Journal of Mathematical Research with Applications
May, 2016, Vol. 36, No. 3, pp. 351–358
DOI:10.3770/j.issn:2095-2651.2016.03.010
Http://jmre.dlut.edu.cn

Viscosity Approximation Methods for Common Fixed Points of a Finite Family of Quasi-Nonexpansive Mappings in Real Hilbert Spaces

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Abstract In this paper, we present two parallel algorithms to compute common fixed points of a finite family of quasi-nonexpansive mappings in real Hilbert spaces, which improve and generalize some known results in this direction.

Keywords quasi-nonexpansive mappings; fixed point; variational inequality; parallel algorithms

MR(2010) Subject Classification 47H05; 47H09; 47H10

1. Introduction

Throughout this paper, we assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$, respectively. Let Ω be a nonempty closed convex subset of H, $T:\Omega\to\Omega$ a self-mapping on Ω with a fixed point set denoted by $\mathrm{Fix}(T)=\{x\in\Omega: Tx=x\}\neq\emptyset$, and $C:\Omega\to\Omega$ a contraction with modulus $\rho\in[0,1)$, i.e.,

$$\|Cx-Cy\|\le\rho\|x-y\|,\quad \forall x,y\in\Omega. \tag{1.1}$$

In this paper, we review the computation of fixed points of such general operators T, by means of the so called viscosity approximation method, which formally consists of the sequence $x_n\in\Omega$ given by the iteration,

$$x_{n+1}=\alpha_nCx_n+(1-\alpha_n)Tx_n, \tag{1.2}$$

where $(\alpha_n)\subset(0,1)$ is a slowly vanishing sequence, i.e., $\lim_{n\to\infty}\alpha_n=0$ and $\sum_n\alpha_n=\infty$.

Recall for instance that one of the main convergence results related to (1.2) goes back to Moudafi [1], regarding the case when T belongs to the class of nonexpansive mappings (with fixed points), i.e., $\|Tx-Ty\|\le\|x-y\|$, for all $x,y\in\Omega$.

It was proved in [1] (also see Xu [2]) that (1.2), under additional conditions on the slowly vanishing parameters (α_n), generates a sequence (x_n) which converges strongly to the unique solution of the variational inequality problem VI(I-C, Fix(T)): find x_* in Fix(T) such that $\forall\nu\in\mathrm{Fix}(T)$,

$$\langle (I-C)(x_*),\nu-x_*\rangle\ge 0, \tag{1.3}$$

Received May 30, 2015; Accepted December 16, 2015
Supported by the National Natural Science Foundation of China (Grant No. 11071053).
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or equivalently

$$x_* = (P_{\mathrm{Fix}(T)} \circ C)(x_*), \tag{1.4}$$

where $P_{\mathrm{Fix}(T)}$ denotes the metric projection from H onto $\mathrm{Fix}(T)$ (see [3]) for more details on the metric projection. Note that, as T is nonexpansive, $\mathrm{Fix}(T)$ is well-known to be a closed and convex subset of H, hence $P_{\mathrm{Fix}(T)}$ is well-defined. Let us mention that the method (1.2) was first considered with regard to the special case when $C = u$ (u being any given element in C), in 1967 by Halpern [4] (for $u = 0$) and in 1977 by Lions [5] (also see [6]).

In 2009, Paul-Emile Maingé had showed how to ensure the strong convergence of the method (1.2) when involving mapping T belongs to the more general class of (possibly discontinuous) quasi-nonexpansive mappings, i.e., $\forall (x,q) \in \Omega \times \mathrm{Fix}(T), \|Tx - q\| \le \|x - q\|$, which are operators commonly encountered in the literature, and proposed a new analysis of the viscosity approximation method, where attention will be focused on the following variant of algorithm:

$$x_{n+1} = \alpha_n C x_n + (1 - \alpha_n) T_\omega x_n, \tag{1.5}$$

where (α_n) is a slowly vanishing sequence. $\omega \in (0, 1]$, and $T_\omega := (1 - \omega)I + \omega T$ (I being the identity mapping on Ω), with two main conditions on T:

(i) T is a quasi-nonexpansive mapping, i.e., $\|Tx - Tq\| \le \|x - q\|, \quad \forall (x, q) \in \Omega \times \mathrm{Fix}(T)$;

(ii) $I - T$ is demiclosed at zero on Ω, that is

$$\{z_k\} \subset \Omega, z_k \rightharpoonup z \text{ weakly}, \ (I - T)(z_k) \to 0 \text{ strongly } \ \Rightarrow z \in \mathrm{Fix}(T). \tag{1.6}$$

Paul-Emile Maingé established the strong convergence of the sequence given by (1.5) to the unique solution in the above setting, no additional conditions are made on the operator T. To be precise he proved the following convergence theorem:

Theorem 1.1 *Let $\{x_n\}$ be the sequence given by (1.5) with T quasi-nonexpansive and demiclosed at zero on Ω, $\omega \in (0, 1)$, and $\{\alpha_n\} \subset (0, 1)$ such that*

$$(C1) \ \lim_{n\to\infty} \alpha_n = 0; \ (C2) \sum_n \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to the unique element x_* in $\mathrm{Fix}(T)$ verifying $x_* = (P_{\mathrm{Fix}(T)} \circ C)x_*$, which equivalently solves the following variational inequality problem:

$$x_* \in \mathrm{Fix}(T) \ (\forall \nu \in \mathrm{Fix}(T)), \quad \langle (I - C)x_*, \nu - x_* \rangle \ge 0. \tag{1.7}$$

At this point, we put forth the following two questions:

Question 1.2 If each quasi-nonexpansive mapping T_i is demiclosed at zero on Ω, is the convex combination of finitely quasi-nonexpansive mappings demiclosed at zero?

Question 1.3 Can above Theorem 1.1 be extended to a finite family of quasi-nonexpansive mappings in a Hilbert space?

Our purpose in this paper is to present two parallel algorithms and establish the strong convergence of the proposed methods. The results presented in this paper improve and generalize some known results by other authors recently.

2. Preliminaries

In order to define our motivations, we recall some definitions of classes of operators often used in fixed point theory. Let $T : \Omega :\to H$ be a mapping. Then

(i) T belongs to the class of firmly nonexpansive mappings if

$$\forall x, y \in \Omega, \|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2. \tag{2.1}$$

(ii) T belongs to the class of firmly quasi-nonexpansive mappings if $\mathrm{Fix}(T) \neq \emptyset$, such that

$$\forall (x, q) \in \Omega \times \mathrm{Fix}(T), \| Tx - q\|^2 \leq \|x - q\|^2 - \|x - Tx\|^2. \tag{2.2}$$

(iii) T is said to be averaged if there exist a real number $\lambda \in (0, 1)$ and a nonexpansive mapping S such that

$$T = (1 - \lambda)I + \lambda S. \tag{2.3}$$

(iv) T is said to be quasi-averaged if $\mathrm{Fix}(T) \neq \emptyset$ and there exist a real number $\lambda \in (0, 1)$ and a quasi-nonexpansive mapping S such that

$$T = (1 - \lambda)I + \lambda S. \tag{2.4}$$

(v) $T : \Omega \to \Omega$ is called strictly pseudocontractive on Ω if there exists a constant $\nu \in [0, 1)$ such that

$$\forall (x, y) \in \Omega \times \Omega, \| Tx - Ty\|^2 \leq \|x - y\|^2 + \nu\|x - y - (Tx - Ty)\|^2. \tag{2.5}$$

(vi) $T : \Omega \to \Omega$ is called demicontractive on Ω if $\mathrm{Fix}(T) \neq \emptyset$ and there exists a constant $\beta < 1$, such that

$$\forall (x, q) \in \Omega \times \mathrm{Fix}(T), \|Tx - q\|^2 \leq \|x - q\|^2 + \beta\|x - Tx\|^2. \tag{2.6}$$

As usual, a mapping satisfying (2.3) is called λ-averaged and a mapping satisfying (2.6) will be referred to as β-demicontractive. It is well known that a firmly nonexpansive mapping is $\frac{1}{2}$-averaged. It is worth noting that the class of demicontractive maps contains important classes of operators such as firmly-quasinonexpansive maps for $\beta = -1$, quasi-nonexpansive for $\beta = 0$ and strictly pseudocontractive maps for $\beta \in (0, 1)$.

