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# Finite $p$-Groups and Normal Closures of Nonnormal Subgroups

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**Abstract** In this paper, finite $p$-groups $G$ with $G/H^G$ being cyclic for every minimal nonnormal subgroup $H$ are classified up to isomorphism, where $H^G$ denotes the normal closure of $H$.

**Keywords** nonnormal subgroups; normal closures; Dedekindian $p$-groups

**MR(2010) Subject Classification** 20D10; 20D15

## 1. Introduction

In this paper, all groups considered are finite groups. Let $G$ be a finite group. Given a nonempty subset $H$ of $G$, the normal closure of $H$ in $G$ is the intersection of all normal subgroups of $G$ which contain $H$, it is denoted by $H^G$. Obviously, $H^G$ is the smallest normal subgroup containing $H$ and it is easy to show that $H^G = \langle h^g | h \in H, g \in G\rangle$. If $H$ is a subgroup of $G$, we notice that $H \le H^G \le G$ and

$$H = H^G \text{ if and only if } H \trianglelefteq G.$$

Thus a subgroup may be regarded as "far normal" if it has "large" normal closure or "nearly normal" if it has "small" normal closure. Finite $p$-groups with "small" normal closure have been investigated in [1–3], respectively. On the other hand, finite $p$-groups with "large" normal closure have also been investigated. For example, Janko [4] classified finite $p$-groups $G$ such that $|G : H^G| = p$ for every nonnormal subgroup $H$ of $G$. Zhao and Guo [5] classified finite $p$-groups $G$ such that $|G : H^G| \le p^2$ for every nonnormal cyclic subgroup $H$ of $G$. As a continuation of Janko, Zhao and Guo's works, we classify finite $p$-groups such that $G/H^G$ is cyclic for every nonnormal subgroup $H$ of $G$ in this paper.

For convenience, a finite $p$-group $G$ is called a $\mathcal{C}_c$-group if $G/H^G$ is cyclic for every minimal nonnormal subgroup $H$.

In Lemma 2.4, we give some equivalent conditions for a finite $p$-group $G$ to be a $\mathcal{C}_c$-group. It turns out that $G$ is a $\mathcal{C}_c$-group if and only if every nonnormal subgroup of $G$ is contained in exactly one maximal subgroup of $G$. In [6], Janko gave a classification of such finite $p$-groups.

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Hence $\mathcal{C}_c$-groups are classified. This paper will give an independent proof of this classification by using central extension. In Theorem 3.1, $\mathcal{C}_c$-groups will be classified up to isomorphism and more detail information of $\mathcal{C}_c$-groups will be given.

Throughout this paper, $p$ is always a prime. Let $G$ be a finite $p$-group. The $n$th term of the lower central series of $G$ is denoted by $G_n$ and $G' = G_2$. We use $c(G)$, $\exp(G)$ and $d(G)$ to denote the nilpotency class, the exponent and the minimal number of generators of $G$ respectively. $o(g)$ denotes the order of $g$, and $M \lessdot G$ denotes that $M$ is a maximal subgroup of $G$. We also use $C_{p^n}$ and $E_{p^n}$ to denote the cyclic group and the elementary abelian group of order $p^n$, respectively, where $C_{p^0} = E_{p^0} = 1$.

We also use the following notation.

$$\Omega_1(G) = \langle g \in G \mid g^p = 1 \rangle \text{ and } \mho_1(G) = \langle g^p \mid g \in G \rangle.$$

$$M_p(n,m) = \langle a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}} \rangle, \text{where } n \geq 2.$$

$$M_p(n,m,1) = \langle a, b; c \mid a^{p^n} = b^{p^m} = c^p = 1, [a,b] = c, [c,a] = [c,b] = 1 \rangle,$$

where $n \geq m$, and $m + n \geq 3$ if $p = 2$.

The other terminology and notations are standard, as in [7].

## 2. Preliminaries

A nonabelian $p$-group is said to be minimal nonabelian if all its proper subgroups are abelian. A $p$-group is said to be Dedekindian if all its subgroups are normal.

