Journal of Mathematical Research with Applications
Sept., 2015, Vol. 35, No. 5, pp. 568–580
DOI:10.3770/j.issn:2095-2651.2015.05.010
Http://jmre.dlut.edu.cn

A Remark on Weighted Cubic Variation of Subfractional Brownian Motion with $H < 1/6$

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Abstract In this paper, we prove by means of Malliavin calculus the convergence in L^2 of some properly renormalized weighted cubic variation of sub-fractional Brownian motion S^H with parameter $H < \frac{1}{6}$.

Keywords Subfractional Brownian motion; Malliavin calculus; weighted cubic variation

MR(2010) Subject Classification 60G15; 60H05; 60H07

1. Introduction

The study of single path behavior of stochastic process is often based on the study of its power variations. A quite extensive literature have been developed on this subject, see e.g. Corcuera et al. [1], Gradinaru et al. [2] for references concerning the power variations of Gaussian and Gaussian-related processes, and Barndorff-Nielsen et al. [3] (and the references therein) for applications of power variation techniques to the continuous time modeling of financial markets. Recall that, a real number $p > 1$ being given, the p-power variation of a process X, with respect to a subdivision $\pi_n = \{0 = t_{n,0} < t_{n,1} < \cdots < t_{n,n} = 1\}$ of $[0, 1]$, is defined to be the sum

$$\sum_{k=0}^{n-1} \left| X_{t_{n,k+1}} - X_{t_{n,k}} \right|^p . \tag{1}$$

For simplicity, consider from now on the case where $t_{n,k} = k/n$, for $n \in N^*$ and $k \in \{0, 1, 2, \dots, n\}$. When weights are introduced in (1), some interesting phenomenon appears. More precisely, consider quantities such as

$$\sum_{k=0}^{n-1} f\left(X_{k/n}\right) \left(X_{(k+1)/n} - X_{k/n}\right)^p , \tag{2}$$

where the function $f : \mathbb{R} \to \mathbb{R}$ is assumed to be smooth enough. Notice that (2) is called weighted power variations because of the presence of the factor $f\left(X_{k/n}\right)$.

Before dwelling on sub-fractional Brownian motion (see Section 2 for precise definition), let us recall some recent results concerning (2) when $X = B^H$, the fractional Brownian motion with

Received October 12, 2014; Accepted November 24, 2014
Supported by the National Natural Science Foundation of China (Grant No. 11401313), China Postdoctoral Science Foundation (Grant Nos. 2014M560368; 2015T80475), the Natural Science Foundation of Jiangsu Educational Committee (Grant No. 14KJB110013), 2014 Qing Lan Project, Jiangsu Planned Projects for Postdoctoral Research Funds (Grant No. 1401011C), 2013 Jiangsu Government Scholarship for Overseas Studies and Priority Academic Program Development of Jiangsu Higher Education Institutions.
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Hurst index $H \in (0,1)$ and when $p \geq 2$ is an integer, the asymptotic behavior of (2) received a lot of attentions (see Gradinaru and Nourdin [2] and Neuenkirich and Nourdin [4]). The analysis of the asymptotic behavior of quantities of type (2) is motivated, for instance, by the study of the exact rates of convergence of some approximation schemes of scalar stochastic differential equations driven by B^H (see [2], [4] and references therein for precise statements), besides, of course, the traditional applications of quadratic variations to parameter estimation problems. However, it turned out that it was also interesting itself because it highlighted new phenomena with respect to some classical results obtained in the seminal works by Breuer and Major [5], Dobrushin and Major [6], Giraitis and Surgailis [7] or Taqqu [8]. Indeed, we know that, for any $0 < H < 3/4$, the convergence

$$\frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \left[n^{2H} \left(B^H_{(l+1)/n} - B^H_{l/n} \right)^2 - 1 \right] \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0, \sigma_H^2), \tag{3}$$

holds, where σ_H denotes a constant depending only H which can be computed explicitly.

Nourdin [9] began the study of asymptotic analysis of (2) with B^H instead of X. If $H < 1/4$, he proved that

$$n^{2H-1} \sum_{l=0}^{n-1} f\left(B^H_{l/n}\right) \left[n^{2H} \left(B^H_{(l+1)/n} - B^H_{l/n} \right)^2 - 1 \right] \xrightarrow[n\to\infty]{L^2} \frac{1}{4} \int_0^1 f''(B^H_s) \mathrm{d}s, \tag{4}$$

and if $H < 1/6$,

$$n^{3H-1} \sum_{l=0}^{n-1} \left[f\left(B^H_{l/n}\right) n^{3H} \left(B^H_{(l+1)/n} - B^H_{l/n} \right)^3 + \frac{3}{2} f\left(B^H_{l/n}\right) n^{-H} \right] \xrightarrow[n\to\infty]{L^2} -\frac{1}{8} \int_0^1 f'''(B^H_s) \mathrm{d}s. \tag{5}$$

As pointed out by Nourdin [9], (4) is somewhat surprising when compared with (3). Indeed, instead of an L^2-convergence, we only have a convergence in law in (3). Observe that, since $2H - 1 < -1/2$ if and only if $H < 1/4$, convergence (3) and (4) are, of course, not contradictory. Let us also stress that the study in Nourdin [9] and Nourdin et al. [10] and [11] has been used in Gradinaru and Nourdin [2] and Neuenkirich and Nourdin [4] to deduce the exact rate of convergence of some approximation schemes of scalar stochastic differential equations driven by fractional Brownian motion. Nourdin et al. [11] proved some central and non-central limit theorem for the (renormalized) weighted power variation of order $q \geq 2$ of fractional Brownian motion with Hurst parameter $H \in (0,1)$, where q is an integer. Moreover, Belfadli [12] extended Nourdin [9] to a more general self-similar Gaussian process, namely the bi-fractional Brownian motion.

