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On Rees Factor S-Posets Satisfying Condition (P_w)

Roghaieh KHOSRAVI
Department of Mathematics, Fasa University, Fasa, Iran

Abstract In this paper, we present some homological classifications of pomonoids by using Rees factor S-posets satisfying condition (P_w).

Keywords Pomonoids; S-posets; condition (P_w)

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1. Introduction

Throughout this paper S will denote a pomonoid that is a monoid S together with a partial order relation which is compatible with the binary operation. A right S-poset A_S is a right S-act A equipped with a partial order $\leq$. And in addition, for all $s,t \in S$ and $a,b \in A$, if $s \leq t$, then $as \leq at$, and if $a \leq b$, then $as \leq bs$. An S-subposet of a right S-poset A is a subset of A that is closed under the S-action.

A right S-poset A_S satisfies condition (P) if, for all $a,b \in A$ and $s,t \in S$, $as \leq bt$ implies $a = a'u$, $b = a'v$ for some $a' \in A$, $u,v \in S$ with $us \leq vt$, and it satisfies condition (E) if, for all $a \in A$ and $s,t \in S$, $as \leq at$ implies $a = a'u$ for some $a' \in A$, $u \in S$ with $us \leq ut$. A right S-poset is called strongly flat if it satisfies both conditions (P) and (E). Projectivity is defined in the standard categorical manner. A right S-poset A_S is weakly po-flat if $a \otimes s \leq a' \otimes t$ in $A_S \otimes {}_SS$ (equivalently, $as \leq a't$) implies that the same inequality holds also in $A_S \otimes {}_S(Ss \cup St)$ for $a,a' \in A_S$, $s,t \in S$. A right S-poset A_S is principally weakly po-flat if $as \leq a's$ implies that $a \otimes s \leq a' \otimes s$ in $A_S \otimes {}_SSs$ for $a,a' \in A_S$, $s \in S$. Weakly flat and principally weakly flat can be defined similarly to the previous by replacing $\leq$ by $=$.

Condition (P_w) was introduced in [1]. An S-poset A_S satisfies condition (P_w) if, for all $a,b \in A$ and $s,t \in S$, $as \leq bt$ implies $a \leq a'u$, $a'v \leq b$ for some $a' \in A$, $u,v \in S$ with $us \leq vt$.

In [2], condition $(PWP)_w$ was introduced. An S-poset A_S is said to satisfy condition $(PWP)_w$ if, for all $a,b \in A_S$ and $s \in S$, $as \leq bs$ implies $a \leq a'u$ and $a'v \leq b$ for some $a' \in A_S$, $u,v \in S$ with $us \leq vs$.

During recent years several articles on flatness properties of S-posets such as projectivity, condition (P), condition (P_w), strong flatness $\cdots$ have appeared. Flatness properties of one element and Rees factor S-posets were discussed in [3,4]. Following Section 1, the coproducts of S-posets satisfying conditions (P_w) or $(PWP)_w$ are studied. In Section 2, we first discuss

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E-mail address: khosravi@fasau.ac.ir

Rees factor S-posets satisfying condition (P_w). Then we consider pomonoids over which flatness properties of Rees factor S-posets imply condition (P_w). In Section 3, we give a classification of pomonoids when Rees factor S-posets satisfying condition (P_w) imply other properties of S-posets.

A subpomonoid K of S is called convex if $K = [K]$ where $[K] = \{x \in S | \exists\, p, q \in K,\ p \leq x \leq q\}$. If K_S is a convex, proper right ideal of the pomonoid S, S/K_S will always stand for $S/\nu(K \times K)$. The following lemma gives an explicit description of the order in S/K_S.

Lemma 1.1 ([5]) *Let K_S be a convex, proper right ideal of the pomonoid S. Then for $x, y \in S$,*

$$[x] \leq [y] \text{ in } S/K_S \Longleftrightarrow (x \leq y) \text{ or } (x \leq k \text{ and } k' \leq y \text{ for some } k, k' \in K).$$

Moreover, $[x] = [y]$ in S/K_S if, and only if, either $x = y$ or else $x, y \in K$.

Similarly to S-acts, coproducts of S-posets are disjoint unions, with S-action and order defined componentwise. In [6] it was shown that most of flatness properties can be transferred from S-posets over a pomonoid S to their coproducts. To complete this, we show that this is also valid for conditions (P_w) and $(PWP)_w$.