**Lemma 2.1** ([8]) *Let $G$ be a minimal nonabelian $p$-group. Then $G$ is $Q_8$, $M_p(m,n)$, or $M_p(m,n,1)$.*

**Lemma 2.2** ([9, Theorem 1.1]) *If $G$ is Dedekindian, then $G$ is either abelian or $G \cong Q_8 \times E_{2^n}$.*

**Lemma 2.3** *Let $G$ be a finite $p$-group. If $|G'| \geq p^2$, then there exists $N \trianglelefteq G$ such that $G/N$ is not a Dedekindian group, where $N \leq G' \cap Z(G)$ and $|K| = p$.*

**Proof** Since $|G'| \geq p^2$, $G$ is not Dedekindian. By [9, Lemma 2.1], there exists $K \trianglelefteq G$ such that $|G' : K| = p$ and $G/K$ is not Dedekindian. Thus there exists $N \leq K \cap Z(G) \leq G' \cap Z(G)$ such that $|N| = p$. Since $G/K \cong (G/N)/(K/N)$, $G/N$ is not Dedekindian. $\square$

A minimal nonnormal subgroup of a finite group $G$ is a nonnormal subgroup whose proper subgroups are normal in $G$. We have the following

**Lemma 2.4** *Let $G$ be a finite $p$-group which is not Dedekindian. Then the following statements are equivalent.*

*(1) The factor group $G/H^G$ is cyclic for every minimal nonnormal subgroup $H$ of $G$.*

*(2) The factor group $G/H^G$ is cyclic for every nonnormal subgroup $H$ of $G$.*

*(3) Every subgroup of $\Phi(G)$ is normal in $G$ and $d(G) = 2$.*

*(4) Every nonnormal subgroup is contained in exactly one maximal subgroup of $G$.*

**Proof** (1)⇒(2). Let $H$ be a nonnormal subgroup of $G$. Then there exists $L \leq H$ such that $L$ is a minimal nonnormal subgroup of $G$. Thus $G/L^G$ is cyclic. It follows that $G/H^G$ is cyclic.

(2)⇒(3). Let $H \leq \Phi(G)$. Then $H^G \leq \Phi(G)$. Since $G/\Phi(G)$ is not cyclic, $G/H^G$ is not cyclic. It follows that $H \trianglelefteq G$. Thus every subgroup of $\Phi(G)$ is normal in $G$.

Let $H$ be a minimal nonnormal subgroup of $G$. By [9, Lemma 1.4], $H$ is cyclic. Let $H = \langle a\rangle$. Since $H^G \leq HG'$ and $G/H^G$ is cyclic, $G/HG'$ is cyclic. Let $G/HG' = \langle \bar{b}\rangle$. Then $G = \langle a, b, G'\rangle = \langle a, b\rangle$. Thus $d(G) = 2$.

(3)⇒(4). Since $d(G) = 2$, $M_1 \cap M_2 = \Phi(G)$ for any two distinct maximal subgroups $M_1$ and $M_2$ of $G$. If $H \leq M_1 \cap M_2 = \Phi(G)$, then $H \trianglelefteq G$. It follows that (4) holds.

(4)⇒(1). Let $H$ be a minimal nonnormal subgroup of $G$. Then $H$ is contained in exactly one maximal subgroup of $G$. It follows that $H\Phi(G)$ is contained in exactly one maximal subgroup of $G$. Hence $G/H\Phi(G)$ is of order $p$ by correspondence theorem. Let $G/H\Phi(G) = \langle \bar{a}\rangle$. Then $G = \langle a, H, \Phi(G)\rangle = \langle a\rangle H^G$. Thus $G/H^G = \langle \bar{a}\rangle$. (1) holds. □

**Lemma 2.5** *Let $G$ be a $\mathcal{C}_c$-group. Then the following statements hold.*

*(1) If $N \trianglelefteq G$, then $G/N$ is a $\mathcal{C}_c$-group.*

*(2) The derived subgroup $G'$ is cyclic.*

*(3) If $H \leq \Phi(G)$ and $H \cap G' = 1$, then $H \leq Z(G)$.*

**Proof** (1) Let $\overline{G} = G/N$ and $\overline{H} = H/N \ntrianglelefteq \overline{G}$. Then $H \ntrianglelefteq G$. Notice that $\overline{G}/(\overline{H}^{\overline{G}}) = \overline{G}/\overline{H^G} \cong G/H^G$. Since $G$ is a $\mathcal{C}_c$-group, $\overline{G}$ is a $\mathcal{C}_c$-group.