Motivated by all these results, we will show in the present note that the convergence (5) still holds for another self-similar Gaussian process, namely the subfractional Brownian motion (see below for a precise definition), which has been proposed by Bojdecki et al. [13]. The sub-fractional Brownian motion has properties analogous to those of fractional Brownian motion (self-similarity, long-range dependence, Hölder paths, the variation and the renormalized variation). However, in comparison with fractional Brownian motion, the sub-fractional Brownian motion has non-stationary increments and the increments over non-overlapping intervals are more weakly correlated and their covariance decays polynomially as a higher rate in comparison with fractional Brownian motion (for this reason in Bojdecki et al. [13] is called sub-fractional Brownian motion). Therefore, it seems interesting to study the weighted cubic variation of sub-fractional Brownian motion. And we need more precise estimates to prove our results because of the non-stationary increments. As in Nourdin [9], our main tool for the proof is based on the integration by parts formula of Malliavin calculus.

This paper is organized as follows. Section 2 contains some preliminaries for sub-fractional Brownian motion. In Section 3, we state and prove our main result convergence similar to (5), but in the case where B^H is replaced by the sub-fractional Brownian motion S^H.

2. Preliminaries and notations

We begin by briefly recalling some basic facts about stochastic calculus with respect to a sub-fractional Brownian motion. Let $S^H = \{S_t^H, t \in [0,1]\}$ be a sub-fractional Brownian motion with parameter $H \in (0,1)$ defined on a completed probability space $(\Omega, \mathscr{F}, P)$. It means that S^H is a centered Gaussian process with the covariance function given by

$$C_H(s,t) \equiv E\left[S_t^H S_s^H\right] = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right]. \tag{6}$$

For $H = 1/2$, S^H coincides with the standard Brownian motion B. S^H being neither a semi-martingale nor a Markov process unless $H = 1/2$, many of the powerful techniques from stochastic analysis are not available when dealing with S^H. The sub-fBm has properties analogous to those of fractional Brownian motion (self-similarity, long-range dependence, Hölder paths), and, for $s \le t$, satisfies the following estimates:

$$[1 \wedge (2 - 2^{2H-1})](t-s)^{2H} \le E\left[\left(S_t^H - S_s^H\right)^2\right] \le [1 \vee (2 - 2^{2H-1})](t-s)^{2H}. \tag{7}$$

So its increments are not stationary. Further works for sub-fractional Brownian motion can be found in Bojdecki et al. [13–16], Dzhaparidze and Zanten [17], Liu-Yan [18], Shen-Yan [19], Tudor [20–23] and Yan et al. [24,25].

We denote by $\mathscr{E}$ the set of step $\mathbb{R}$-valued functions on $[0,1]$. Let $\mathscr{H}$ be the Hilbert space defined as the closure of $\mathscr{E}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathscr{H}} = C_H(s,t). \tag{8}$$

We denote by $\|\cdot\|_{\mathscr{H}}$ the associated norm. The mapping $1_{[0,t]} \to S_t^H$ can be extended to an isometry between $\mathscr{H}$ and the Gaussian space $\mathfrak{H}_1(S^H)$ associated with S^H. We denote this isometry by $\xi \to S^H(\xi)$.

Let $\mathscr{S}$ be the set of smooth cylindrical functionals of the form

$$F = h\left(S^H(\xi_1), \ldots, S^H(\xi_n)\right),$$

where $n \ge 1$ and $h \in C_b^\infty(\mathbb{R}^n)$ and $\xi_i \in \mathscr{H}$. The Malliavin derivative of a functional F defined as above is given by

$$DF = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}\left(S^H(\xi_1), \ldots, S^H(\xi_n)\right)\xi_i$$

and this operator can be extended to the closure $\mathbb{D}^{m,2}(m \geq 1)$ of $\mathscr{S}$ with respect to the norm

$$\|F\|_{m,2}^2 = E|F|^2 + E\|DF\|_{\mathscr{H}}^2 + \cdots + E\|D^m F\|_{\mathscr{H}^{\odot m}}^2$$

where $\mathscr{H}^{\odot m}$ denotes the m fold symmetric tensor product of $\mathscr{H}$ and the m-th derivative D^m is defined by iteration. The Malliavin derivative satisfies the following chain rule. For every random variable $F = (F_1, F_2, \ldots, F_n)$ with components in $\mathbb{D}^{1,2}$ and for every continuously differentiable function $\mu : \mathbb{R}^n \to \mathbb{R}$ with bounded partial derivative, we obtain $\mu(F_1, \ldots, F_n) \in D^{1,2}$ and for any $s \in [0, 1]$:

$$D_s \mu(F_1, \ldots, F_n) = \sum_{i=1}^{n} \frac{\partial \mu}{\partial x_i}(F_1, \ldots, F_n) D_s F_i.$$

The divergence integral I is the adjoint operator of D. Concretely, a random variable $u \in L^2(\Omega, \mathscr{H})$ belongs to the domain of the divergence operator I (in symbol $\mathrm{Dom}(I)$) if

$$E|\langle DF, u \rangle_{\mathscr{H}}| \leq c\|F\|_{L^2(\Omega)}$$

for every $F \in \mathscr{S}$. In this case $I(u)$ is given by the duality relationship

$$E(FI(u)) = E\langle DF, u \rangle_{\mathscr{H}}$$

for any $F \in \mathbb{D}^{1,2}$, and we have the following integration by parts:

$$FI(u) = I(Fu) + \langle DF, u \rangle_{\mathscr{H}} \tag{9}$$

for any $u \in \mathrm{Dom}(I)$, $F \in \mathbb{D}^{1,2}$ such that $Fu \in L^2(\Omega, \mathscr{H})$. Moreover, one can see Nualart [26] and references therein for more details about the Malliavin calculus.

