus \mathcal{E}_{+1}^-\mathcal{E}_{+2}\mathcal{E}_{+2}$;

(v) $\mathcal{E}_{+i}\mathcal{E}_{+i}\mathcal{E}_{+j} \oplus \mathcal{E}_{+j}\mathcal{E}_{+i}\mathcal{E}_{+i} \simeq (\mathcal{E}_{+i}\mathcal{E}_{+j}\mathcal{E}_{+i})^{\oplus 2}$, *where* $(i,j) \in \{(2,3),(2,4),(3,2),(4,2)\}$;

(vi) $\mathcal{E}_{+3}\mathcal{E}_{+4} \simeq \mathcal{E}_{+4}\mathcal{E}_{+3}$;

(6) *(i)* $\mathcal{F}_{+1}^+\mathcal{F}_{+1}^+\mathcal{F}_{+2} \oplus \mathcal{F}_{+1}^-\mathcal{F}_{+1}^-\mathcal{F}_{+2} \oplus \mathcal{F}_{+1}^+\mathcal{F}_{+2}\mathcal{F}_{+1}^- \oplus \mathcal{F}_{+1}^-\mathcal{F}_{+2}\mathcal{F}_{+1}^- \oplus \mathcal{F}_{+2}\mathcal{F}_{+1}^+\mathcal{F}_{+1}^+ \oplus \mathcal{F}_{+2}\mathcal{F}_{+1}^-\mathcal{F}_{+1}^- \simeq \mathcal{F}_{+1}^+\mathcal{F}_{+1}^-\mathcal{F}_{+2} \oplus \mathcal{F}_{+1}^-\mathcal{F}_{+1}^+\mathcal{F}_{+2} \oplus \mathcal{F}_{+1}^+\mathcal{F}_{+2}\mathcal{F}_{+1}^+ \oplus \mathcal{F}_{+1}^-\mathcal{F}_{+2}\mathcal{F}_{+1}^- \oplus \mathcal{F}_{+2}\mathcal{F}_{+1}^+\mathcal{F}_{+1}^- \oplus \mathcal{F}_{+2}\mathcal{F}_{+1}^-\mathcal{F}_{+1}^+$;

(ii) $\mathcal{F}_{+1}^+\mathcal{F}_{+i} \oplus \mathcal{F}_{+i}\mathcal{F}_{+1}^- \simeq \mathcal{F}_{+1}^-\mathcal{F}_{+i} \oplus \mathcal{F}_{+i}\mathcal{F}_{+1}^+$, *where* $i = 3, 4$;

(iii) $\mathcal{F}_{+i}\mathcal{F}_{+1}^+ \oplus \mathcal{F}_{+1}^-\mathcal{F}_{+i} \simeq \mathcal{F}_{+i}\mathcal{F}_{+1}^- \oplus \mathcal{F}_{+1}^+\mathcal{F}_{+i}$, *where* $i = 3, 4$;

(iv) $\mathcal{F}_{+2}\mathcal{F}_{+2}\mathcal{F}_{+1}^+ \oplus (\mathcal{F}_{+2}\mathcal{F}_{+1}^-\mathcal{F}_{+2})^{\oplus 2} \oplus \mathcal{F}_{+1}^+\mathcal{F}_{+2}\mathcal{F}_{+2} \simeq \mathcal{F}_{+2}\mathcal{F}_{+2}\mathcal{F}_{+1}^- \oplus (\mathcal{F}_{+2}\mathcal{F}_{+1}^+\mathcal{F}_{+2})^{\oplus 2} \oplus \mathcal{F}_{+1}^-\mathcal{F}_{+2}\mathcal{F}_{+2}$;

(v) $\mathcal{F}_{+i}\mathcal{F}_{+i}\mathcal{F}_{+j} \oplus \mathcal{F}_{+j}\mathcal{F}_{+i}\mathcal{F}_{+i} \simeq (\mathcal{F}_{+i}\mathcal{F}_{+j}\mathcal{F}_{+i})^{\oplus 2}$, *where* $(i,j) \in \{(2,3),(2,4),(3,2),(4,2)\}$;

(vi) $\mathcal{F}_{+3}\mathcal{F}_{+4} \simeq \mathcal{F}_{+4}\mathcal{F}_{+3}$;

Proof Here some cases will be proved, the rest cases are similar to prove.