Proposition 1.2 *Let S be a pomonoid. Then a right S-poset $A_S = \coprod_{i \in I} A_i$ satisfies condition (P_w) if and only if every strongly convex indecomposable component A_i satisfies condition (P_w).*

Proof Necessity. Suppose that $A_S = \coprod_{i \in I} A_i$ satisfies condition (P_w). Fix $i \in I$, and let $as \leq bt$ for some $a, b \in A_i$, $s, t \in S$. Since A_S satisfies condition (P_w), there exist $a' \in A$, $u, v \in S$ such that $a \leq a'u$, $a'v \leq b$, $vs \leq vt$. Now since S-action and order are defined componentwise for $A_S = \coprod_{i \in I} A_i$, $a' \in A_i$, and so the result follows.

Sufficiency. Let $as \leq bt$ for some $a, b \in A$, $s, t \in S$. Since S-action and order of A are defined componentwise, there exists $i \in I$ such that $a, b \in A_i$. Now, since A_i satisfies condition (P_w), there exist $a' \in A_i$, $u, v \in S$ such that $a \leq a'u$, $a'v \leq b$, $us \leq vt$, and we are done. $\square$

By a similar argument, one can show the following.

Proposition 1.3 *Let S be a pomonoid. Then a right S-poset A_S satisfies condition $(PWP)_w$ if and only if its strongly convex indecomposable components satisfy condition $(PWP)_w$.*

2. When flatness properties imply condition (P_w)

In this section, we first focus our attention on Rees factor S-posets satisfying condition (P_w). Then, we mainly consider the case when flatness properties for Rees factor S-posets imply condition (P_w).

Let K be a convex, proper right ideal of a pomonoid S. K is called left stabilizing if $k \in [Kk]$ for every $k \in K$. Moreover, K is called strongly left stabilizing if, $(\forall k \in K)(\forall s \in S)$

$$(k \leq s \Rightarrow (\exists k' \in K)(k's \leq s),\ \text{and } s \leq k \Rightarrow (\exists k'' \in K)(s \leq k''s)).$$

The following result which was proved in [7] will be our main tool in what follows.

Lemma 2.1 *Let S be a pomonoid. The Rees factor S-poset S/K by a convex, proper right ideal K_S satisfies condition (P_w) if and only if for any $k, l \in K$ one of the following three conditions is satisfied:*

(a) $k \leq l$;

(b) There exist $p, q \in K$ such that $k \leq pl$ and $q < 1$;

(c) There exist $p, q \in K$ such that $1 < p$ and $qk \leq l$.

For a subpomonoid X of a pomonoid S let $[X) = \{s \in S | \exists\, p \in X, p \leq s\}$ and $(X] = \{p \in S | \exists\, x \in X, p \leq x\}$.

Definition 2.2 *Let K be a convex, proper right ideal of a pomonoid S. We say that K has property (X) if it satisfies one of the following conditions:*

(a) $q < 1$ for some $q \in K$, and $K = (Kl]$ for each $l \in K$;

(b) $q > 1$ for some $q \in K$, and $K = [Kl)$ for each $l \in K$.

Now we paraphrase the previous lemma as follows.

Lemma 2.3 *Let K be a convex, proper right ideal of a pomonoid S. Then S/K satisfies condition (P_w) if, and only if, either $|K| = 1$ or K satisfies property (X).*

Proof Necessity. Let K be a convex, proper right ideal of a pomonoid S and $|K| > 1$. Then there exists $q \in K$ such that $q \nparallel 1$. Otherwise, if 1 is incomparable with all elements of K, then by Lemma 2.1 for each $k, l \in K$, $k \leq l$. So $|K| = 1$. Now, suppose that $q < 1$ for some $q \in K$. Then since K is a convex proper right ideal, there is no $p \in K$ with $1 \leq p$, thus fix $l \in K$ by part (b) of Lemma 2.1 for each $k \in K$ there exists $p \in K$ such that $k \leq pl$. Then $k \in (Kl]$, and so $K \subseteq (Kl]$. On the other hand, suppose that $t \in (Kl]$, that is, $t \leq pl$ for some $p \in K$. Thus, $qt \leq t \leq pl$, and by convexity of K, $t \in K$. Therefore, $K = (Kl]$. If $q > 1$ for some $q \in K$, with the similar argument for each $l \in K$, $K = [Kl)$.

Sufficiency. Suppose that K is a convex, proper right ideal of a pomonoid S. If $|K| = 1$, then $S/K = S$ satisfies condition (P_w). Otherwise, by assumption the parts (b) or (c) of Lemma 2.1 holds. Thus S/K satisfies condition (P_w). $\square$

A pomonoid S is called weakly right reversible in case $Ss \cap (St] \neq \emptyset$ for all $s, t \in S$. By [5, Theorem 1], Θ satisfies condition (P_w) if and only if S is weakly right reversible.