3. Main results and proofs

In this section, we assume that $H \in (0, 1/6)$. For simplicity, we denote

$$\Delta S_{k/n}^H = S_{(k+1)/n}^H - S_{k/n}^H, \quad \delta_{k/n} = 1_{[k/n,(k+1)/n]} \quad \text{and} \quad \varepsilon_{k/n} = 1_{[0,k/n]}.$$

Also C will denote a generic constant independent of k, l, n that can be different from line to line.

We will make use of the following assumption on the weight function f.

Assumption ($\mathrm{H_m}$): $f : \mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{C}^m$ and, for any $p > 0$ and any $i = 1, 2, \ldots, m$,

$$\sup_{s \in [0,1]} E[|f^{(i)}(S_s^H)|^p] < \infty. \tag{10}$$

The main result of this note is the following

Theorem 3.1 *Let S^H be a sub-fractional Brownian motion with parameter H such that $0 < H < \frac{1}{6}$, and let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying (H_6). Then we have,*

$$n^{3H-1} \sum_{k=0}^{n-1} \left[f\left(S_{k/n}^H\right) n^{3H} \left(\Delta S_{k/n}^H\right)^3 + \frac{3}{2} f'(S_{k/n}^H) n^{-H} \right] \xrightarrow[n\to\infty]{L^2} -\frac{1}{8} \int_0^1 f'''\left(S_s^H\right) \mathrm{d}s. \tag{11}$$

We will need several lemmas. The proof of the first two lemmas being immediate to check, the details are left to the readers.

Lemma 3.2 *For any $x \geq 0$, we have $0 \leq (x+1)^{2H} - x^{2H} \leq 1$.*

Lemma 3.3 *(1) If $2H < 1$, then the sequence φ defined by*

$$\varphi(k) := 2(k+1)^{2H} + (2^{2H} - 2)k^{2H} - (2k+1)^{2H},$$

satisfies

$$\varphi(k) \sim Ck^{2H-1}, \quad k \to \infty.$$

In particular, φ is bounded.

(2) If $2H < 1$, then the sequence ϕ defined by

$$\phi(k) := (2k+1)^{2H} - 2^{2H-1}(k+1)^{2H} - 2^{2H-1}k^{2H},$$

satisfies

$$\phi(k) \sim Ck^{2H-2}, \quad k \to \infty.$$

In particular, $\sum_{k\geq 0} |\phi(k)| < \infty$.

Lemma 3.4 *(1) If $0 < H < 1/2$, then as n tends to infinity,*

$$\sum_{k,l=0}^{n-1} \left|\langle \varepsilon_{l/n}, \delta_{k/n}\rangle_{\mathscr{H}}\right| = o\left(n^{2-2H}\right).$$

(2) If $0 < H < 1/6$, for $k, l = 0, 1, \ldots, n-1$, then as n tends to infinity,

$$\sum_{k,l=0}^{n-1} \left|\langle \delta_{k/n}, \delta_{l/n}\rangle_{\mathscr{H}}\right| = o\left(n^{2-6H}\right).$$

Proof The first point has been proved in Liu-Yan [18]. We only prove the second point. For $0 \leq k, l \leq n-1$, we have

$$\begin{aligned}\langle \delta_{k/n}, \delta_{l/n}\rangle_{\mathscr{H}} =& \frac{1}{2}n^{-2H}[2(k+l+1)^{2H} - (k+l+2)^{2H} - (k+l)^{2H} + \\ & |k-l-1|^{2H} + |k-l+1|^{2H} - 2|k-l|^{2H}],\end{aligned}$$

and therefore

$$\begin{aligned}\sum_{k,l=0}^{n-1} \left|\langle \delta_{k/n}, \delta_{l/n}\rangle_{\mathscr{H}}\right| \leq& \frac{1}{2}n^{-2H} \sum_{k,l=0}^{n-1} \big[|2(k+l+1)^{2H} - (k+l+2)^{2H} - (k+l)^{2H}| + \\ & ||k-l-1|^{2H} + |k-l+1|^{2H} - 2|k-l|^{2H}|\big] \\ \leq& Cn = o(n^{2-6H}), \quad \text{since} \quad H < 1/6. \quad \square\end{aligned}$$

Lemma 3.5 *If $H < \frac{1}{2}$, f and g are two functions satisfying the condition (H_3), then*

$$\begin{aligned}& n^{3H} \sum_{k,l=0}^{n-1} E\Big\{ f\left(S^H_{k/n}\right) g\left(S^H_{l/n}\right) \left(\Delta S^H_{k/n}\right)^3 \Big\} \\ &= -\frac{3}{2}n^{-H} \sum_{k,l=0}^{n-1} E\left\{ f'\left(S^H_{k/n}\right) g\left(S^H_{l/n}\right)\right\} -\end{aligned}$$

$$\frac{1}{8}n^{-3H}\sum_{k,l=0}^{n-1}E\left\{f'''\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\right\}+o\left(n^{2-3H}\right).$$