(2) Since $G$ is a $\mathcal{C}_c$-group, $d(G) = 2$ by Lemma 2.4(3). Let $G = \langle a, b\rangle$. Then $G' = \langle [a, b]^g \mid g \in G\rangle$. Since $\langle [a, b]\rangle \leq \Phi(G)$, $\langle [a, b]\rangle \trianglelefteq G$ by Lemma 2.4(3). It follows that $G' = \langle [a, b]\rangle$.

(3) If $H \leq \Phi(G)$, then $H \trianglelefteq G$ by Lemma 2.4(3). It follows that $[H, G] \leq H \cap G' = 1$. Thus $H \leq Z(G)$. □

**Lemma 2.6** *Let $G$ be a finite $p$-group. Then $G$ is a $\mathcal{C}_c$-group if $G$ is one of following groups.*

(1) *$G$ is a 2-group of maximal class.*

(2) *$G$ is a minimal nonabelian $p$-group.*

**Proof** (1) Let $H$ be a nonnormal subgroup of $G$. Since $G$ is a 2-group of maximal class, $G_i$ is the unique normal subgroup of order $2^{n-i}$. It follows that $H^G = G_i$ or a maximal subgroup of $G$. If $H^G = G_i$, which is cyclic, then $H$ *char* $G_i \lhd G$. It follows that $H \lhd G$, a contradiction. Thus $H^G$ is a maximal subgroup of $G$ and so $G/H^G$ is cyclic. It follows that $G$ is a $\mathcal{C}_c$-group.

(2) Let $H$ be a nonnormal subgroup of $G$. Notice that $G$ is a minimal nonabelian $p$-group, we get $d(G) = 2$, $|G'| = p$ and $Z(G) = \Phi(G)$. It follows that $H \nleq \Phi(G)$ and $H\Phi(G)$ is of index at most $p$. Since $H$ is not normal in $G$, $H < H^G \leq HG'$ and $|HG' : H| = p$. Thus $H^G = HG'$. Let $\overline{G} = G/H^G$. Then

$$|\overline{G}/\Phi(\overline{G})| = |\overline{G}/\overline{\Phi(G)}| = |G/\Phi(G)H| = p.$$

It follows that $G/H^G$ is cyclic. So $G$ is a $\mathcal{C}_c$-group. □

## 3. The main results and their proofs

If all subgroups of $G$ are normal, then $G$ is a Dedekindian group. The groups have been classified. Thus we consider the $\mathcal{C}_c$-groups with nonnormal subgroups in this section.

**Theorem 3.1** *Let $G$ be a finite $p$-group with nonnormal subgroups. Then $G$ is a $\mathcal{C}_c$-group if and only if one of the following occurs.*

*I. $M_p(n,m)$ or $M_p(n,m,1)$.*

*II. 2-groups of maximal class of order $\geq 2^4$.*

*III. Nonmetacyclic 2-groups of order $2^{n+2}$, where $n \geq 3$.*

*(1)* $\langle a,b,c \mid a^{2^n}=b^2=1,[a,b]=c,c^2=a^{-4},[c,a]=1,[c,b]=c^{-2}\rangle$.

*(2)* $\langle a,b,c \mid a^{2^n}=1,b^2=a^{2^{n-1}},[a,b]=c,c^2=a^{-4},[c,a]=1,[c,b]=c^{-2}\rangle$.

*In groups (1) and (2) of III,* $G'=\langle c\rangle \cong C_{2^{n-1}}, \Phi(G)=\langle a^2\rangle \times \langle a^2c\rangle \cong C_{2^{n-1}}\times C_2, Z(G)=\langle a^{2^{n-1}}\rangle \times \langle a^2c\rangle \cong C_2^2$.

*(3)* $\langle a,b,c \mid a^{2^n}=b^4=1,[a,b]=c,c^2=a^{-4},[c,a]=1,[c,b]=c^{-2}\rangle$, *where* $G'=\langle c\rangle \cong C_{2^{n-1}}, \Phi(G)=\langle a^2\rangle \times \langle a^2c\rangle \times \langle b^2\rangle \cong C_{2^{n-1}}\times C_2\times C_2, Z(G)=\langle a^{2^{n-1}}\rangle \times \langle a^2c\rangle \times \langle b^2\rangle \cong C_2^3$.

