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Mixed Generalized Jacobi and Chebyshev Collocation Method for Time-Fractional Convection-Diffusion Equations

Tao SUN
School of Statistics and Mathematics, Shanghai Lixin University of Accounting and Finance, Shanghai 201209*, P. R. China*

Abstract In this paper, we study an efficient higher order numerical method to time-fractional partial differential equations with temporal Caputo derivative. A collocation method based on shifted generalized Jacobi functions is taken for approximating the solution of the given time-fractional partial differential equation in time and a shifted Chebyshev collocation method based on operational matrix in space. The derived numerical solution can approximate the non-smooth solution in time of given equations well. Some numerical examples are presented to illustrate the efficiency and accuracy of the proposed method.

Keywords time-fractional convection-diffusion equations; collocation methods; shifted generalized Jacobi functions; shifted Chebyshev polynomials

MR(2010) Subject Classification 65M70; 65N35; 35R11

1. Introduction

Fractional differential operators appear in many systems in science and engineering. Because of the nonlocal property of fractional derivative, they can be utilized for modeling of memory-dependent phenomena and complex media such as porous media and anomalous diffusion [1–4], fractured media [5], electrochemical processes [6], and viscoelastic materials [7,8]. They also appear in modeling diverse physical problems involving e.g., viscous fluid flows subject to wall-friction effects [9–11], bioengineering applications [12], and even finance [13,14].

In the last two decades, extensive research has been carried out on the development of numerical methods for fractional partial differential equations (FPDEs), such as finite difference methods [15–18], finite element methods [19–23], and spectral methods [24–27] for numerically solving the FPDEs. Spectral methods are efficient approaches for numerical solution of FPDEs, due to the being global of fractional operator and the being global of basis functions of the methods. Li and Xu [24] developed a space-time spectral method for time-fractional diffusion equation, which is a fundamental work on spectral methods for FPDEs. Hanert and Piret [28] developed a pseudospectral scheme to discretize the space-time fractional diffusion equation with exponential tempering in both space and time. The model solution was expanded in both space

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E-mail address: taosun80@yeah.net

and time in terms of Chebyshev polynomials and the discrete equations were obtained with the Galerkin method.

Galerkin projection type schemes in general have difficulties in the treatment of nonlinear or multiterm FPDEs, and even in treating FPDEs with variable coefficients, since no straightforward variational form can be efficiently obtained for such problems. The collocation schemes for fractional equations are relatively easy to implement and they can overcome the aforementioned challenges. The idea of collocation was proposed by Khader in [29], who presented a Chebyshev collocation method for the discretization of the space-fractional diffusion equation. More recently, Zayernouri and Karniadakis in [30] and [31] developed an exponentially accurate fractional spectral collocation method for solving steady-state and time-dependent FPDEs, and linear/nonlinear FPDEs with field-variable order, respectively.

In this paper, we consider the following time-fractional convection-diffusion equation with variable coefficients:

$$ {}_0\mathcal{D}_t^{\alpha}U(x,t) + a(x)\partial_x U(x,t) + b(x)\partial_x^2 U(x,t) = f(x,t), \quad x \in (0,L), \ t \in (0,T], \tag{1.1} $$

with initial and Dirichlet boundary conditions

$$ U(x,0) = g(x), \quad x \in (0,L), \tag{1.2} $$

$$ U(0,t) = h_1(t), \quad U(L,t) = h_2(t), \ t \in (0,T], \tag{1.3} $$

where $\alpha \in (0,1]$, $a(x)$ and $b(x)$ are continuous functions, which satisfy $(a,b) \neq (0,0)$. Here, the time-fractional derivative is defined as the Caputo fractional derivative. Saadatmandi et al. [32] used the Sinc-Legendre collocation method for the solution of (1.1) with homogeneous boundary conditions, and Uddin and Haq [33] applied radial basis functions for solving this problem with constant coefficients. Authors of [34] developed finite difference/element approaches to solve (1.1) on the condition that $a(x) = 0$, $b(x) = -c$, $c \in \mathbb{R}^+$. For the case of $a(x) = 0$ and $b(x) = -1$, numerical methods based on finite difference and finite element approaches can be found in [35]. The author of [22] developed implicit unconditionally stable numerical methods to solve (1.1) with $a(x) = 0$, $b(x) = -1$ and $f(x,t) = 0$. More recently, Mohammad and Jafar [36] proposed the Gegenbauer spectral method to derive the numerical solution of (1.1) with $L = 1$.