Proof For $0\le k,l\le n-1$, we use the integration by parts formula to have

$$\begin{aligned}
&n^{3H}E\Big\{f\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^3\Big\}\\
&=n^{3H}E\Big\{f'''\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\langle\varepsilon_{k/n},\delta_{k/n}\rangle^3_{\mathscr{H}}+\\
&\quad 3f''\left(S^H_{k/n}\right)g'\left(S^H_{l/n}\right)\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathscr{H}}\langle\varepsilon_{k/n},\delta_{k/n}\rangle^2_{\mathscr{H}}+\\
&\quad 3f'\left(S^H_{k/n}\right)g''\left(S^H_{l/n}\right)\langle\varepsilon_{l/n},\delta_{k/n}\rangle^2_{\mathscr{H}}\langle\varepsilon_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}+\\
&\quad 3f'\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\langle\varepsilon_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}\langle\delta_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}+\\
&\quad f\left(S^H_{k/n}\right)g'''\left(S^H_{l/n}\right)\langle\varepsilon_{l/n},\delta_{k/n}\rangle^3_{\mathscr{H}}+3f\left(S^H_{k/n}\right)g'\left(S^H_{l/n}\right)\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathscr{H}}\langle\delta_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}\Big\}\\
&\equiv\sum_{i=1}^{6}A^i_{k,l,n}.
\end{aligned}\tag{12}$$

We claim that $\sum_{i=1}^6 A^i_{k,l,n}=o\left(n^{2-3H}\right)$ for $i=2,3,5,6$. Let us first consider the cases where $i=2$ and $i=6$. It is easy to check that

$$\langle\varepsilon_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}=\frac{1}{2n^{2H}}\left(2(k+1)^{2H}+(2^{2H}-2)k^{2H}-(2k+1)^{2H}-1\right),\tag{13}$$

and

$$\begin{aligned}\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathscr{H}}=&\frac{1}{n^{2H}}\Big[(k+1)^{2H}-k^{2H}+\frac{1}{2}(k+l)^{2H}-\frac{1}{2}(k+l+1)^{2H}+\\&\frac{1}{2}(|k-l|^{2H}-|k-l+1|^{2H})\Big].\end{aligned}\tag{14}$$

Using Lemma 3.2, (13) and (14), we have

$$|\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathscr{H}}|\le Cn^{-2H}\left[(k+1)^{2H}-k^{2H}+||k-l|^{2H}-|k-l+1|^{2H}|\right],\tag{15}$$

and

$$\left|\langle\varepsilon_{k/n},\delta_{k/n}\rangle^2_{\mathscr{H}}\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathscr{H}}\right|\le Cn^{-6H}\left[(k+1)^{2H}-k^{2H}+||k-l|^{2H}-|k-l+1|^{2H}|\right].\tag{16}$$

This yields, under (H_3),

$$\sum_{k,l=0}^{n-1}|A^2_{k,l,n}|\le Cn^{1-H}=o\left(n^{2-3H}\right),\quad\text{since}\quad H<1/6,$$

$$\sum_{k,l=0}^{n-1}|A^6_{k,l,n}|\le Cn^{1+H}=o\left(n^{2-3H}\right),\quad\text{since}\quad H<1/6.$$

Similarly, we prove that $\sum_{k,l=0}^{n-1}|A^i_{k,l,n}|=o\left(n^{2-3H}\right)$ for $i=3$ and 5. It remains to consider the terms $A^1_{k,l,n}$ and $A^4_{k,l,n}$. From Lemma 3.2, (13) and (14), we deduce

$$\left|\langle\varepsilon_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}+\frac{1}{2n^{2H}}\right|\le\frac{1}{2}n^{-2H}\left[2|(k+1)^{2H}-k^{2H}|+|(2k)^{2H}-(2k+1)^{2H}|\right],\tag{17}$$

and

$$\left|\langle\varepsilon_{k/n},\delta_{k/n}\rangle^3_{\mathscr{H}}+\frac{1}{8n^{6H}}\right|\le Cn^{-6H}\left[2|(k+1)^{2H}-k^{2H}|+|(2k)^{2H}-(2k+1)^{2H}|\right]. \tag{18}$$

Thus, since $H<1/6$,

$$n^{3H}\sum_{k,l=0}^{n-1}\left|\langle\varepsilon_{k/n},\delta_{k/n}\rangle^3_{\mathscr{H}}+\frac{1}{8n^{6H}}\right|\le Cn^{1-H}=o\left(n^{2-3H}\right). \tag{19}$$

Moreover

$$\begin{aligned}\left|\langle\varepsilon_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}\langle\delta_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}+\frac{1}{2n^{2H}}\right|&\le C|\langle\varepsilon_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}+\frac{1}{2n^{2H}}|\\ &\le Cn^{-2H}\left[2|(k+1)^{2H}-k^{2H}|+|(2k)^{2H}-(2k+1)^{2H}|\right].\end{aligned} \tag{20}$$

Thus, for $H<1/6$,

$$n^{3H}\sum_{k,l=0}^{n-1}\left|\langle\varepsilon_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}\langle\delta_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}+\frac{1}{2n^{2H}}\right|\le Cn^{1+H}=o\left(n^{2-3H}\right). \tag{21}$$

This yields, under (H_3),

$$\sum_{k,l=0}^{n-1}|A^4_{k,l,n}|=-\frac{3}{2}n^{-H}\sum_{k,l=0}^{n-1}E\left\{f'\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\right\}+o\left(n^{2-3H}\right),$$

$$\sum_{k,l=0}^{n-1}|A^1_{k,l,n}|=-\frac{1}{8}n^{-3H}\sum_{k,l=0}^{n-1}E\left\{f'''\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\right\}+o\left(n^{2-3H}\right).$$