*IV. Metacyclic 2-groups of order $2^{n+2}$, where $n\geq 3$.*

*(1)* $\langle a,b \mid a^{2^n}=b^4=1,[a,b]=a^{-2}\rangle$.

*(2)* $\langle a,b \mid a^{2^n}=b^4=1,[a,b]=a^{-2+2^{n-1}}\rangle$.

*In groups (1) and (2) of IV,* $G'=\langle a^2\rangle \cong C_{2^{n-1}}$, $\Phi(G)=\langle a^2\rangle \times \langle b^2\rangle \cong C_{2^{n-1}}\times C_2, Z(G)=\langle a^{2^{n-1}}\rangle \times \langle b^2\rangle \cong C_2^2$.

**Proof** Let $G$ be a $\mathcal{C}_c$-group with nonnormal subgroups. By Lemma 2.4(3), every subgroup of $\Phi(G)$ is normal in $G$ and $d(G)=2$. If $p>2$, then $\Phi(G)\leq Z(G)$ by [7, §4, Exercise 8]. It follows that $G$ is a minimal nonabelian $p$-group. Since $G$ has nonnormal subgroups, $G$ is not Dedekindian. By Lemma 2.1, we get $G$ is isomorphic to $M_p(n,m)$ or $M_p(n,m,1)$.

Next, we complete the proof by induction on $|G'|$ for $p=2$. Assume $|G'|=2$. By Lemma 2.4(3), we get $d(G)=2$. It follows that $G$ is a minimal nonabelian $p$-group. By Lemma 2.1, group $G$ is $M_p(n,m)$ or $M_p(n,m,1)$. Assume $|G'|\geq 2^2$. Then there exists a normal subgroup $N$ of order 2 of $G$ such that $N\leq G'\cap Z(G)$. Let $\overline{G}=G/N$. Then $\overline{G}$ is a $\mathcal{C}_c$-group by Lemma 2.5(1), $\overline{G}$ has nonnormal subgroups by Lemma 2.3 and $|\overline{G}'|<|G'|$. By induction, $\overline{G}$ is one of groups of Theorem.

**Case 1** $\overline{G}$ is a 2-group of maximal class or $\overline{G}\cong M_p(n,1)$, where $n\geq 2$.

Let $\overline{G}=\langle \overline{a},\overline{b}\rangle$ and $N=\langle x\rangle$. Then $G$ is metacyclic by [10, Theorem 1]. Thus $N=\langle a^{2^n}\rangle$ and $G=\langle a,b\rangle$. It follows that $\langle a\rangle$ is a cyclic maximal subgroup of $G$. Since $|G'|\geq p^2$, $G$ is a 2-group of maximal class by [7, Theorem 1.2]. Thus $G$ is one of groups of type II.

**Case 2** $\overline{G}\cong M_p(n,m)$, where $n,m\geq 2$.

Let $\overline{G}=\langle \overline{a},\overline{b} \mid \overline{a}^{2^n}=\overline{b}^{2^m}=\overline{1},[\overline{a},\overline{b}]=\overline{a}^{2^{n-1}}\rangle$ and $N=\langle x\rangle$. Then $G$ is metacyclic by [10, Theorem 1]. Thus $N=\langle a^{2^n}\rangle$ and $G=\langle a,b\rangle$ with relations

$$a^{2^{n+1}}=1,b^{2^m}=a^{j2^n},[a,b]=a^{2^{n-1}+k2^n},$$

where $j, k \in \{0, 1\}$.

We can assume that $j = 0$. In fact, if $j = 1$ and $m \geq n$, then $(b^{-2^{m-n+1}} \cdot a^2)^{2^{n-1}} = b^{-2^m} \cdot a^{2^n} = 1$. It follows that $\langle b^{-2^{m-n+1}} \cdot a^2 \rangle \cap G' = 1$. Notice that $a^2 \notin Z(G)$, then $b^{-2^{m-n+1}} \cdot a^2 \notin Z(G)$. Hence $\langle b^{-2^{m-n+1}} \cdot a^2 \rangle \ntrianglelefteq G$. This contradicts Lemma 2.4(3). It follows that $m < n$ if $j = 1$. Since $m \geq 2$, $n \geq 3$. Let $b_1 = ba^{-2^{n-m}}$. Then $b_1^{2^m} = 1$. So we can assume that $j = 0$.