(1) For $1 \le i, j \le 4$, it suffices to prove that

$$\begin{aligned}&[\mathcal{H}_i^+][\mathcal{H}_j^+] + [\mathcal{H}_j^+][\mathcal{H}_i^-] + [\mathcal{H}_j^-][\mathcal{H}_i^+] + [\mathcal{H}_i^-][\mathcal{H}_j^-]\\ &\quad = [\mathcal{H}_j^+][\mathcal{H}_i^+] + [\mathcal{H}_i^+][\mathcal{H}_j^-] + [\mathcal{H}_i^-][\mathcal{H}_j^+] + [\mathcal{H}_j^-][\mathcal{H}_i^-].\end{aligned}$$

Since γ_n is an isomorphism and $h_ih_j = h_jh_i$, we have $h_ih_j\gamma_n = h_jh_i\gamma_n$. By Theorem 3.4,

$$\begin{aligned}h_ih_j\gamma_n &= h_i\gamma_n([\mathcal{H}_j^+] - [\mathcal{H}_j^-]) = \gamma_n([\mathcal{H}_i^+] - [\mathcal{H}_i^-])([\mathcal{H}_j^+] - [\mathcal{H}_j^-]),\\ h_jh_i\gamma_n &= h_j\gamma_n([\mathcal{H}_i^+] - [\mathcal{H}_i^-]) = \gamma_n([\mathcal{H}_j^+] - [\mathcal{H}_j^-])([\mathcal{H}_i^+] - [\mathcal{H}_i^-]).\end{aligned}$$

Hence $([\mathcal{H}_i^+] - [\mathcal{H}_i^-])([\mathcal{H}_j^+] - [\mathcal{H}_j^-]) = ([\mathcal{H}_j^+] - [\mathcal{H}_j^-])([\mathcal{H}_i^+] - [\mathcal{H}_i^-])$. That is,

$$[\mathcal{H}_i^+][\mathcal{H}_j^+] + [\mathcal{H}_j^+][\mathcal{H}_i^-] + [\mathcal{H}_j^-][\mathcal{H}_i^+] + [\mathcal{H}_i^-][\mathcal{H}_j^-]$$

$$= [\mathcal{H}_j^+][\mathcal{H}_i^+] + [\mathcal{H}_i^+][\mathcal{H}_j^-] + [\mathcal{H}_i^-][\mathcal{H}_j^+] + [\mathcal{H}_j^-][\mathcal{H}_i^-].$$

(2) (i) It suffices to prove that

$$\begin{aligned}&[\mathcal{E}_{+1}^+][\mathcal{F}_{+1}^+] + [\mathcal{E}_{+1}^-][\mathcal{F}_{+1}^-] + [\mathcal{F}_{+1}^+][\mathcal{E}_{+1}^-] + [\mathcal{F}_{+1}^-][\mathcal{E}_{+1}^+] + [\mathcal{H}_1^-]\\ &\quad = [\mathcal{E}_{+1}^+][\mathcal{F}_{+1}^-] + [\mathcal{E}_{+1}^-][\mathcal{F}_{+1}^+] + [\mathcal{F}_{+1}^+][\mathcal{E}_{+1}^+] + [\mathcal{F}_{+1}^-][\mathcal{E}_{+1}^-] + [\mathcal{H}_1^+].\end{aligned}$$

Since γ_n is an isomorphism and $e_1f_1 - f_1e_1 = h_1$, we have $(e_1f_1 - f_1e_1)\gamma_n = h_1\gamma_n$. By Theorem 3.4,

$$(e_1f_1 - f_1e_1)\gamma_n = \gamma_n(([\mathcal{E}_{+1}^+] - [\mathcal{E}_{+1}^-])([\mathcal{F}_{+1}^+] - [\mathcal{F}_{+1}^-]) - ([\mathcal{F}_{+1}^+] - [\mathcal{F}_{+1}^-])([\mathcal{E}_{+1}^+] - [\mathcal{E}_{+1}^-])),$$

$$h_1\gamma_n = \gamma_n([\mathcal{H}_1^+] - [\mathcal{H}_1^-]).$$

Hence

$$([\mathcal{E}_{+1}^+] - [\mathcal{E}_{+1}^-])([\mathcal{F}_{+1}^+] - [\mathcal{F}_{+1}^-]) - ([\mathcal{F}_{+1}^+] - [\mathcal{F}_{+1}^-])([\mathcal{E}_{+1}^+] - [\mathcal{E}_{+1}^-]) = [\mathcal{H}_1^+] - [\mathcal{H}_1^-].$$