Thus the proof of Lemma 3.5 is completed. □

Lemma 3.6 *If $0<H<1/6$, and f,g are two functions satisfying the condition (H_6), then*

$$\begin{aligned}&n^{6H}\sum_{k,l=0}^{n-1}E\Big\{f\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^3\left(\Delta S^H_{l/n}\right)^3\Big\}\\ &=\frac{9}{4n^{2H}}\sum_{k,l=0}^{n-1}E\left\{f'\left(S^H_{k/n}\right)g'\left(S^H_{l/n}\right)\right\}+\frac{3}{16n^{4H}}\sum_{k,l=0}^{n-1}E\left\{f'\left(S^H_{k/n}\right)g'''\left(S^H_{l/n}\right)\right\}+\\ &\quad\frac{3}{16n^{4H}}\sum_{k,l=0}^{n-1}E\left\{f'''\left(S^H_{k/n}\right)g'\left(S^H_{l/n}\right)\right\}+\frac{1}{64n^{6H}}\sum_{k,l=0}^{n-1}E\left\{f'''\left(S^H_{k/n}\right)g'''\left(S^H_{l/n}\right)\right\}+\\ &\quad o\left(n^{2-6H}\right).\end{aligned} \tag{22}$$

Proof Using the integration by parts formula, for $0\le k,l\le n-1$, we can write

$$\begin{aligned}&n^{6H}E\Big\{f\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^3\left(\Delta S^H_{l/n}\right)^3\Big\}\\ &=n^{6H}E\Big\{f'''\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^3\langle\varepsilon_{k/n},\delta_{l/n}\rangle^3_{\mathscr{H}}+\\ &\quad 3f''\left(S^H_{k/n}\right)g'\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^3\langle\varepsilon_{k/n},\delta_{l/n}\rangle^2_{\mathscr{H}}\langle\varepsilon_{l/n},\delta_{l/n}\rangle_{\mathscr{H}}+\\ &\quad 9f''\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^2\langle\varepsilon_{k/n},\delta_{l/n}\rangle^2_{\mathscr{H}}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}+\end{aligned}$$

$$
\begin{aligned}
&3f'\left(S^H_{k/n}\right)g''\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^3\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}\langle\varepsilon_{l/n},\delta_{l/n}\rangle^2_{\mathscr{H}}+\\
&12f'\left(S^H_{k/n}\right)g'\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^2\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}\langle\varepsilon_{l/n},\delta_{l/n}\rangle_{\mathscr{H}}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}+\\
&12f'\left(S^H_{k/n}\right)g'\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}\langle\delta_{k/n},\delta_{l/n}\rangle^2_{\mathscr{H}}+\\
&3f'\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^3\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}\langle\delta_{l/n},\delta_{l/n}\rangle_{\mathscr{H}}+\\
&f\left(S^H_{k/n}\right)g'''\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^3\langle\varepsilon_{l/n},\delta_{l/n}\rangle^3_{\mathscr{H}}+\\
&9f\left(S^H_{k/n}\right)g''\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^2\langle\varepsilon_{l/n},\delta_{l/n}\rangle^2_{\mathscr{H}}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}+\\
&18f\left(S^H_{k/n}\right)g'\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)\langle\varepsilon_{l/n},\delta_{l/n}\rangle_{\mathscr{H}}\langle\delta_{k/n},\delta_{l/n}\rangle^2_{\mathscr{H}}+\\
&3f\left(S^H_{k/n}\right)g'\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^3\langle\varepsilon_{l/n},\delta_{l/n}\rangle_{\mathscr{H}}\langle\delta_{l/n},\delta_{l/n}\rangle_{\mathscr{H}}+\\
&6f'\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}\langle\delta_{k/n},\delta_{l/n}\rangle^2_{\mathscr{H}}+\\
&6f\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^2\langle\delta_{k/n},\delta_{l/n}\rangle^3_{\mathscr{H}}+\\
&9f\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^2\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}\langle\delta_{l/n},\delta_{l/n}\rangle_{\mathscr{H}}\Big\}\equiv\sum_{i=1}^{14}B^i_{k,l,n}. \qquad (23)
\end{aligned}
$$

To obtain Lemma 3.6, we develop the right-hand side of the previous identity in the same way as for the obtention of (12) in the proof of Lemma 3.5. Then, only the terms containing $\langle\varepsilon_{k/n},\delta_{k/n}\rangle^\alpha_{\mathscr{H}}\langle\varepsilon_{l/n},\delta_{l/n}\rangle^\beta_{\mathscr{H}}$, for $\alpha,\beta\geq 1$, have a contribution to (23), as we can check by using (13), (14), (17) and (18). The other terms are $o\left(n^{2-6H}\right)$.

Consequently, the proof of this lemma will be deduced after the study of the asymptotic behavior of $\sum_{i=1}^{14}B^i_{k,l,n}$, as $n\to\infty$, for each $i\in\{1,2,\ldots,14\}$.

