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Two Elementary Applications of the Lagrange Expansion Formula

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Abstract The present paper offers two likely neglected applications of the classical Lagrange expansion formula. One is a unified approach to some age-old derivative identities originally due to Pfaff and Cauchy. Another is two explicit matrix inversions which may serve as common generalizations of some known inverse series relations.

Keywords matrix inversion; Lagrange-Bürmann expansion formula; derivative identities; Pfaff; Cauchy; Leibniz product rule

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1. Introduction

The following Lagrange expansion formula can be found in the book [20, Chap.7, Eqs.(2)-(3)]. It now stands as a landmark discovery in the history of Analysis concerning the problems of solving implicit equations and, more generally, expanding analytic functions into power series just like the classical Taylor's formula does.

Lemma 1.1 (The Lagrange expansion formula) *Assume that $F(x)$ and $\phi(x)$ are analytic around $x=a$ and $\phi(a)\neq 0$. Then for $\epsilon=0,1$, we have*

$$\frac{F(x)}{\big(1-(x-a)\,\phi'(x)/\phi(x)\big)^{\epsilon}}=\sum_{n=0}^{\infty}\chi_{n,\epsilon}(F)(\frac{x-a}{\phi(x)})^{n}, \tag{1.1}$$

where the coefficient $\chi_{n,\epsilon}(F)$ is given by

$$\chi_{n,\epsilon}(F)=\frac{1}{n!}\,\mathbf{D}_a^{n-1+\epsilon}[F^{(1-\epsilon)}(x)\phi^{n}(x)]. \tag{1.2}$$

Hereafter, $\mathbf{D}=\mathrm{d}/\mathrm{d}x$ denotes the usual derivative operator and, as conventions, we assume

$$\mathbf{D}_a^n\big[F(x)\big]=(\frac{\mathrm{d}}{\mathrm{d}x})^nF(x)\big|_{x=a}=F^{(n)}(a),$$

$$\mathbf{D}_xF(x)=F'(x),\quad \mathbf{D}_x^{-1}[F'(x)]=F(x).$$

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We refer the reader to [3,5,12,13,20] for more comprehensive knowledge on the Lagrange expansion formula and its various q-analogues, especially to [8] of Hofbauer and [12] of Merlini et al. on their typical applications in Analysis and Combinatorics.

In the present paper, we are mainly concerned with two applications of the Lagrange expansion formula: one is a unified proof for many derivative identities derived by various ad hoc techniques, another is two matrix inversions in Combinatorial Analysis. As far as we know, all these have eluded many mathematicians in the past, even not been mentioned in several excellent surveys on the Lagrange inversion such as [8,13].

2. Derivative identities

In his paper [10] which contains many historical anecdotes, Johnson brought the following derivative identities [10, Eqs.(1.1)–(1.4)] to public. Found by Pfaff, Cauchy and Walker more than two centuries ago, these identities seem very likely unknown for a long time. For this reason, similar identities have now and then been rediscovered as "new" ones.

Theorem 2.1 *For $n \in \mathbb{N}$, the set of nonnegative integers, and any analytic functions $F(x), G(x)$ and $\phi(x)$, the following identities hold:*

(i) (Pfaff's derivative identity (1795))

$$\mathbf{D}_x^n\big[F(x)G(x)\phi^n(x)\big] = \sum_{k=0}^n \binom{n}{k} \mathbf{D}_x^{k-1}\big[F'(x)\phi^k(x)\big]\ \mathbf{D}_x^{n-k}\big[G(x)\phi^{n-k}(x)\big]. \tag{2.1}$$

(ii) (Cauchy's derivative identity (1826))

$$\mathbf{D}_x^{n-1}\big[\{F(x)G(x)\}'\phi^n(x)\big] = \sum_{k=0}^n \binom{n}{k} \mathbf{D}_x^{k-1}\big[F'(x)\phi^k(x)\big]\ \mathbf{D}_x^{n-k-1}\big[G'(x)\phi^{n-k}(x)\big]. \tag{2.2}$$