That is,

$$\begin{aligned}&[\mathcal{E}_{+1}^+][\mathcal{F}_{+1}^+] + [\mathcal{E}_{+1}^-][\mathcal{F}_{+1}^-] + [\mathcal{F}_{+1}^+][\mathcal{E}_{+1}^-] + [\mathcal{F}_{+1}^-][\mathcal{E}_{+1}^+] + [\mathcal{H}_1^-]\\ &\quad = [\mathcal{E}_{+1}^+][\mathcal{F}_{+1}^-] + [\mathcal{E}_{+1}^-][\mathcal{F}_{+1}^+] + [\mathcal{F}_{+1}^+][\mathcal{E}_{+1}^+] + [\mathcal{F}_{+1}^-][\mathcal{E}_{+1}^-] + [\mathcal{H}_1^+].\end{aligned}$$

(3) (iii) Take $i = 3$ for example, it suffices to prove

$$\begin{aligned}&[\mathcal{H}_3^+][\mathcal{E}_{+1}^+] + [\mathcal{H}_3^-][\mathcal{E}_{+1}^-] + [\mathcal{E}_{+1}^+][\mathcal{H}_3^-] + [\mathcal{E}_{+1}^-][\mathcal{H}_3^+]\\ &\quad = [\mathcal{E}_{+1}^+][\mathcal{H}_3^+] + [\mathcal{E}_{+1}^-][\mathcal{H}_3^-] + [\mathcal{H}_3^+][\mathcal{E}_{+1}^-] + [\mathcal{H}_3^-][\mathcal{E}_{+1}^+].\end{aligned}$$

Since γ_n is an isomorphism and $h_3e_1 - e_1h_3 = 0$, we have $(h_3e_1 - e_1h_3)\gamma_n = 0$. By Theorem 3.4,

$$(h_3e_1 - e_1h_3)\gamma_n = \gamma_n([\mathcal{H}_3^+] - [\mathcal{H}_3^-])([\mathcal{E}_{+1}^+] - [\mathcal{E}_{+1}^-]) - \gamma_n([\mathcal{E}_{+1}^+] - [\mathcal{E}_{+1}^-])([\mathcal{H}_3^+] - [\mathcal{H}_3^-]).$$

Hence

$$([\mathcal{H}_3^+] - [\mathcal{H}_3^-])([\mathcal{E}_{+1}^+] - [\mathcal{E}_{+1}^-]) = ([\mathcal{E}_{+1}^+] - [\mathcal{E}_{+1}^-])([\mathcal{H}_3^+] - [\mathcal{H}_3^-]).$$

That is,

$$\begin{aligned}&[\mathcal{H}_3^+][\mathcal{E}_{+1}^+] + [\mathcal{H}_3^-][\mathcal{E}_{+1}^-] + [\mathcal{E}_{+1}^+][\mathcal{H}_3^-] + [\mathcal{E}_{+1}^-][\mathcal{H}_3^+]\\ &\quad = [\mathcal{E}_{+1}^+][\mathcal{H}_3^+] + [\mathcal{E}_{+1}^-][\mathcal{H}_3^-] + [\mathcal{H}_3^+][\mathcal{E}_{+1}^-] + [\mathcal{H}_3^-][\mathcal{E}_{+1}^+].\end{aligned}$$

(6) (vi) Take $(i, j) = (2, 3)$ for example, it suffices to prove that

$$[\mathcal{F}_{+2}]^2[\mathcal{F}_{+3}] + [\mathcal{F}_{+3}][\mathcal{F}_{+2}]^2 = 2[\mathcal{F}_{+2}][\mathcal{F}_{+3}][\mathcal{F}_{+2}].$$