The paper is devoted to a higher order numerical method for problem (1.1)–(1.3). We extend the application of spectral methods with shifted generalized Jacobi functions and Chebyshev polynomials for solution of time-fractional partial differential equations. The rest of this paper is arranged as follows. In Section 2, we introduce the Riemann-Liouville and Caputo type of fractional derivative and their relations, the shifted Chebyshev polynomials and the shifted generalized Jacobi functions, and their properties are discussed, too. In Section 3, we introduce the mixed generalized Jacobi and Chebyshev collocation method and use it to solve the time-fractional convection-diffusion equations. In Section 4, we describe the numerical implementations of the proposed method and present some numerical results to show the efficiency and high accuracy of proposed numerical method. The final section is for some concluding remarks.

2. Preliminaries

2.1. Notation and definitions

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. We give some definitions and properties of the fractional calculus.

Definition 2.1 ([37]) *The left-sided Riemann-Liouville fractional derivative of order $\nu \in (m-1, m)$ with $m \in \mathbb{N}$, is defined as*

$$ {}_0^{RL}\mathcal{D}_x^{\nu} f(x) = \frac{1}{\Gamma(m-\nu)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \int_0^x \frac{f(t)}{(x-t)^{\nu-m+1}} \mathrm{d}t, \quad x > 0, \tag{2.1}$$

and the left-sided Caputo fractional derivative of order $\nu \in (m-1, m)$ with $m \in \mathbb{N}$, is defined as

$$ {}_0^{C}\mathcal{D}_x^{\nu} f(x) = \frac{1}{\Gamma(m-\nu)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\nu-m+1}} \mathrm{d}t, \quad x > 0. \tag{2.2}$$

Due to the definition of the left-sided fractional integral of order $\rho > 0$ (see [37]):

$$ {}_0\mathcal{I}_x^{\rho} f(x) = \frac{1}{\Gamma(\rho)} \int_0^x \frac{f(t)}{(x-t)^{1-\rho}} \mathrm{d}t, \quad x > 0, \tag{2.3}$$

we know that for $\nu \in (m-1, m)$ with $m \in \mathbb{N}$, there hold

$$ {}_0^{RL}\mathcal{D}_x^{\nu} f(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m} ({}_0\mathcal{I}_x^{m-\nu} f(x)), \quad {}_0^{C}\mathcal{D}_x^{\nu} f(x) = {}_0\mathcal{I}_x^{m-\nu} f^{(m)}(x). \tag{2.4}$$

Note that, if $\nu = m \in \mathbb{N}$, we have ${}_0^{RL}\mathcal{D}_x^{\nu} = {}_0^{C}\mathcal{D}_x^{\nu} = \mathrm{d}^m/\mathrm{d}x^m$.

Generally speaking, the Caputo derivative and the Riemman-Liouville derivative of the same order $\nu \in (m-1, m)$ are not equivalent. However, under certain smooth conditions, there exists a link between them, which is given below

$$ {}_0^{RL}\mathcal{D}_x^{\nu} f(x) = {}_0^{C}\mathcal{D}_x^{\nu} f(x) + \sum_{k=0}^{m-1} \frac{f^{(k)}(0) x^{k-\nu}}{\Gamma(k+1-\nu)}. \tag{2.5}$$

The Caputo's fractional derivative is a linear operation, i.e., for any constants κ and λ,

$$ {}_0^{C}\mathcal{D}_x^{\nu} (\kappa f_1(x) + \lambda f_2(x)) = \kappa {}_0^{C}\mathcal{D}_x^{\nu} f_1(x) + \lambda {}_0^{C}\mathcal{D}_x^{\nu} f_2(x). \tag{2.6}$$

For the Caputo derivative we have, ${}_0^{C}\mathcal{D}_x^{\nu} C = 0$ for $C \in \mathbb{R}$, and

$$ {}_0^{C}\mathcal{D}_x^{\nu} x^{\beta} = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil \nu \rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)} x^{\beta-\nu}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \geq \lceil \nu \rceil \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > \lfloor \nu \rfloor, \end{cases} \tag{2.7}$$

where the ceiling function $\lceil \nu \rceil$ denotes the smallest integer greater than or equal to ν and the floor function $\lfloor \nu \rfloor$ denotes the largest integer less than or equal to ν.