(iii) (Olver's derivative identity (1992))

$$\frac{a+c}{a+c+n}\,\mathbf{D}_x^n\big[\phi^{a+c+n}(x)\big] = \sum_{k=0}^n \frac{a\ c}{(a+k)(c+n-k)}\binom{n}{k} \mathbf{D}_x^{k}\big[\phi^{a+k}(x)\big]\ \mathbf{D}_x^{n-k}\big[\phi^{c+n-k}(x)\big]. \tag{2.3}$$

(iv) (U. Abel's derivative identity (2013))

$$\sum_{|\mathbf{k}|=n}\binom{n}{\mathbf{k}}\prod_{i=1}^r \mathbf{D}_x^{k_i}[\phi^{k_i}(x)f_i(x))] = \mathbf{D}_x^n\Big[\frac{\phi^n(t)\prod_{i=1}^r f_i(t)}{(1-(t-x)\,\phi'(t)/\phi(t))^{r-1}}\Big], \tag{2.4}$$

where the vector $\mathbf{k} = (k_1, k_2, \ldots, k_r) \in \mathbb{N}^r, |\mathbf{k}| = k_1 + k_2 + \cdots + k_r$, and the usual multinomial coefficients

$$\binom{n}{\mathbf{k}} = \binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1!k_2!\cdots k_r!} \quad \textit{for} \quad |\mathbf{k}| = n.$$

Here are some necessary comments on these identities.

Remark 2.2 Treating them as "curious" extensions of the Leibniz product rule, Johnson proved (i)–(iii) by induction [10, §2] and so filled out a proof of Lemma 1.1 which had been sketched by Lagrange but still unknown to the literature for centuries. Before Johnson, Olver [16] found identity (2.3) via the method of differential operators, thereby obtaining many interesting identities. As the main result of very recent paper [1, Theorem 3], U. Abel showed (2.4) by induction in an effort to generalize the Leibniz rule.

With the help of Lemma 1.1, we shall first show Theorem 2.1 in a unified but to certain extent self-evident manner.

Proof Without loss of generality, we assume that $\phi(x) \neq 0$. Otherwise, we may show all identities above by choosing $\phi(x)+c$ with $c \neq 0$ and then taking the limit $c \to 0$. Under this assumption, by Lemma 1.1 we are able to expand three functions of the relation

$$\frac{F(t)G(t)}{\big(1-(t-x)\phi'(t)/\phi(t)\big)^{\epsilon}} = F(t)\,\frac{G(t)}{\big(1-(t-x)\phi'(t)/\phi(t)\big)^{\epsilon}} \tag{2.5}$$

into power series in $(t-x)/\phi(t)$ simultaneously. Upon equating the coefficients of $(t-x)^n/\phi^n(t)$ on both sides of (2.5), we immediately obtain that

$$\chi_{n,\epsilon}(FG) = \sum_{k=0}^{n} \chi_{k,0}(F)\ \chi_{n-k,\epsilon}(G), \tag{2.6}$$

where $\chi_{n,\epsilon}(FG)$ denotes the coefficient of $(t-x)^n/\phi^n(t)$ in the expansion (1.1) with the replacement of $F(t)$ with $F(t)G(t)$. Obviously, identity (2.1) is the case $\epsilon = 0$ of (2.6) and the case $\epsilon = 1$ of (2.2). Identity (2.3) is the special case of (2.6) with $F(t) = \phi^a(t)$ and $G(t) = \phi^c(t)$. To achieve (2.4), we only need to apply (1.1) to each function

$$F_i(t) = \frac{f_i(t)}{1-(t-x)\phi'(t)/\phi(t)}, \quad i = 1, 2, \ldots, r$$

and further to

$$G(t) = \frac{\prod_{i=1}^{r} f_i(t)}{(1-(t-x)\phi'(t)/\phi(t))^{r-1}}.$$

Finally, by equating the coefficients of $(t-x)^n/\phi^n(t)$ on both sides of the relation

$$\prod_{i=1}^{r} F_i(t) = \frac{G(t)}{1-(t-x)\phi'(t)/\phi(t)},$$

we obtain identity (2.4). This completes the proof of Theorem 2.1. □

Evidently, Pfaff's derivative identity (2.1) contains the notable Abel identity as a special case.