Since γ_n is an isomorphism and $f_2^2f_3 - 2f_2f_3f_2 + f_3f_2^2 = 0$, we can get

$$f_2^2f_3\gamma_n - 2f_2f_3f_2\gamma_n + f_3f_2^2\gamma_n = 0.$$

By Theorem 3.4, $\gamma_n[\mathcal{F}_{+2}]^2[\mathcal{F}_{+3}] - 2\gamma_n[\mathcal{F}_{+2}][\mathcal{F}_{+3}][\mathcal{F}_{+2}] + \gamma_n[\mathcal{F}_{+3}][\mathcal{F}_{+2}]^2 = 0$. That is,

$$[\mathcal{F}_{+2}]^2[\mathcal{F}_{+3}] + [\mathcal{F}_{+3}][\mathcal{F}_{+2}]^2 = 2[\mathcal{F}_{+2}][\mathcal{F}_{+3}][\mathcal{F}_{+2}].$$

The proof is completed. □

By the definition of spin module $V_{-\mathrm{sp}}$, we can categorify $V_{-\mathrm{sp}}^{\otimes n}$ in a way similar to $V_{+\mathrm{sp}}^{\otimes n}$.

References

[1] J. N. BERNSTEIN, I. FRENKEL, M. KHOVANOV. *A categorification of the Temperley-Lieb algebra and schur quotients of $U(\mathrm{sl}_2)$ via projective and Zukerman functors*. Selecta Math. (N.S.), 1999, **5**(2): 199–241.

[2] J. N. BERNSTEIN, S. I. GELFAND. *Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras*. Compositio Math., 1980, **41**(2): 245–285.

[3] J. N. BERNSTEIN, I. M. GELFAND, S. I. GELFAND. *Category of g-modules*. Funkcional Anal. Appl., 1976, **10**(2): 87–92.

[4] R. W. CARTER. *Lie Algebras of Finite and Affine Type*. Cambridge University Press, Cambridge, 2005.

[5] L. CRANE, I. FRENKEL. *Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases*. J. Math. Phys., 1994, **35**(10): 5136–5154.

[6] L. CRANE. *Clock and category: is quantum gravity algebraic*. J. Math. Phys., 1995, **36**(11): 6180–6193.

[7] J. DIXMIER. *Enveloping Algebras*. Amer. Math. Soc, Providence, RI, 1996.

[8] I. FRENKEL, C. STROPPEL, J. SUSSAN. *A categorification of finite-dimensional irreducible representations of quantum and their tensor product*. Selecta Math. (N.S.), 2006, **12**(3-4): 379–431.

[9] Jin HONG, S. J. KANG. *Introduction to Quantum Groups and Crystal Bases*. American Mathematical Society, Providence, RI, 2002.

[10] J. E. HUMPHREYS. *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*. American Mathematical Society, Providence, RI, 2008.

[11] M. KHOVANOV, V. MAZORCHUK, C. STROPPEL. *A brief review of abelian categorifications*. Theory Appl. Categ., 2009, **22**(19): 479–508.

[12] M. KHOVANOV, V. MAZORCHUK, C. STROPPEL. *A categorification of integral Specht modules*. Proc. Amer. Math. Soc., 2008, **136**(4): 1163–1169.

[13] V. MAZORCHUK. *Lectures on algebraic categorification*. European Mathematical Society (EMS), Zürich, 2012.

[14] V. MAZORCHUK, C. STROPPEL. *Categorification of (induced) cell modules and the rough structure of generalised Verma modules*. Adv. Math., 2008, **219**(4): 1363–1426.

[15] V. MAZORCHUK, C. STROPPEL. *Categorification of Wedderburn's basis for $\mathbb{C}[S_n]$*. Arch. Math. Basel, 2008, **91**(1): 1–11.

[16] C. STROPPEL. *Categorification of the Temperley-Lieb category, tangles, and cobordisms via projective functors*. Duke Math. J., 2005, **126**(3): 547–596.

[17] J. SUSSAN. *Category $\mathcal{O}$ and sl_k-link invariants*. arXiv:math.QA/0701045.

[18] Yongjun XU, Shilin YANG. *A categorification of the spin representation of $U(\mathrm{so}(7,\mathcal{C}))$ via projective functors*. Comm. Algebra, 2013, **41**(10): 3868–3888.