Since $\langle b^2 \rangle \leq \Phi(G)$, by Lemma 2.4(3), we get $\langle b^2 \rangle \trianglelefteq G$. Notice that $\langle b^2 \rangle \cap G' = 1$, then $[a, b^2] = 1$. Thus

$$1 = [a, b^2] = [a, b]^2 [a, b, b] = a^{2^n} [a^{2^{n-1}}, b] = a^{2^n} [a, b]^{2^{n-1}} = a^{2^n} a^{2^{2n-2}}.$$

It follows that $n - 2 \equiv 0 \pmod{n+1}$, which implies that $n = 2$.

If $m \geq 3$, then $(a^2 b^{2^{m-2}})^{2^2} = (a^{2^2} b^{2^{m-1}})^2 = a^{2^3} b^{2^m} = 1$. It follows that $\langle a^2 b^{2^{m-2}} \rangle \cap G' = 1$. Notice that $a^2 \notin Z(G)$, then $a^2 b^{2^{m-2}} \notin Z(G)$. Hence $\langle a^2 b^{2^{m-2}} \rangle \ntrianglelefteq G$. This contradicts Lemma 2.4(3). It follows that $m = 2$.

Now $G$ is one of groups of IV(1) or IV(2).

**Case 3** $\overline{G} \cong M_2(n, m, 1)$, where $n \geq m$ and $m + n \geq 3$.

Let $\overline{G} = \langle \overline{a}, \overline{b}, \overline{c} \mid \overline{a}^{2^n} = \overline{b}^{2^m} = \overline{c}^2 = \overline{1}, [\overline{a}, \overline{b}] = \overline{c}, [\overline{a}, \overline{c}] = [\overline{b}, \overline{c}] = \overline{1} \rangle$. By Lemma 2.5(2), we get $G'$ is cyclic. It follows that $N = \langle c^2 \rangle$. Thus

$$G = \langle a, b, c \mid a^{2^n} = c^{i2}, b^{2^m} = c^{j2}, c^{2^2} = 1, [a, b] = c^{1+k2}, [a, c] = c^{s2}, [b, c] = c^{t2} \rangle,$$

where $i, j, k, s, t \in \{0, 1\}$.

We can assume $k = 0$ by letting $c_1 = c^{1+k2}$.

We may assume $s = 0$, that is, $[a, c] = 1$. If $st = 0$, without loss of generality, we can let $s = 0$. If $st = 1$, letting $a_1 = ab$, then $[a_1, c] = [a, c][b, c] = c^4 = 1$. Thus we can assume that $s = 0$.

Since $[a^2, b] = [a, b]^2 = c^2 \neq 1$, $a^2 \notin Z(G)$. It follows that $n \geq 2$. Notice that $\langle a^2 \rangle \leq \Phi(G)$, by Lemma 2.4(3), $\langle a^2 \rangle \trianglelefteq G$. Thus $[a^2, b] = c^2 \in \langle a^2 \rangle$. It follows that $i = 1$.

If $t = 0$, that is $[b, c] = 1$, then $[a, b^2] = [a, b]^2 = c^2$. Thus $b^2 \notin Z(G)$. It follows that $m \geq 2$. Since $\langle b^2 \rangle \leq \Phi(G)$, by Lemma 2.5(3), we can get $j = 1$. If $n \geq m$, then $(b^2 a^{2^{n-m+1}})^{2^{m-1}} = b^{2^m} a^{2^n} = 1$. It follows that $\langle b^2 a^{2^{n-m+1}} \rangle \cap G' = 1$. Notice that $\langle b^2 a^{2^{n-m+1}} \rangle \leq \Phi(G)$, then $\langle b^2 a^{2^{n-m+1}} \rangle \leq Z(G)$, which contradicts that $[b^2 a^{2^{n-m+1}}, a] = [b^2, a] = c^2 \neq 1$. If $m \geq n$, consider $\langle a^2 b^{2^{m-n+1}} \rangle$, we can get a contradiction too. Thus $t = 1$.

Since $(a^{2^{n-1}} c)^2 = a^{2^n} c^2 = c^4 = 1$, $\langle a^{2^{n-1}} c \rangle \cap G' = 1$. Notice that $\langle a^{2^{n-1}} c \rangle \leq \Phi(G)$, then $\langle a^{2^{n-1}} c \rangle \leq Z(G)$. It follows that

$$1 = [a^{2^{n-1}} c, b] = [a, b]^{2^{n-1}} [c, b] = c^{2^{n-1}} c^2 = c^{2(1+2^{n-2})}.$$

We obtain that $t + 2^{n-2} \equiv 0 \pmod 2$. Thus $n = 2$.