Example 2.3 (Abel identity) For $n \in \mathbb{N}$ and complex numbers a, b, z, we have

$$\sum_{k=0}^{n} \binom{n}{k} a(a+kz)^{k-1}(b+(n-k)z)^{n-k} = (a+b+nz)^n. \tag{2.7}$$

3. Jensen identity and its generalization

As Johnson pointed out in [10], the following identity is attributed to Walker (1880) but may date back to Pfaff (1795) and Cauchy (1826). Later as we shall see, it is much more general than the well–known Jensen identity published in 1902. Now we proceed to show that this identity also lies in the realm of the Lagrange expansion formula.

Theorem 3.1 ([10, Eq.(1.5)]) *For $n \in \mathbb{N}$ and arbitrary analytic functions $F(x), G(x)$ and $\phi(x)$, we have*

$$\sum_{k=0}^{n} \frac{n!}{k!}\, \mathbf{D}_x^k \big[F(x)G(x)\phi^k(x)\{\phi'(x)\}^{n-k}\big] = \sum_{k=0}^{n} \binom{n}{k}\, \mathbf{D}_x^k \big[F(x)\phi^k(x)\big]\, \mathbf{D}_x^{n-k}\big[G(x)\phi^{n-k}(x)\big]. \quad (3.1)$$

Proof As previously, we assume that $\phi(x) \neq 0$. Clearly, for $|(t-x)\phi'(t)/\phi(t)| < 1$, there holds

$$\frac{1}{1-(t-x)\phi'(t)/\phi(t)} = \sum_{k\geq 0}\{\phi'(t)\}^k \big(\frac{t-x}{\phi(t)}\big)^k.$$

Consequently, we have

$$\sum_{k\geq 0} \frac{F(t)G(t)\{\phi'(t)\}^k}{1-(t-x)\phi'(t)/\phi(t)} \big(\frac{t-x}{\phi(t)}\big)^k = \frac{F(t)}{1-(t-x)\phi'(t)/\phi(t)}\frac{G(t)}{1-(t-x)\phi'(t)/\phi(t)}.$$

Next, using Lemma 1.1 with $\epsilon = 1$, we expand the first member on the left-hand sum and these two on the right-hand side into power series in $(t-x)/\phi(t)$. By equating the coefficients of $(t-x)^n/\phi^n(t)$, we readily find that

$$\sum_{k=0}^{n} \chi_{k,1}(FG\{\phi'(t)\}^{n-k}) = \sum_{k=0}^{n} \chi_{k,1}(F)\chi_{n-k,1}(G),$$

which is consistent with (3.1). $\square$

It is noteworthy that from identity (3.1) we easily deduce the classical Jensen identity and its exponential analogue as follows. A first glance at [9] shows that Jensen's method to his result is by induction, wherein he also used the Lagrange expansion formula but unlike our own.

Example 3.2 For $n \in \mathbb{N}$ and complex numbers a, b, z, we have

(i) (Jensen identity: Gould [6, Eq.(3.144)])

$$\sum_{k=0}^{n} \binom{a+kz}{k}\binom{b+(n-k)z}{n-k} = \sum_{k=0}^{n} \binom{a+b+nz-k}{n-k} z^k. \quad (3.2)$$

(ii) (Exponential Jensen identity: Egorychev [3, Exer. 2.4.2(d)])

$$\sum_{k=0}^{n} \binom{n}{k}(a+kz)^k(b+(n-k)z)^{n-k} = \sum_{k=0}^{n} \binom{n}{k}(a+b+nz)^{n-k}\, k!z^k. \quad (3.3)$$