Remark 2.1 Let T be a β-demicontractive mapping on Ω with $\mathrm{Fix}(T) \neq \emptyset$ and set $T_\omega := (1 - \omega)I + \omega T$ for $\omega \in (0, \infty)$:

(i) T β-demicontractive is equivalent to

$$\langle x - Tx, x - q\rangle \geq (\frac{1}{2})(1 - \beta)\|x - Tx\|^2, \quad \forall (x, q) \in \Omega \times \mathrm{Fix}(T);$$

(ii) $\mathrm{Fix}(T) = \mathrm{Fix}(T_\omega)$ if $\omega \neq 0$;

(iii) T_ω is quasi-nonexpansive for $\omega \in [0, 1 - \beta]$ and satisfies

$$\|T_\omega x - q\|^2 \leq \|x - q\|^2 - \omega(1 - \beta - \omega)\|Tx - x\|^2, \quad \forall (x, q) \in \Omega \times \mathrm{Fix}(T);$$

(iv) $\mathrm{Fix}(T)$ is a closed convex subset of H.

Remark 2.2 (As shown by Remark 2.1) When $\beta = 0$, we can immediately deduce the following conclusions:

(i) $\mathrm{Fix}(T) = \mathrm{Fix}(T_\omega)$;

(ii) T_ω is quasi-averaged, if $\omega \in (0, 1)$;

(iii) $\|T_\omega x - q\|^2 \le \|x - q\|^2 - \omega(1 - \omega)\|Tx - x\|^2$, $\forall(x, q) \in \Omega \times \mathrm{Fix}(T)$;

(iv) $\langle x - T_\omega x, x - q\rangle \ge \frac{1}{2}\omega\|x - Tx\|^2$, $\forall(x, q) \in \Omega \times \mathrm{Fix}(T)$.

Lemma 2.3 *For all $x, y \in H$ and $\lambda \in [0, 1]$,*

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \tag{2.7}$$

which can be extended to the more general situation: For all $x_1, x_2, \ldots, x_n \in H$, $\lambda_i \in [0, 1]$, and $\sum_{i=1}^n \lambda_i = 1$, we have

$$\|\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n\|^2 = \lambda_1\|x_1\|^2 + \lambda_2\|x_2\|^2 + \cdots + \lambda_n\|x_n\|^2 - \sum_{1 \le i < j \le n} \lambda_i\lambda_j\|x_i - x_j\|^2. \tag{2.8}$$

Now we are in a proposition to prove the main results of this paper.

3. Main results

Theorem 3.1 *Let T_i $(i = 1, 2, \ldots, r)$ be r quasi-nonexpansive mappings on Ω such that $F = \bigcap_{i=1}^r \mathrm{Fix}(T_i) \ne \emptyset$ and $I - T_i$ are demiclosed at zero. Put $T = \sum_{i=1}^r \lambda_i T_i$, where $\sum_{i=1}^r \lambda_i = 1$. Let $(\alpha_n) \subset (0, 1)$ be a real sequence of numbers such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_n \alpha_n = \infty$. Define a sequence $\{x_n\}$ in Ω by the following algorithm:*

$$x_1 \in \Omega,\ x_{n+1} = \alpha_n C x_n + (1 - \alpha_n) T_\omega x_n, \quad \forall n \ge 1,$$

where $\omega \in (0, 1)$. Then (x_n) converges strongly to the unique element x_* in F verifying $x_* = P_F \circ C x_*$, which equivalently solves the following variational inequality problem:

$$x_* \in F,\ \forall \nu \in F,\ \langle (I - C)x_*, \nu - x_*\rangle \ge 0. \tag{3.1}$$

Proof We split the proof into four steps.

Step 1. Show that $F = \mathrm{Fix}(T) = \bigcap_{i=1}^r \mathrm{Fix}(T_i)$.