The term ${}_0\mathcal{D}_t^{\alpha} U(x,t)$ in (1.1) is the Caputo fractional derivative of order $\alpha \in (0, 1]$ in time and is defined as

$$ {}_0\mathcal{D}_t^{\alpha} U(x,t) := {}_0^{C}\mathcal{D}_t^{\alpha} U(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial U(x,s)}{\partial s} \frac{\mathrm{d}s}{(t-s)^{\alpha}}. \tag{2.8}$$

2.2. Properties of shifted Chebyshev polynomials

The Chebyshev polynomials $T_i(\xi)$, $i = 0, 1, \ldots$ are defined on the interval $[-1, 1]$. In order to use these polynomials on the interval $[0, L]$, we define the shifted Chebyshev polynomials by introducing the change of variable $\xi = \frac{2x}{L} - 1$. Denote by $T_{L,i}(x) := T_i(\frac{2x}{L} - 1)$ the shifted Chebyshev polynomials, which satisfy the orthogonality relation

$$\int_0^L T_{L,j}(x)T_{L,k}(x)\omega_L(x)\mathrm{d}x = \hbar_j\delta_{jk}, \tag{2.9}$$

where δ_{jk} is the Kronecker symbol, $\omega_L(x) = \frac{1}{\sqrt{Lx - x^2}}$ and $\hbar_j = \frac{\epsilon_j}{2}\pi$, $\epsilon_0 = 2, \epsilon_j = 1$, $j \geq 1$.

The analytic form of the shifted Chebyshev polynomials $T_{L,i}(x)$ of degree i is given by

$$T_{L,i}(x) = i\sum_{k=0}^{i}(-1)^{i-k}\frac{(i+k-1)!2^{2k}}{(i-k)!(2k)!L^k}x^k, \tag{2.10}$$

where $T_{L,i}(0) = (-1)^i$ and $T_{L,i}(L) = 1$. The shifted Chebyshev polynomials $T_{L,i}(x)$ satisfy the following recurrence formula:

$$\begin{aligned} &T_{L,i+1}(x) = 2(\frac{2x}{L} - 1)T_{L,i}(x) - T_{L,i-1}(x), \quad i \geq 1, \\ &T_{L,0}(x) = 1, \quad T_{L,1}(x) = \frac{2x}{L} - 1. \end{aligned} \tag{2.11}$$

We now introduce the weighted Sobolev space

$$L^2_{\omega_L}(0, L) = \{v \,|\, v \text{ is measurable on } \ (0, L) \text{ and } \|v\|_{\omega_L} < \infty\}, \tag{2.12}$$

equipped with the following norm,

$$\|v\|_{\omega_L} = \Big(\int_0^L v^2(x)\omega_L(x)\mathrm{d}x\Big)^{\frac{1}{2}}. \tag{2.13}$$

For any function $u \in L^2_{\omega_L}(0, L)$, which may be expressed in terms of the shifted Chebyshev polynomials as

$$u(x) = \sum_{i=0}^{\infty} c_i T_{L,i}(x), \tag{2.14}$$

where the coefficients c_i are given by

$$c_i = \frac{1}{\hbar_i}\int_0^L u(x)T_{L,i}(x)\omega_L(x)\mathrm{d}x, \quad i = 0, 1, \ldots. \tag{2.15}$$

We set

$$u_M(x) = \sum_{i=0}^{M} c_i T_{L,i}(x) = \vec{\mathbf{C}}\vec{\Phi}(x), \tag{2.16}$$

where

$$\vec{\mathbf{C}} = [c_0, c_1, \ldots, c_M], \quad \vec{\Phi}(x) = [T_{L,0}(x), T_{L,1}(x), \ldots, T_{L,M}(x)]^T. \tag{2.17}$$

Lemma 2.2 ([38]) *The derivative of the vector $\vec{\Phi}(x)$ can be expressed by*

$$\frac{\mathrm{d}\vec{\Phi}(x)}{\mathrm{d}x} = \mathbf{D}^{(1)}\vec{\Phi}(x), \tag{2.18}$$

where $\mathbf{D}^{(1)}$ *is the* $(M+1)(M+1)$ *operational matrix of derivative given by*

$$\mathbf{D}^{(1)} = (d_{i,j}^{(1)}) = \begin{cases} \frac{4i}{\epsilon_j L}, & j = 0,1,\dots, i = j+k, \\ 0, & \textit{otherwise}, \end{cases} \tag{2.19}$$

with

$$k = \begin{cases} 1,3,5,\dots,M, & \textit{if } M \textit{ is odd}, \\ 1,3,5,\dots,M-1, & \textit{if } M \textit{ is even}. \end{cases}$$