We assert that $m \leq 2$. Otherwise, if $m \geq 3$, then $a^2 b^{2^{m-1}}, a^2 b^{2^{m-2}} \in \Phi(G) - Z(G)$. By Lemma 2.5(3), we can get $\langle a^2 b^{2^{m-1}} \rangle \cap G' \neq 1$ and $\langle a^2 b^{2^{m-2}} \rangle \cap G' \neq 1$. If $j = 1$, then $(a^2 b^{2^{m-1}})^2 = 1$. It follows that $\langle a^2 b^{2^{m-1}} \rangle \cap G' = 1$, a contradiction. If $j = 0$, then $(a^2 b^{2^{m-2}})^{2^2} = (a^{2^2} b^{2^{m-1}})^2 = 1$. It follows that $\langle a^2 b^{2^{m-2}} \rangle \cap G' = 1$, a contradiction too. Thus $m \leq 2$.

If $m=1$, then $G$ is one of groups III(1) or (2).

If $m=2$, we can get $j=0$. If $j=1$, then $\langle b^2c\rangle \le \Phi(G)$ and $\langle b^2c\rangle \not\le Z(G)$. By Lemma 2.5(3), we can get $\langle b^2c\rangle \cap G' \ne 1$. Since $(b^2c)^2=b^4c^2=1$, $\langle b^2c\rangle\cap G'=1$, a contradiction. Thus $j=0$ if $m=2$, and $G$ is one of groups III(3).

**Case 4** $\overline{G}$ is one of groups III(1) of Theorem.

Let $\overline{G}=\langle \bar a,\bar b \mid \bar a^{2^n}=1,\bar b^2=1,[\bar a,\bar b]=\bar c,\bar c^2=\bar a^{-4},[\bar c,\bar a]=1,[\bar c,\bar b]=\bar c^{-2}\rangle$ and $N=\langle x\rangle$. Then

$$G=\langle a,b \mid a^{2^n}=x^i,b^2=x^j,[a,b]=cx^k,c^2=a^{-4}x^l,[c,a]=x^s,[c,b]=c^{-2}x^t\rangle,$$

where $i,j,k,l,s,t\in\{0,1\}$.

By Lemma 2.5(2), $G'$ is cyclic. It follows that $G'=\langle c\rangle$ and $N=\langle c^{2^{n-1}}\rangle$. If $a^{2^n}=1$, notice that $n\ge 3$, then $1=(a^{-4}x^l)^{2^{n-2}}=(c^2)^{2^{n-2}}=c^{2^{n-1}}$, a contradiction. Thus $i=1$. Since $b^2\in N\le Z(G)$, by computation, we get $1=[a,b^2]=[a,b]^2[a,b,b]=c^2c^{-2}x^t=x^t$. Thus $t=0$. We can assume $k=0$ by letting $c_1=cx^k$. It follows that

$$G=\langle a,b \mid a^{2^{n+1}}=1,b^2=c^{j2^{n-1}},[a,b]=c,c^2=a^{-4}c^{l2^{n-1}},[c,a]=c^{s2^{n-1}},[c,b]=c^{-2}\rangle,$$

where $j,l,s\in\{0,1\}$.

If $s=1$, then

$$(c^{1+l2^{n-2}}a^2)^2=c^2c^{l2^{n-1}}a^4=a^{-4}c^{l2^{n-1}}c^{l2^{n-1}}a^4=1.$$

It follow that $\langle c^{1+l2^{n-2}}a\rangle\cap G'=1$. Notice that $\langle c^{1+l2^{n-2}}a\rangle\le\Phi(G)$, by Lemma 2.5(3), we get $\langle c^{1+l2^{n-2}}a\rangle\le Z(G)$. Since $n\ge 3$ and $[c,a]\in Z(G)$, by computation, we get $[c^{1+l2^{n-2}}a^2,a]=[c^{1+l2^{n-2}},a]=[c,a]=c^{s2^{n-1}}\ne 1$, a contradiction. Thus $s=0$.