Theorem 2.4 *For any pomonoid S, the following statements are equivalent:*

(i) All principally weakly flat right Rees factor S-posets satisfy condition (P_w);

(ii) S is weakly right reversible and each proper left stabilizing convex right ideal K with $|K| > 1$ satisfies property (X).

Proof (i)$\Rightarrow$ (ii). Since Θ is principally weakly flat, Θ satisfies condition (P_w) and so S is weakly right reversible. Now, suppose that K is a proper left stabilizing convex right ideal with $|K| > 1$. Then by [5, Proposition 9], S/K is principally weakly flat, and by assumption S/K satisfies condition (P_w). Thus in light of Lemma 2.3, K has property (X).

(ii)$\Rightarrow$ (i). As we mentioned, Θ is principally weakly flat, and since S is weakly right reversible

Θ satisfies condition (P_w). Now, suppose that S/K is principally weakly flat for a proper convex right ideal K. Using [1, Proposition 9], K is a proper left stabilizing ideal. If $|K| = 1$, then $S/K = S$ satisfies condition (P_w). Otherwise, by Lemma 2.3, S/K satisfies condition (P_w). $\square$

In [5], it was shown that S/K is principally weakly po-flat for a proper convex right ideal K if and only if K is strongly left stabilizing. Recall that Θ is also principally weakly po-flat. Therefore, similar to the previous argument one can prove the following.

Proposition 2.5 *For any pomonoid S, the following statements are equivalent:*

(i) All principally weakly po-flat right Rees factor S-posets satisfy condition (P_w);

(ii) S is weakly right reversible and each proper strongly left stabilizing convex right ideal K with $|K| > 1$ satisfies property (X).

Theorem 2.6 *For any pomonoid S, the following statements are equivalent:*

(i) All weakly flat right Rees factor S-posets satisfy condition (P_w);

(ii) If S is weakly right reversible, then each proper left stabilizing convex right ideal K with $|K| > 1$ satisfies property (X).

Proof (i)$\Rightarrow$ (ii). Suppose that S is weakly right reversible and K is a proper left stabilizing convex right ideal with $|K| > 1$. Then by [1, Proposition 14], S/K is weakly flat, and by assumption S/K satisfies condition (P_w). Hence, by Lemma 2.3, K has property (X).

(ii)$\Rightarrow$ (i). Suppose that S/K is weakly flat for a proper convex right ideal K. Using [1, Proposition 14], K is a proper left stabilizing ideal and S is weakly right reversible. So by assumption $|K| = 1$ or K has property (X). Thus by Lemma 2.3, S/K satisfies condition (P_w). $\square$

By [1, Proposition 13], S/K is weakly po-flat for a proper convex right ideal K if and only if K is strongly left stabilizing and S is weakly right reversible. Hence, similarly the next proposition holds.

Proposition 2.7 *For any pomonoid S, the following statements are equivalent:*

(i) All weakly po-flat right Rees factor S-posets satisfy condition (P_w);

(ii) If S is weakly right reversible, then each proper strongly left stabilizing convex right ideal K with $|K| > 1$ satisfies property (X).

3. When condition (P_w) implies flatness properties

In this section, we consider Rees factor S-posets when condition (P_w) implies some other flatness properties. Recall that for a proper, convex right ideal K, S/K satisfies condition (P) if and only if $|K| = 1$.

Lemma 3.1 *If a pomonoid S has a convex, proper right ideal K such that S/K satisfies condition (P_w), then S is weakly right reversible.*

Proof Suppose that $s, t \in S$ and $k \in K$. Then $[k]s = [k]t$ in S/K_S, by condition (P_w), there exist $[a] \in S/K_S,\ u, v \in S$ such that $[k] \leq [a]u,\ [a]v \leq [k],\ us \leq vt$, and we are done. □

Theorem 3.2 *For any pomonoid S, the following statements are equivalent:*

(i) All right Rees factor S-posets satisfying condition (P_w) also satisfy condition (P);

(ii) If S is a weakly right reversible pomonoid, then S has no proper, convex right ideal K having property (X) with $|K| > 1$.

Proof (i)⇒ (ii). Suppose that S is weakly right reversible, and S has a proper convex right ideal K with $|K| > 1$ and property (X). Then by Lemma 2.3, S/K satisfies condition (P_w) and by assumption S/K satisfies condition (P). So $|K| = 1$, a contradiction is obtained.