First, we show that $\bigcap_{i=1}^r \mathrm{Fix}(T_i) \subseteq \mathrm{Fix}(T)$. For $\forall p \in \bigcap_{i=1}^r \mathrm{Fix}(T_i)$, we have $T_i p = p$ $(i = 1, 2, \ldots, r)$ and $Tp = \sum_{i=1}^r \lambda_i T_i p = \sum_{i=1}^r \lambda_i p = p$, hence $\bigcap_{i=1}^r \mathrm{Fix}(T_i) \subseteq \mathrm{Fix}(T)$.

Next, we show that $\mathrm{Fix}(T) \subseteq \bigcap_{i=1}^r \mathrm{Fix}(T_i)$. For $\forall x \in \mathrm{Fix}(T)$ and $p \in \bigcap_{i=1}^r \mathrm{Fix}(T_i)$, we have

$$\|x - p\| = \|Tx - p\| = \|\sum_{i=1}^r \lambda_i T_i x - p\|.$$

Note that T_i $(i = 1, 2, \ldots, r)$ are quasi-nonexpansive, it follows immediately that

$$\|x - p\| = \|\sum_{i=1}^r \lambda_i (T_i x - p)\| \le \sum_{i=1}^r \lambda_i \|T_i x - p\| \le \sum_{i=1}^r \lambda_i \|x - p\| = \|x - p\|.$$

By using this inequality, we have

$$\|\sum_{i=1}^r \lambda_i (T_i x - p)\| = \sum_{i=1}^r \lambda_i \|T_i x - p\| = \|x - p\|.$$

Since H is strictly convex, there must be $T_i x = x$ $(i = 1, 2, \ldots, r)$, which means that

$$x \in \bigcap_{i=1}^{r} \mathrm{Fix}(T_i), \quad i = 1, 2, \ldots, r$$

so $\mathrm{Fix}(T) \subseteq \bigcap_{i=1}^{r} \mathrm{Fix}(T_i)$, and the desired result follows.

Step 2. Show that $T = \sum_{i=1}^{r} \lambda_i T_i$ is a quasi-nonexpansive mapping. Let $p \in \mathrm{Fix}(T)$. Then $p \in \bigcap_{i=1}^{r} \mathrm{Fix}(T_i) \neq \emptyset$ by Step 1. Noting that $\forall x \in \Omega$, $\sum_{i=1}^{r} \lambda_i p = p$ for $\forall p \in \Omega$. We have

$$\begin{aligned} \|Tx - p\| &= \|\sum_{i=1}^{r} \lambda_i T_i x - \sum_{i=1}^{r} \lambda_i p\| = \|\sum_{i=1}^{r} \lambda_i (T_i x - p)\| \\ &\leq \sum_{i=1}^{r} \lambda_i \|T_i x - p\| \leq \sum_{i=1}^{r} \lambda_i \|x - p\| = \|x - p\|, \end{aligned}$$

that is the desired result.

Step 3. Show that if T_i $(i = 1, 2, \ldots, r)$ are demiclosed at zero, then $T = \sum_{i=1}^{r} \lambda_i T_i$ with $\sum_{i=1}^{r} \lambda_i = 1$ is also demiclosed at zero. We first show if $\|x_n - Tx_n\| \to 0$, then $\|x_n - T_i x_n\| \to 0$ $(i = 1, 2, \ldots, r)$.

From $p \in \bigcap_{i=1}^{r} F(T_i)$, by using the equality (2.8), and noting that T_i $(i = 1, 2, \ldots, r)$ are quasi-nonexpansive mappings, we have

$$\begin{aligned} \|Tx_n - p\|^2 &= \|\lambda_1 (T_1 x_n - p) + \lambda_2 (T_2 x_n - p) + \cdots + \lambda_r (T_r x_n - p)\|^2 \\ &= \sum_{i=1}^{r} \lambda_i \|T_i x_n - p\|^2 - \sum_{1 \leq i < j \leq r} \lambda_i \lambda_j \|u_i - v_j\|^2 \\ &\leq \|x_n - p\|^2 - \sum_{1 \leq i < j \leq r} \lambda_i \lambda_j \|u_i - v_j\|^2, \end{aligned} \tag{3.2}$$

where $u_i = T_i x_n - p, v_j = T_j x_n - p$ $(i, j = 1, 2, \ldots, r)$, it follows from (3.2) that

$$\begin{aligned} \sum_{1 \leq i < j \leq r} \lambda_i \lambda_j \|u_i - v_j\|^2 &\leq \|x_n - p\|^2 - \|Tx_n - p\|^2 \\ &\leq 2\|x_n - p\| \|x_n - Tx_n\| \longrightarrow 0. \end{aligned}$$