If $l=1$, then $(ca^{2-2^{n-1}})^2=c^2a^{2^2-2^n}=1$. It follows that $\langle ca^{2-2^{n-1}}\rangle\cap G'=1$. Notice that $\langle ca^{2-2^{n-1}}\rangle\le\Phi(G)$, by Lemma 2.5(3), $\langle ca^{2-2^{n-1}}\rangle\le Z(G)$. Since $n\ge 3$ and $[c,a]=1$, by computation, we get

$$[ca^{2-2^{n-1}},b]=[c,b][a^{2-2^{n-1}},b]=c^{-2}[a^2,b][a^{-2^{n-1}},b]=c^{-2}[a,b]^2[a,b,a][a,b]^{-2^{n-1}}=c^{-2^{n-1}}\ne 1,$$

a contradiction. Thus $l=0$.

If $j=0$, then $G$ is one of groups III(1). If $j=1$, then $G$ is one of groups III(2).

**Case 5** $\overline{G}$ is one of groups III(2) of Theorem.

Let $\overline{G}=\langle \bar a,\bar b \mid \bar a^{2^n}=1,\bar b^2=\bar a^{2^{n-1}},[\bar a,\bar b]=\bar c,\bar c^2=\bar a^{-4},[\bar c,\bar a]=1,[\bar c,\bar b]=(\bar c)^{-2}\rangle$ and $N=\langle x\rangle$. Then

$$G=\langle a,b \mid a^{2^n}=x^i,b^2=a^{2^{n-1}}x^j,[a,b]=cx^k,c^2=a^{-4}x^l,[c,a]=x^s,[c,b]=c^{-2}x^t\rangle,$$

where $i,j,k,l,s,t\in\{0,1\}$.

By a similar argument as in case 4, we can get $G'=\langle c\rangle$ and $x=c^{2^{n-1}}=a^{2^n}$. Since $[a^2,b]=[a,b]^2[a,b,a]$ and $[a^2,b,a^2]=[c,a^2][x^s,a^2]=[c,a]^2=1$, noticing that $n\ge 3$, we can get

$$1=[b^2,b]=[a^{2^{n-1}}x,b]=[(a^2)^{2^{n-2}},b]=[a^2,b]^{2^{n-2}}=[a,b]^{2^{n-1}}=c^{2^{n-1}},$$

a contradiction. Thus $G/N$ is not one of groups III(2).

**Case 6** $\overline{G}$ is one of groups III(3) of Theorem.

Let $\overline{G} = \langle \bar{a}, \bar{b} \mid \bar{a}^{2^n} = 1, \bar{b}^4 = 1, [\bar{a}, \bar{b}] = \bar{c}, \bar{c}^2 = \bar{a}^{-4}, [\bar{c}, \bar{a}] = 1, [\bar{c}, \bar{b}] = (\bar{c})^{-2} \rangle$ and $N = \langle x \rangle$. Then

$$G = \langle a, b \mid a^{2^n} = x^i, b^4 = x^j, [a, b] = cx^k, c^2 = a^{-4}x^l, [c, a] = x^s, [c, b] = c^{-2}x^t \rangle,$$

where $i, j, k, l, s, t \in \{0, 1\}$.

We can assume $k = 0$ by letting $c_1 = cx^k$. By a similar argument as in Case 4, we can get $G' = \langle c \rangle$, and $x = c^{2^{n-1}} = a^{2^n}$ and $s = 0$.

If $j = 1$, that is $b^4 = x$, then $(b^2 c^{2^{n-2}})^2 = b^4 c^{2^{n-1}} [b^2, c^{2^{n-2}}] = x^2 = 1$. It follows that $\langle b^2 c^{2^{n-2}} \rangle \cap G' = 1$. Notice that $\langle b^2 c^{2^{n-2}} \rangle \leq \Phi(G)$, by Lemma 2.5(3), we get $\langle b^2 c^{2^{n-2}} \rangle \leq Z(G)$. By computation,

$$1 = [b^2 c^{2^{n-2}}, b] = [c, b]^{2^{n-2}} = c^{-2^{n-1}},$$

a contradiction. Thus $j = 0$, that is $b^4 = 1$.

Since $\langle b^2 \rangle \leq \Phi(G)$ and $\langle b^2 \rangle \cap G' = 1$, by Lemma 2.5(3), we get $\langle b^2 \rangle \leq Z(G)$. Thus $1 = [a, b^2] = [a, b]^2 [a, b, b] = c^2 [c, b] = c^2 c^{-2} x^t = x^t$. It follows that $t = 0$.