(ii)⇒ (i). Let K be a convex right ideal and S/K satisfies condition (P_w). If $K = S$, $S/K = \Theta$ satisfies condition (P_w) if and only if it satisfies condition (P). Now, suppose that K is a proper convex right ideal of the pomonoid S, and S/K satisfies condition (P_w). By previous lemma, S is weakly right reversible, and by Lemma 2.3, K is a proper, convex right ideal such that $|K| = 1$ or K satisfies property (X). By assumption $|K| = 1$ and S/K satisfies condition (P). □

Notice that Θ is strongly flat if and only if S is left collapsible. Moreover, if K is a proper convex right ideal such that S/K is strongly flat, then $|K| = 1$. Hence in a similar way we have the following.

Proposition 3.3 *For any pomonoid S, the following statements are equivalent:*

(i) All right Rees factor S-posets satisfying condition (P_w) are strongly flat;

(ii) If S is weakly right reversible, then S is left collapsible, and S has no proper, convex right ideal K with property (X) and $|K| > 1$.

If K is a convex, proper right ideal of S, then S/K is projective if and only if $|K| = 1$. But if $K = S$, $S/K = \Theta$ is projective if and only if S has a left zero element. Thus by the similar argument we can deduce the next result.

Proposition 3.4 *For any pomonoid S, the following statements are equivalent:*

(i) All right Rees factor S-posets satisfying condition (P_w) are projective;

(ii) If S is weakly right reversible, then S contains a left zero, and S has no proper convex right ideal K with property (X) and $|K| > 1$.

A convex, proper right ideal K of a pomonoid S is called w-strongly left annihilating, if $[x]_{\rho_K} t \leq [y]_{\rho_K} t$ for any $x, y \in S \setminus K$ and $t \in S$, there exist $u, v \in S$, and $k, k', l, l' \in K$ such that one of the following four conditions is satisfied:

(a) $x \leq u,\ v \leq y$ and $ut \leq vt$;

(b) $x \leq u,\ v \leq l,\ l' \leq y$ and $ut \leq vt$;

(c) $x \leq k,\ k' \leq u,\ v \leq y$ and $ut \leq vt$;

(d) $x \leq k,\ k' \leq u,\ v \leq l,\ l' \leq y$ and $ut \leq vt$.

In [8], it was proved that the Rees factor S-poset S/K satisfies condition $(PWP)_w$ if and only if K is strongly left stabilizing and w-strongly left annihilating. The following lemma gives a slightly different condition.

Lemma 3.5 *Let K be a convex, proper right ideal of the pomonoid S. Then S/K satisfies condition $(PWP)_w$ if, and only if,*

(i) K is strongly left stabilizing;

(ii) For each $t \in S$ and $x, y \in S \setminus K$ with $(\forall k \in K)$ $(x \not\leq k,\ y \not\geq k)$, there exist $u, v \in S$ such that $x \leq u,\ v \leq y$, and $ut \leq vt$.

Proof Necessity. Suppose that S/K satisfies condition $(PWP)_w$ for a proper convex right ideal K. Let $t \in S$ and $x, y \in S \setminus K$ be such that $(\forall k \in K)(x \not\leq k,\ y \not\geq k)$. Then since parts (b), (c) and (d) of definition of w-strongly left annihilating are not valid for x, y, by part (a) of that definition, there exist $u, v \in S$ such that $x \leq u,\ v \leq y$, and $ut \leq vt$, as required.

Sufficiency. Suppose that $[xt]_{\rho K} \leq [yt]_{\rho K}$ for $x, y, t \in S$. Then we have $xt \leq yt$, or $xt \leq k$ and $k' \leq yt$ for $k, k' \in K$. If $xt \leq yt$, then take $u = x, y = v$. Otherwise, there are the following four cases:

(1) If $x, y \in K$, then take $u = v = x$.

(2) Let $x \in K,\ y \notin K$. Since $k' \leq yt$, by (i), there exists $k'' \in K$ such that $k''yt \leq yt$, and so take $u = k''y$ and $v = y$.

(3) If $x \notin K, y \in K$, it is similar to (2).

(4) Let $x, y \notin K$. Now, we have three cases:

(a) Assume that there exists $l \in K$ with $x \leq l$. Since $k' \leq yt$, by (i), there exists $k'' \in K$ such that $k''yt \leq yt$. Take $u = k''y$ and $v = y$. We have $x \leq l,\ k''y \leq k''y$, so $[x]_{\rho K} \leq [u]_{\rho K}$, and the result follows.

(b) If there exists $l \in K$ with $l \leq y$, it is analogous to (a).

(c) If $x, y \in S \setminus K$ such that $(\forall k \in K)(x \not\leq k,\ y \not\geq k)$, then by (ii) there exist $u, v \in S$ such that $x < u,\ v < y$, and $ut < vt$. Thus we are done. $\square$

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