Noting that $\lambda_i \lambda_j > 0$, we have

$$\|u_i - v_j\| = \|T_i x_n - T_j x_n\| \to 0. \tag{3.3}$$

On the other hand, we have

$$\begin{aligned} Tx_n - T_1 x_n &= (\lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_r T_r) x_n - T_1 x_n \\ &= -(\lambda_2 + \lambda_3 + \cdots + \lambda_r) T_1 x_n + \lambda_2 T_2 x_n + \cdots + \lambda_r T_r x_n \\ &= -\lambda_2 (T_1 x_n - T_2 x_n) - \lambda_3 (T_1 x_n - T_3 x_n) - \cdots - \lambda_r (T_1 x_n - T_r x_n). \end{aligned}$$

Consequently, we have

$$\|Tx_n - T_1 x_n\| \leq \lambda_2 \|T_1 x_n - T_2 x_n\| + \lambda_3 \|T_1 x_n - T_3 x_n\| + \cdots + \lambda_r \|T_1 x_n - T_r x_n\| \to 0. \tag{3.4}$$

In a similar way, we can obtain

$$\|Tx_n - T_2 x_n\|, \|Tx_n - T_3 x_n\|, \ldots, \|Tx_n - T_r x_n\| \to 0. \tag{3.5}$$

Since $x_n - T_1x_n = x_n - Tx_n + Tx_n + T_1x_n$, we have

$$\|x_n - T_1x_n\| \leq \|x_n - Tx_n\| + \|Tx_n - T_1x_n\| \to 0. \tag{3.6}$$

Similarly, we have

$$\|x_n - T_2x_n\|, \|x_n - T_3x_n\|, \ldots, \|x_n - T_rx_n\| \to 0. \tag{3.7}$$

Noting that T_i $(i = 1, 2, \ldots, r)$ are demiclosed at zero, and assume that $x_n \rightharpoonup x$, we obtain that

$$x = T_1x, x = T_2x, \ldots, x = T_rx, \tag{3.8}$$

which implies that $x \in \bigcap_{i=1}^{r} \mathrm{Fix}(T_i)$, then we entail the desired result.

Step 4. Show that $x_n \to x$, as $n \to \infty$. Indeed, by using Theorem 1.1, we conclude that $\{x_n\}$ converges strongly to the unique element x_* in $\mathrm{Fix}(T)$ verifying $x_* = P_{\mathrm{Fix}(T)} \circ Cx_*$, which equivalently solves the following variational inequality problem:

$$x_* \in \mathrm{Fix}(T),\ \forall \nu \in \mathrm{Fix}(T)), \langle (I - C)x_*, \nu - x_* \rangle \geq 0.$$

This completes the proof. $\square$

In order to establish another strong convergence theorem, we first prove the following result.

Lemma 3.2 *Let $\{T_i\}_{i=1}^{r} : \Omega \to \Omega$ be r quasi-nonexpansive mappings such that $F = \bigcap_{i=1}^{r} \mathrm{Fix}(T_i) \neq \emptyset$, let $\{\lambda_i\}_{i=1}^{r}$ be a real sequence in $(0, 1)$. Define r new mappings as follows:*

$$U_1 = \lambda_1T_1 + (1 - \lambda_1)I,$$

$$U_2 = \lambda_2T_2U_1 + (1 - \lambda_2)I,$$

$$U_3 = \lambda_3T_3U_2 + (1 - \lambda_3)I,$$

$$\cdots$$

$$U_r = \lambda_rT_rU_{r-1} + (1 - \lambda_r)I.$$

Then

(i) $F = \bigcap_{i=1}^{r} \mathrm{Fix}(U_i)$;

(ii) $\{U_i\}_{i=1}^{r}$ *is a finite family of quasi-averaged mappings.*

(iii) If T_i $(i = 1, 2, \ldots, r)$ are demiclosed at zero, then so are U_i $(i = 1, 2, \ldots, r)$.