If $l = 1$, that is $c^2 = a^{-4} a^{2^n}$, then $(ca^{2^{n-1}+2})^2 = c^2 a^{2^n + 2^2} = 1$. It follows that $\langle ca^{2^{n-1}+2} \rangle \cap G' = 1$. Notice that $\langle ca^{2^{n-1}+2} \rangle \leq \Phi(G)$, by Lemma 2.5(3), we get $\langle ca^{2^{n-1}+2} \rangle \leq Z(G)$. Thus $[ca^{2^{n-1}+2}, a] = [ca^{2^{n-1}+2}, b] = 1$. By computation, we get

$$1 = [ca^{2^{n-1}+2}, a] = [c, a]$$

and so

$$1 = [ca^{2^{n-1}+2}, b] = [c, b][a^2, b]^{2^{n-2}+1} = c^{-2} x^t (c^2)^{2^{n-2}+1} = x^t c^{2^{n-1}}.$$

This implies that $t = 1$, a contradiction. Thus $l = 0$ and $G$ is one of groups III(3).

**Case 7** $\overline{G}$ is one of groups IV(1) of Theorem.

Let $\overline{G} = \langle \bar{a}, \bar{b} \mid \bar{a}^{2^n} = 1, \bar{b}^4 = 1, [\bar{a}, \bar{b}] = \bar{a}^{-2} \rangle$ and $N = \langle x \rangle$. Then

$$G = \langle a, b \mid a^{2^n} = x^i, b^4 = x^j, [a, b] = a^{-2} x^k \rangle, \quad \text{where} \quad i, j, k \in \{0, 1\}.$$

By Lemma 2.5(2), we get $G'$ is cyclic. Notice that $\overline{G}' = \langle \bar{a} \rangle$, then $G' = \langle a, x \rangle = \langle a \rangle$. It follows that $i = 1$, that is $x = a^{2^n}$.

If $j = 1$, then $(b^2 a^{-2^{n-1}})^2 = b^4 a^{-2^n} = 1$. It follows that $\langle b^2 a^{-2^{n-1}} \rangle \cap G' = 1$. Notice that $\langle b^2 a^{-2^{n-1}} \rangle \leq \Phi(G)$, by Lemma 2.5(3), we get $\langle b^2 a^{-2^{n-1}} \rangle \leq Z(G)$. So $1 = [b^2 a^{-2^{n-1}}, b] = [a^{-2^{n-1}}, b] = [a, b]^{-2^{n-1}} = a^{2^n} \neq 1$, a contradiction. Thus $j = 0$.

If $k = 0$, then $G$ is one of groups IV(1). If $k = 1$, then $G$ is one of groups IV(2).

**Case 8** $\overline{G}$ is one of groups IV(2) of Theorem.

Let $\overline{G} = \langle \bar{a}, \bar{b} \mid \bar{a}^{2^n} = 1, \bar{b}^4 = 1, [\bar{a}, \bar{b}] = \bar{a}^{-2+2^{n-1}} \rangle$ and $N = \langle x \rangle$. Then

$$G = \langle a, b \mid a^{2^n} = x^i, b^4 = x^j, [a, b] = a^{-2+2^{n-1}} x^k \rangle, \quad \text{where} \quad i, j, k \in \{0, 1\}.$$

By a similar argument as in Case 7, we can get $x = a^{2^n}$ and $b^4 = 1$. It follows that $\langle b^2\rangle \cap G' = 1$. Notice that $\langle b^2\rangle \le \Phi(G)$, by Lemma 2.5(3), we get $\langle b^2\rangle \le Z(G)$. Since $n \ge 3$, by computation, we get

$$[a, b^2] = [a, b]^2[a, b, b] = [a, b]^2[a^{-2+2^{n-1}}, b] = [a, b]^2[a, b]^{-2+2^{n-1}} = [a, b]^{2^{n-1}} = a^{2^n} \ne 1,$$

a contradiction too. Thus $G/N$ is not one of groups IV(2).

Conversely, if $G$ is one of the groups in the Theorem, we can get easily $G$ is a $\mathcal{C}_c$-group by Lemmas 2.6 and 2.4(3).

The proof is completed. □

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