Proof Jensen identity (3.2) is precisely identity (3.1) with $(\phi(x), F(x), G(x)) = ((1+x)^z, (1+x)^a, (1+x)^b)$ while identity (3.3) results from identity (3.1) by specifying $(\phi(x), F(x), G(x)) = (\exp(zx), \exp(ax), \exp(bx))$. Both are evaluated at $x = 0$. $\square$

4. Matrix inversions

To proceed further, we need to recall some concepts from Combinatorial Analysis [15]. As customary, a pair of infinite-dimensional lower-triangular matrices, say, $F = (f_{n,k})_{n,k\in\mathbb{N}}$ and

$G=(g_{n,k})_{n,k\in\mathbb{N}}$ subject to $f_{n,k}=g_{n,k}=0$ unless $n\geq k$, satisfying

$$\sum_{n\geq i\geq k} f_{n,i}g_{i,k}=\sum_{n\geq i\geq k} g_{n,i}f_{i,k}=\delta_{n,k} \ \text{ for all } n,\, k\in\mathbb{N},$$

is said to be a matrix inversion, where $\delta_{n,k}$ denotes the usual Kronecker symbol. In this context, we employ F^{-1} to denote the inverse G of F. Note that such F together with F^{-1} is often called an inversion formula or a pair of inverse (reciprocal) series relations.

As of today, the use of matrix inversions has become an important tool for deriving summation and transformation formulas of hypergeometric series. For more details, we refer the reader to [3,5,15,17]. The most useful application may be summarized like this: given $F=(f_{n,k})_{n\geq k\geq 0}$ and $F^{-1}=(g_{n,k})_{n\geq k\geq 0}$, then for two arbitrary sequences $\{x(n)\}_{n\geq 0}$ and $\{y(n)\}_{n\geq 0}$, the system of equations

$$x(n)=\sum_{k=0}^{n} f_{n,k}y(k), \quad n=0,1,2,\ldots$$

is equivalent to the system of equations

$$y(n)=\sum_{k=0}^{n} g_{n,k}x(k), \quad n=0,1,2,\ldots.$$

As far as this topic is concerned, we would like to reiterate an insightful observation of P. Henrici [7, Chap. 1] here: the Lagrange expansion formula is essentially equivalent to matrix inversions. Indeed, behind this classical theorem lie two matrix inversions which may hitherto have never been investigated from such a viewpoint.

First, making use of Lemma 1.1 with $\epsilon=0$, we readily obtain that

Theorem 4.1 *Let $h(x),\phi(x)$ and $\psi(x)\in\mathcal{L}_0$. Then we have*

$$\begin{aligned}&\Big(\frac{1}{(n-k)!}\ \mathbf{D}_0^{n-k}\big[(1-\Delta(\phi(x)))h(x)\psi^k(x)\phi^n(x)\big]\Big)^{-1}_{n\geq k\geq 0}\\ &=\Big(\frac{1}{(n-k)!}\ \mathbf{D}_0^{n-k}\Big[\frac{1+\Delta(\psi(x))}{h(x)\phi^k(x)\psi^n(x)}\Big]\Big)_{n\geq k\geq 0},\end{aligned} \tag{4.1}$$

where the mapping $\Delta: f(x)\mapsto xf'(x)/f(x)$ and $\mathcal{L}_0$ denotes the set of functions $f(x)$ which are analytic around $x=0$ with $f(0)\neq 0$.

Proof It suffices to set $F(x)=h(x)(x\ \psi(x))^k$ and $a=0$ in (1.1). So we have

$$h(x)(x\ \psi(x))^k=\sum_{n=k}^{\infty} A_{n,k}\Big(\frac{x}{\phi(x)}\Big)^n. \tag{4.2}$$