Lemma 3.7 *We have, as n tends to infinity*

(1)

$$\sum_{k,l=0}^{n-1}B^i_{k,l,n}=o\left(n^{2-6H}\right),\quad \textit{for every}\ \ i\neq 8,\ i\neq 11;$$

(2)

$$\sum_{k,l=0}^{n-1}B^8_{k,l,n}=\frac{3}{16n^{4H}}\sum_{k,l=0}^{n-1}E\left\{f'\left(S^H_{k/n}\right)g'''\left(S^H_{l/n}\right)\right\}+\frac{1}{64n^{6H}}\sum_{k,l=0}^{n-1}E\left\{f'''\left(S^H_{k/n}\right)g'''\left(S^H_{l/n}\right)\right\}+o\left(n^{2-6H}\right).$$

(3)

$$\sum_{k,l=0}^{n-1}B^{11}_{k,l,n}=\frac{9}{4n^{2H}}\sum_{k,l=0}^{n-1}E\left\{f'\left(S^H_{k/n}\right)g'\left(S^H_{l/n}\right)\right\}+$$

$$\frac{3}{16n^{4H}}\sum_{k,l=0}^{n-1}E\left\{f'''\left(S^H_{k/n}\right)g'\left(S^H_{l/n}\right)\right\}+o\left(n^{2-6H}\right).$$

Proof We first consider the term $B^1_{k,l,n}$, the study of $B^2_{k,l,n}, B^4_{k,l,n}, B^7_{k,l,n}$ being similar. Using Malliavin integration by parts formula, we can write

$$\begin{aligned}B^1_{k,l,n}=&n^{6H}E\Big\{f'''\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^2 I(\delta_{k/n})\langle\varepsilon_{k/n},\delta_{l/n}\rangle^3_{\mathscr{H}}\Big\}\\=&n^{6H}E\Big\{f^{(6)}\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\langle\varepsilon_{k/n},\delta_{k/n}\rangle^3_{\mathscr{H}}\langle\varepsilon_{k/n},\delta_{l/n}\rangle^3_{\mathscr{H}}+\\&3f^{(5)}\left(S^H_{k/n}\right)g'\left(S^H_{l/n}\right)\langle\varepsilon_{k/n},\delta_{k/n}\rangle^2_{\mathscr{H}}\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathscr{H}}\langle\varepsilon_{k/n},\delta_{l/n}\rangle^3_{\mathscr{H}}+\\&3f^{(4)}\left(S^H_{k/n}\right)g''\left(S^H_{l/n}\right)\langle\varepsilon_{l/n},\delta_{k/n}\rangle^2_{\mathscr{H}}\langle\varepsilon_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}\langle\varepsilon_{k/n},\delta_{l/n}\rangle^3_{\mathscr{H}}+\\&3f^{(4)}\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\langle\varepsilon_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}\langle\delta_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}\langle\varepsilon_{k/n},\delta_{l/n}\rangle^3_{\mathscr{H}}+\\&f'''\left(S^H_{k/n}\right)g'''\left(S^H_{l/n}\right)\langle\varepsilon_{l/n},\delta_{k/n}\rangle^3_{\mathscr{H}}\langle\varepsilon_{k/n},\delta_{l/n}\rangle^3_{\mathscr{H}}+\\&3f'''\left(S^H_{k/n}\right)g'\left(S^H_{l/n}\right)\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathscr{H}}\langle\delta_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}\langle\varepsilon_{k/n},\delta_{l/n}\rangle^3_{\mathscr{H}}\Big\}.\end{aligned}\tag{24}$$

Using that $n^{2H}\left|\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}\right|$ and $\phi(k)$ are bounded with respect to k,l,n, and using the condition (H_6), we have

$$\left|B^1_{k,l,n}\right|\le C\cdot n^{-4H}\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}.$$

According to Lemma 3.4, we deduce

$$\sum_{k,l=0}^{n-1}\left|B^1_{k,l,n}\right|=o\left(n^{2-6H}\right).$$

Now, let us consider the term $B^3_{k,l,n}$. The study of the cases $B^i_{k,l,n}$, $i=5,6,12$, is similar because each of these terms contains the factor $\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}$.

As previously, by Malliavin integration by parts formula, we can write

$$\begin{aligned}B^3_{k,l,n}=&9n^{6H}E\left\{f''\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\Delta\left(S^H_{k/n}\right)I(\delta_{k/n})\langle\varepsilon_{k/n},\delta_{l/n}\rangle^2_{\mathscr{H}}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}\right\}\\=&9n^{6H}E\Big\{f^{(4)}\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\langle\varepsilon_{k/n},\delta_{k/n}\rangle^2_{\mathscr{H}}\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}+\\&2f'''\left(S^H_{k/n}\right)g'\left(S^H_{l/n}\right)\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathscr{H}}\langle\varepsilon_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}+\\&f''\left(S^H_{k/n}\right)g''\left(S^H_{l/n}\right)\langle\varepsilon_{l/n},\delta_{k/n}\rangle^2_{\mathscr{H}}\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}+\\&f''\left(S^H_{k/n}\right)g\left(S^H_{l/n}\right)\langle\delta_{k/n},\delta_{k/n}\rangle_{\mathscr{H}}\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}\Big\}.\end{aligned}\tag{25}$$

Hence, using again that $n^{2H}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}$ is bounded and the condition (H_4), we obtain

$$\sum_{k,l=0}^{n-1}\left|B^3_{k,l,n}\right|\le\sum_{k,l=0}^{n-1}\left|\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathscr{H}}\right|,$$

which is $o\left(n^{2-6H}\right)$ by using the point (2) of Lemma 3.4.