(iv) $F = \mathrm{Fix}(U_r)$.

Proof (i) It is obvious that $F \subset \bigcap_{i=1}^{r} \mathrm{Fix}(U_i)$. We show the converse inclusion relation.

Assume that $x = U_ix$ $(i = 1, 2, \ldots, r)$; then we have $x = T_ix$ $(i = 1, 2, \ldots, r)$, and therefore $x \in \bigcap_{i=1}^{r} \mathrm{Fix}(U_i) = F$.

(ii) Since T_iU_{i-1} is quasi-nonexpansive, and $\lambda_i \in (0, 1)$, we know that U_i is quasi-averaged.

(iii) Next we shall prove that if $x_n - U_1x_n \to 0$ and $x_n \rightharpoonup x$, then $x = U_1x$.

Indeed, from $x_n - U_1x_n = \lambda_1(I - T_1)x_n \to 0$, and $\lambda_1 \in (0, 1)$, we have $x_n - T_1x_n \to 0$. Since T_1 is demi-closed at zero, we assert that $x = T_1x$. Assume that $x_n - U_2x_n \to 0$ and $x_n \rightharpoonup x$, then $\forall p \in F$,

$$\|U_2x_n - p\|^2 = \|\lambda_2T_2U_1x_n + (1 - \lambda_2)x_n - p\|^2$$

$$
\begin{aligned}
&=\lambda_2\|T_2U_1x_n-p\|^2+(1-\lambda_2)\|x_n-p\|^2-\lambda_2(1-\lambda_2)\|T_2U_1x_n-x_n\|^2\\
&\le\lambda_2\|U_1x_n-p\|^2+(1-\lambda_2)\|x_n-p\|^2-\lambda_2(1-\lambda_2)\|T_2U_1x_n-x_n\|^2\\
&=\lambda_2[\lambda_1\|T_1x_n-p\|^2+(1-\lambda_1)(x_n-p)^2-\lambda_1(1-\lambda_1)\|x_n-T_1x_n\|^2]+\\
&\quad(1-\lambda_2)\|x_n-p\|^2-\lambda_2(1-\lambda_2)\|T_2U_1x_n-x_n\|^2\\
&\le\lambda_2\|x_n-p\|^2-\lambda_2\lambda_1(1-\lambda_1)\|x_n-T_1x_n\|^2+(1-\lambda_2)\|x_n-p\|^2-\\
&\quad\lambda_2(1-\lambda_2)\|T_2U_1x_n-x_n\|^2\\
&=\|x_n-p\|^2-\lambda_2\lambda_1(1-\lambda_1)\|x_n-T_1x_n\|^2-\lambda_2(1-\lambda_2)\|T_2U_1x_n-x_n\|^2,
\end{aligned}
$$

which implies that

$$\lambda_2\lambda_1(1-\lambda_1)\|x_n-T_1x_n\|^2\le\|x_n-p\|^2-\|U_2x_n-p\|^2\le M_1\|x_n-U_2x_n\|\to 0.$$

Then we have that $x_n-T_1x_n\to 0$ and $x_n-U_1x_n\to 0$, which implies that $x=U_1x$. We also get that

$$\lambda_2(1-\lambda_2)\|T_2U_1x_n-x_n\|^2\le\|x_n-P\|^2-\|U_2x_n-p\|^2\le M_2\|x_n-U_2x_n\|\to 0.$$

Then we deduce that $T_2U_1x_n-x_n\to 0$ and $T_2U_1x_n-U_1x_n\to 0$.