From there, by Lemma 1.1 subject to $\epsilon=0$ and the Leibniz product rule, it is immediate that

$$\begin{aligned}A_{n,k}&=\frac{1}{n!}\ \mathbf{D}_0^{n-1}\big[\{h(x)\big(x\ \psi(x)\big)^k\}'\phi^n(x)\big]\\ &=\frac{1}{n!}\ \mathbf{D}_0^{n-1}\big[\{h(x)\big(x\ \psi(x)\big)^k\phi^n(x)\}'-n\,h(x)(x\ \psi(x))^k\phi^{n-1}(x)\phi'(x)\big]\\ &=\frac{1}{(n-k)!}\ \mathbf{D}_0^{n-k}\big[h(x)\psi^k(x)\phi^n(x)\big]-\frac{1}{(n-k)!}\ \mathbf{D}_0^{n-k}\Big[h(x)\psi^k(x)\phi^n(x)\frac{x\phi'(x)}{\phi(x)}\Big]\end{aligned}$$

$$= \frac{1}{(n-k)!}\ \mathbf{D}_0^{n-k}\big[h(x)\psi^k(x)\phi^n(x)(1-\Delta(\phi(x)))\big]. \tag{4.3}$$

Apparently, owing to $A_{n,n} = h(0)\big(\psi(0)\phi(0)\big)^n \neq 0$, the corresponding matrix $(A_{n,k})$ is not only lower–triangular but also invertible. Thus we may assume $A^{-1} = (B_{n,k})$ and then use it to invert (4.2). The result is as follows:

$$\frac{1}{h(x)}\Big(\frac{x}{\phi(x)}\Big)^k = \sum_{n=k}^{\infty} B_{n,k}\big(x\ \psi(x)\big)^n. \tag{4.4}$$

This expression reminds us that the coefficients $B_{n,k}$ can still be evaluated with the use of Lemma 1.1. By doing so, we have

$$\begin{aligned} B_{n,k} &= \frac{1}{n!}\ \mathbf{D}_0^{n-1}\Big[\Big\{\frac{1}{h(x)}\Big(\frac{x}{\phi(x)}\Big)^k\Big\}'\frac{1}{\psi^n(x)}\Big] \\ &= \frac{1}{(n-k)!}\ \mathbf{D}_0^{n-k}\Big[\frac{1}{h(x)\phi^k(x)\psi^n(x)}\Big] + \frac{1}{(n-k)!}\ \mathbf{D}_0^{n-k}\Big[\frac{x\ \psi'(x)}{h(x)\phi^k(x)\psi^{n+1}(x)}\Big] \\ &= \frac{1}{(n-k)!}\ \mathbf{D}_0^{n-k}\Big[\frac{1+\Delta(\psi(x))}{h(x)\phi^k(x)\psi^n(x)}\Big]. \end{aligned}$$

Note that the penultimate equality is based on the Leibniz product rule. Thus, we have finished the complete proof of the theorem. □

In much the same way, using Lemma 1.1 with $\epsilon = 1$, we readily establish another matrix inversion.

Theorem 4.2 *Let $h(x), \phi(x)$ and $\psi(x) \in \mathcal{L}_0$. Then we have*

$$\begin{aligned} &\Big(\frac{1}{(n-k)!}\ \mathbf{D}_0^{n-k}\big[h(x)\psi^k(x)\phi^n(x)\big]\Big)^{-1}_{n\geq k\geq 0} \\ &\quad = \Big(\frac{1}{(n-k)!}\ \mathbf{D}_0^{n-k}\Big[\frac{(1-\Delta(\phi(x)))(1+\Delta(\psi(x)))}{h(x)\psi^n(x)\phi^k(x)}\Big]\Big)_{n\geq k\geq 0}. \end{aligned} \tag{4.5}$$

A few comments on Theorems 4.1 and 4.2 are in order.

Remark 4.3 Both inversion (4.1) and inversion (4.5) are valid at any point $x = a$, not only $x = 0$.

Remark 4.4 Inversion (4.5) first appeared in Egorychev [3, Theorem 3.1.2 and Corollary] and recently in Egorychev and Zima [4, Theorem 2(b)], being derived by the method of coefficients. By contrast, the foregoing argument seems more straightforward and elementary. It is also worthy of note that in his Ph.D. thesis [19, §3.3], Rosenkranz sketched a result similar to Theorem 4.1 with the use of residue operator over the ring of formal power series.