For the term $B^8_{k,l,n}$, we use (18) and the point (1) of Lemma 3.3 to write

$$\langle\varepsilon_{k/n},\delta_{k/n}\rangle^3_{\mathscr{H}}=-\frac{1}{8n^{6H}}+o\Big(\frac{1}{n^{6H}}\Big).\tag{26}$$

Substituting into the expression of $B^8_{k,l,n}$, yields

$$\sum_{k,l=0}^{n-1} B^8_{k,l,n} = \sum_{k,l=0}^{n-1} n^{6H} E\Big\{ f\left(S^H_{k/n}\right) g'''\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^3 \langle \varepsilon_{l/n}, \delta_{l/n}\rangle^3_{\mathscr{H}} \Big\}$$

$$= -\frac{1}{8n^{3H}} \sum_{k,l=0}^{n-1} E\Big\{ f\left(S^H_{k/n}\right) g'''\left(S^H_{l/n}\right)\left(\Delta S^H_{k/n}\right)^3 \Big\} + o\Big(\frac{1}{n^{3H}}\Big). \tag{27}$$

Therefore, using Lemma 3.5, with g''' instead of g, we obtain

$$\sum_{k,l=0}^{n-1} B^8_{k,l,n} = \frac{3}{16n^{4H}} \sum_{k,l=0}^{n-1} E\left\{ f'\left(S^H_{k/n}\right) g'''\left(S^H_{l/n}\right)\right\} +$$

$$\frac{1}{64n^{6H}} \sum_{k,l=0}^{n-1} E\left\{ f'''\left(S^H_{k/n}\right) g'''\left(S^H_{l/n}\right)\right\} + o\left(n^{2-6H}\right).$$

Now, let us consider the term $B^9_{k,l,n}$. The study of the cases $B^i_{k,l,n}$, $i = 10, 13, 14$, are similar because each of these terms contains the factor $\langle \delta_{k/n}, \delta_{l/n}\rangle_{\mathscr{H}}$. As previously, by Malliavin integration by parts formula, we can write

$$\begin{aligned}
B^9_{k,l,n} =& 9n^{6H} E\Big\{ f\left(S^H_{k/n}\right) g''\left(S^H_{l/n}\right) \Delta\left(S^H_{k/n}\right) I(\delta_{k/n}) \langle \varepsilon_{l/n}, \delta_{l/n}\rangle^2_{\mathscr{H}} \langle \delta_{k/n}, \delta_{l/n}\rangle_{\mathscr{H}} \Big\} \\
=& 9n^{6H} E\Big\{ f''\left(S^H_{k/n}\right) g''\left(S^H_{l/n}\right) \langle \varepsilon_{k/n}, \delta_{k/n}\rangle^2_{\mathscr{H}} \langle \varepsilon_{l/n}, \delta_{l/n}\rangle^2_{\mathscr{H}} \langle \delta_{k/n}, \delta_{l/n}\rangle_{\mathscr{H}} + \\
& 2f'\left(S^H_{k/n}\right) g'''\left(S^H_{l/n}\right) \langle \varepsilon_{l/n}, \delta_{k/n}\rangle_{\mathscr{H}} \langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathscr{H}} \langle \varepsilon_{l/n}, \delta_{l/n}\rangle^2_{\mathscr{H}} \langle \delta_{k/n}, \delta_{l/n}\rangle_{\mathscr{H}} + \\
& f\left(S^H_{k/n}\right) g^{(4)}\left(S^H_{l/n}\right) \langle \varepsilon_{l/n}, \delta_{k/n}\rangle^2_{\mathscr{H}} \langle \delta_{l/n}, \delta_{l/n}\rangle^2_{\mathscr{H}} \langle \delta_{k/n}, \delta_{l/n}\rangle_{\mathscr{H}} + \\
& f\left(S^H_{k/n}\right) g''\left(S^H_{l/n}\right) \langle \delta_{k/n}, \delta_{k/n}\rangle_{\mathscr{H}} \langle \varepsilon_{l/n}, \delta_{l/n}\rangle^2_{\mathscr{H}} \langle \delta_{k/n}, \delta_{l/n}\rangle_{\mathscr{H}} \Big\} \\
:=& (a)_{k,l,n} + (b)_{k,l,n} + (c)_{k,l,n} + (d)_{k,l,n}.
\end{aligned} \tag{28}$$

We claim that

$$\sum_{k,l=0}^{n-1} \left|B^9_{k,l,n}\right| = o\left(n^{2-6H}\right).$$

Indeed, using again that $n^{2H}\langle \delta_{k/n}, \delta_{l/n}\rangle_{\mathscr{H}}$ is bounded and the condition (H_4), we obtain

$$\sum_{k,l=0}^{n-1} |(a)_{k,l,n}| \leq C\Big(n^{2H} \sum_{k=0}^{n-1} E\left\{ f''\left(S^H_{k/n}\right)\right\} \langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathscr{H}} \Big)^2$$

$$\leq C n^{-4H} \Big(\sum_{k=0}^{n-1} n^{2H} \left|\langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathscr{H}}\right| \Big)^2.$$

Since $n^{2H}\langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathscr{H}} = \frac{1}{2}[\varphi(k) - 1]$, $6H < 1$ and Lemma 3.3, we have

$$n^{-4H} \sum_{l=0}^{n-1} |\phi(l)| = o\left(n^{2-6H}\right).$$

Combining with $n^{1-4H} = o\left(n^{2-6H}\right)$ since $2H < 1$, it follows that

$$\sum_{k,l=0}^{n-1} |(a)_{k,l,n}| = o\left(n^{2-6H}\right).$$

Also, it is easy to check

$$\sum_{k,l=0}^{n-1} \left(|(b)_{k,l,n}| + |(c)_{k,l,n}| + |(d)_{k,l,n}|\right) = o\left(n^{2-6H}\right),$$

following the same lines as that for $\sum_{k,l=0}^{n-1} |(a)_{k,l,n}|$.