Since $U_1x_n\rightharpoonup x$ and T_2 is demiclosed at zero, $T_2x=x$, also $U_2x=x$. Assume U_{r-1} is demiclosed at zero for some $r\ge 1$, we want to show U_r is demiclosed at zero. To this end, assume that $x_n-U_rx_n\to 0$ and $x_n\rightharpoonup x$, we plan to show $x=U_rx$. Indeed, $\forall p\in F$ and from (2.2) we get that

$$
\begin{aligned}
\|U_rx_n-p\|^2&=\lambda_r\|T_rU_{r-1}x_n-p\|^2+(1-\lambda_r)\|x_n-p\|^2-\lambda_r(1-\lambda_r)\|T_rU_{r-1}x_n-x_n\|^2\\
&\le\lambda_r\|U_{r-1}x_n-p\|^2+(1-\lambda_r)\|x_n-p\|^2-\lambda_r(1-\lambda_r)\|T_rU_{r-1}x_n-x_n\|^2\\
&\le\|x_n-p\|^2-\lambda_r\lambda_{r-1}(1-\lambda_{r-1})\|x_n-T_{r-1}U_{r-2}x_n\|^2-\\
&\quad\lambda_r(1-\lambda_r)\|T_rU_{r-1}x_n-x_n\|^2.
\end{aligned}
$$

Then we could conclude that $T_rU_{r-1}x_n-x_n\to 0$ and $T_{r-1}U_{r-2}x_n-x_n\to 0$, which implies that $U_{r-1}x_n-x_n\to 0$ and $T_rU_{r-1}x_n-U_{r-1}x_n\to 0$.

Since U_{r-1} is demiclosed at zero, then $U_{r-1}x_n\rightharpoonup x$, for the results we have got $T_rU_{r-1}x_n-U_{r-1}x_n\to 0$, we see that $x=U_rx$ immediately, as claimed.

(iv) The inclusion relation that $F\subset \mathrm{Fix}(U_r)$ is obvious, we only show the reverse inclusion relation. Assume that $x=U_rx$; then $\forall p\in F$, by the definition of U_r, we get that

$$
\begin{aligned}
\|x-p\|^2&=\|U_rx-p\|^2\\
&=\lambda_r\|T_rU_{r-1}x-p\|^2+(1-\lambda_r)\|x-p\|^2-\lambda_r(1-\lambda_r)\|T_rU_{r-1}x-x\|^2\\
&\le\lambda_r\|U_{r-1}x-p\|^2+(1-\lambda_r)\|x-p\|^2-\lambda_r(1-\lambda_r)\|T_rU_{r-1}x-x\|^2\\
&\le\|x-p\|^2-\lambda_r\lambda_{r-1}(1-\lambda_{r-1})\|x-T_{r-1}U_{r-2}x\|^2-\\
&\quad\lambda_r(1-\lambda_r)\|T_rU_{r-1}x-x\|^2,
\end{aligned}
$$

which implies that $x=T_rU_{r-1}x$ and $x=T_{r-1}U_{r-2}x$. It follows the definition of U_{r-1} that $x=U_{r-1}x$, which turns out that $x=T_rx$. In a similar way, we can show that $x=T_{r-1}x=$

$T_{r-2}x = \cdots = T_1 x$, thus, we have $x \in F$. This completes the proof. □

Theorem 3.3 *Let T_i $(i = 1, 2, \ldots, r)$ be r quasi-nonexpansive mappings Ω such that $I - T_i$ are demiclosed at zero and $F = \bigcap_{i=1}^{r} \mathrm{Fix}(T_i) \neq \emptyset$. Let U_i $(i = 1, 2, \ldots, r)$ be defined as in Lemma 3.1. Let $(\alpha_n) \subset (0, 1)$ be a real sequence of numbers such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_n \alpha_n = \infty$. Let $\{x_n\}$ be generated by the following algorithm:*

$$x_1 \in \Omega,\ x_{n+1} = \alpha_n C x_n + (1 - \alpha_n) U_r x_n, \quad \forall n \geq 1.$$

Then (x_n) converges strongly to the unique element x_* in F verifying $x_* = (P_F \circ C)x_*$, which equivalently solves the following variational inequality problem:

$$x_* \in F,\ \forall \nu \in F,\ \langle (I - C)x_*, \nu - x_* \rangle \geq 0.$$

Proof By Lemma 3.2, we know that $F = \mathrm{Fix}(U_r)$, U_r is averaged on Ω and $I - U_r$ is demiclosed at zero. Now the conclusion of Theorem 3.3 follows from Theorem 1.1 immediately. This completes the proof. □.

Acknowledgements I would like to show my deepest gratitude to my tutor, Professor Haiyun ZHOU, who has provided me with valuable guidance in every stage of the writing of this thesis.

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