Undoubtedly, putting $(h(x), \phi(x), \psi(x)) = (\phi^c(x), \phi^p(x), \phi^q(x))$ in Theorem 4.1, we thus obtain

Corollary 4.5 *For $\phi(x) \in \mathcal{L}_0$ and complex numbers p, q, c, it holds*

$$\Big(\frac{1}{(n-k)!}\frac{c+kp+kq}{c+np+kq}\ \mathbf{D}_0^{n-k}\big[\phi(x)^{c+np+kq}\big]\Big)^{-1}_{n\geq k\geq 0}$$

$$= \Big(\frac{1}{(n-k)!}\frac{c+kp+kq}{c+kp+nq}\ \mathbf{D}_0^{n-k}\big[\phi(x)^{-c-kp-nq}\big]\Big]\Big)_{n\geq k\geq 0}. \tag{4.6}$$

It is quite clear that this inversion contains the following inverse series relation due to Chu as the special case $q=0$.

Example 4.6 ([2, Theorem 1]) Let $\phi(x)$ be an infinitely differentiable function with respect to x. Then the system of equations

$$x(n)=\sum_{k=0}^{n}\binom{n}{k}\frac{c+kp}{c+np}\ \mathbf{D}_x^{n-k}\big[\phi^{c+np}(x)\big]y(k),\quad n=0,1,2,\ldots \tag{4.7}$$

is equivalent to the system

$$y(n)=\sum_{k=0}^{n}\binom{n}{k}\ \mathbf{D}_x^{n-k}\big[\phi^{-c-kp}(x)\big]x(k),\quad n=0,1,2,\ldots. \tag{4.8}$$

We end this paper by examining some easy-to-use specializations of Theorem 4.2 and Corollary 4.5.

Example 4.7 Write $\binom{x}{n}$ for $x(x-1)(x-2)\cdots(x-n+1)/n!$. Then the following matrix inversions hold:

$$\left(\frac{c+kp+kq}{c+np+kq}\binom{c+np+kq}{n-k}\right)^{-1}=\left(\frac{c+kp+kq}{c+kp+nq}\binom{-c-kp-nq}{n-k}\right), \tag{4.9}$$

$$\left(\binom{c+np+kq}{n-k}\right)^{-1}=\left(\frac{\mathcal{E}(n,k)}{(c+kp+nq)(c+kp+nq+1)}\binom{-c-kp-nq}{n-k}\right), \tag{4.10}$$

where $\mathcal{E}(n,k)=\{c+n(p+q)\}\{c+k(p+q)\}+(n-k)pq+(c+np+kq)$.

$$\Big(\frac{(c+np+kq)^{n-k}}{(n-k)!}\Big)^{-1}=\Big(\mathcal{F}(n,k)\frac{(-c-kp-nq)^{n-k-2}}{(n-k)!}\Big), \tag{4.11}$$

$$\Big(\frac{(c+kp+kq)(c+np+kq)^{n-k-1}}{(n-k)!}\Big)^{-1}=\Big(\frac{-(c+kp+kq)(-c-kp-nq)^{n-k-1}}{(n-k)!}\Big), \tag{4.12}$$

where $\mathcal{F}(n,k)=\{c+n(p+q)\}\{c+k(p+q)\}+(n-k)pq$.

Proof All these are deducible from Theorem 4.2 and Corollary 4.5 under the choices $\phi(x)=1+x$ and $\phi(x)=e^x$, correspondingly. □

Remark 4.8 It should be mentioned that inversion (4.11) was first established by Merlini et al. via the technique of Riordan array [14, p.483, Example]. In [11], we derived (4.9)–(4.12) from the classical Hagen–Rothe formula and investigated their applications to Combinatorics and Special Functions. As demonstrated in [11], (4.9)–(4.12) unify all inverse series relations collected by Riordan [18, Chapters II and III]. Some q-analogues of these cases are given therein.

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