Finally we consider the last term $\sum_{k,l=0}^{n-1} B_{k,l,n}^{11}$. We use (17) and the point (1) of Lemma 3.3 to write

$$\langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathscr{H}} = -\frac{1}{2n^{2H}} + o\Big(\frac{1}{n^{2H}}\Big). \tag{29}$$

Substituting into the expression of $B_{k,l,n}^{11}$, yields

$$\begin{aligned}\sum_{k,l=0}^{n-1} B_{k,l,n}^{11} &= 3\sum_{k,l=0}^{n-1} n^{6H} E\Big\{f\left(S_{k/n}^H\right) g'\left(S_{l/n}^H\right)\left(\Delta S_{k/n}^H\right)^3 \langle \varepsilon_{l/n}, \delta_{l/n}\rangle_{\mathscr{H}} \langle \delta_{l/n}, \delta_{l/n}\rangle_{\mathscr{H}}\Big\} \\ &= -\frac{3}{2n^{2H}} \sum_{k,l=0}^{n-1} E\Big\{f\left(S_{k/n}^H\right) g'''\left(S_{l/n}^H\right)\left(\Delta S_{k/n}^H\right)^3\Big\} + o\Big(\frac{1}{n^{3H}}\Big).\end{aligned} \tag{30}$$

Therefore, using Lemma 3.5, with g''' instead of g, we obtain

$$\begin{aligned}\sum_{k,l=0}^{n-1} B_{k,l,n}^{11} =& \frac{9}{4n^{2H}} \sum_{k,l=0}^{n-1} E\left\{f'\left(S_{k/n}^H\right) g'\left(S_{l/n}^H\right)\right\} + \\ & \frac{3}{16n^{4H}} \sum_{k,l=0}^{n-1} E\left\{f'''\left(S_{k/n}^H\right) g'\left(S_{l/n}^H\right)\right\} + o\left(n^{2-6H}\right).\end{aligned}$$

This finished the proof of Lemma 3.7, and thus the proof of Lemma 3.6. □

Proof of Theorem 3.1 Combined with these two lemmas, the proof of the theorem can be completed along the same lines as in Nourdin [9]. Indeed, by Lemmas 3.5 and 3.6, we have on the one hand

$$\begin{aligned}& E\Big\{\Big(n^{3H-1}\sum_{k=0}^{n-1}\Big[f\left(S_{k/n}^H\right) n^{3H}\left(\Delta S_{k/n}^H\right)^3 + \frac{3}{2} f'\left(S_{k/n}^H\right) n^{-H}\Big]\Big)^2\Big\} \\ &= n^{6H-2}\sum_{k,l=0}^{n-1} E\Big\{\Big[f\left(S_{k/n}^H\right) n^{3H}\left(\Delta S_{k/n}^H\right)^3 + \frac{3}{2} f'\left(S_{k/n}^H\right) n^{-H}\Big]\times \\ &\quad \Big[f\left(S_{l/n}^H\right) n^{3H}\left(\Delta S_{l/n}^H\right)^3 + \frac{3}{2} f'\left(S_{l/n}^H\right) n^{-H}\Big]\Big\} \\ &= \frac{1}{64n^2}\sum_{k,l=0}^{n-1} E\left\{f'''\left(S_{k/n}^H\right) f'''\left(S_{l/n}^H\right)\right\} + o(1).\end{aligned} \tag{31}$$

On the other hand, we have by Lemma 3.5

$$E\Big\{n^{3H-1}\sum_{k=0}^{n-1}\Big[f\left(S^H_{k/n}\right)n^{3H}\left(\Delta S^H_{k/n}\right)^3+\frac{3}{2}n^{-H}f'\left(S^H_{k/n}\right)\Big]\Big(-\frac{1}{8n}\Big)\sum_{l}f'''\left(S^H_{l/n}\right)\Big\}$$
$$=-\frac{1}{8}n^{3H-2}\sum_{k,l=0}^{n-1}E\Big\{f\left(S^H_{k/n}\right)f'''\left(S^H_{l/n}\right)n^{3H}\left(\Delta S^H_{k/n}\right)^3+\frac{3}{2n^H}f'\left(S^H_{k/n}\right)f'''\left(S^H_{l/n}\right)\Big\}$$
$$=\frac{1}{64n^2}\sum_{k,l=0}^{n-1}E\left\{f'''\left(S^H_{k/n}\right)f'''\left(S^H_{l/n}\right)\right\}+o(1). \tag{32}$$

Now, we easily deduce (11). Indeed, thanks to (31) and (32), we obtain, by developing the square and by remembering that $H < 1/6$, that

$$E\Big\{\Big(n^{3H-1}\sum_{k=0}^{n-1}\Big[f\left(S^H_{k/n}\right)n^{3H}\left(\Delta S^H_{k/n}\right)^3+\frac{3}{2n^H}f'\left(S^H_{k/n}\right)\Big]+$$
$$\frac{1}{8}n^{-1}\sum_{k=0}^{n-1}f'''\left(S^H_{k/n}\right)\Big)^2\Big\}\to 0, \tag{33}$$

as n tends to infinity. Since

$$-\frac{1}{8}n^{-1}\sum_{k=0}^{n-1}f'''\left(S^H_{k/n}\right)\xrightarrow[n\to\infty]{L^2}-\frac{1}{8}\int_0^1 f'''(S^H_s)\mathrm{d}s,$$

we finally prove that (11) holds.

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