Corollary 2.3 *For* $n \in \mathbb{N}$, *there hold*

$$\frac{\mathrm{d}^n u_M(x)}{\mathrm{d}x^n} = \vec{\mathbf{C}}\mathbf{D}^{(n)}\vec{\Phi}(x), \ \textit{with } \mathbf{D}^{(n)} = (\mathbf{D}^{(1)})^n. \tag{2.20}$$

Proof By using (2.18), it is clear that

$$\frac{\mathrm{d}^n \vec{\Phi}(x)}{\mathrm{d}x^n} = (\mathbf{D}^{(1)})^n \vec{\Phi}(x),$$

thus inserting the above equation into (2.16) leads to

$$\frac{\mathrm{d}^n u_M(x)}{\mathrm{d}x^n} = \sum_{i=0}^{M} c_i \frac{\mathrm{d}^n T_{L,i}(x)}{\mathrm{d}x^n} = \vec{\mathbf{C}}\frac{\mathrm{d}^n \vec{\Phi}(x)}{\mathrm{d}x^n} = \vec{\mathbf{C}}\mathbf{D}^{(n)}\vec{\Phi}(x). \ \Box$$

Remark 2.4 For notational convenience, we denote the entries of the operational matrix of nth-order derivative as $d_{i,j}^{(n)}$, i.e., $\mathbf{D}^{(n)} = (d_{i,j}^{(n)})$, $n \in \mathbb{N}$ which will be used in Section 3.

2.3. Properties of shifted generalized Jacobi functions

We introduce the following generalized Jacobi functions.

Definition 2.5 ([30,41]) *For all* $\eta \in [-1,1]$ *and* $j \in \mathbb{N}_0$, *the generalized Jacobi functions (GJFs) are defined as*

$$\mathcal{J}_j^{\sigma,-\lambda}(\eta) := (1+\eta)^\lambda J_j^{\sigma,\lambda}(\eta), \ \textit{for } \sigma,\lambda > -1, \tag{2.21}$$

where $J_j^{\sigma,\lambda}(\eta)$ *is the standard Jacobi polynomial of order* j *on* $[-1,1]$.

Through the transformation $\eta = \frac{2t}{T} - 1$, one easily obtains the shifted generalized Jacobi functions (SGJFs) on $[0,T]$:

$$\mathcal{J}_{T,j}^{\sigma,-\lambda}(t) = (\frac{2}{T})^\lambda t^\lambda J_{T,j}^{\sigma,\lambda}(t), \tag{2.22}$$

where $J_{T,j}^{\sigma,\lambda}(t)$ is the shifted Jacobi polynomial of order j on $[0,T]$. Obviously, $\mathcal{J}_{T,j}^{\sigma,-\lambda}(0) = 0$, if $\lambda > 0$.

The SGJFs $\mathcal{J}_{T,j}^{\sigma,-\lambda}(t)$ satisfy the following recurrence formula:

$$\begin{aligned} &a_j^{\sigma,\lambda}\mathcal{J}_{T,j+1}^{\sigma,-\lambda}(t) = \big(b_j^{\sigma,\lambda}(\frac{2t}{T}-1\big) - c_j^{\sigma,\lambda})\mathcal{J}_{T,j}^{\sigma,-\lambda}(t) - d_j^{\sigma,\lambda}\mathcal{J}_{T,j-1}^{\sigma,-\lambda}(t), \quad j \geq 1, \\ &\mathcal{J}_{T,0}^{\sigma,-\lambda}(t) = (\frac{2t}{T})^\lambda, \quad \mathcal{J}_{T,1}^{\sigma,-\lambda}(t) = (\frac{2t}{T})^\lambda((\sigma+\lambda+2)\frac{t}{T} - (\lambda+1)), \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} a_j^{\sigma,\lambda} &= 2(j+1)(j+\sigma+\lambda+1)(2j+\sigma+\lambda), \\ b_j^{\sigma,\lambda} &= (2j+\sigma+\lambda)(2j+\sigma+\lambda+1)(2j+\sigma+\lambda+2), \end{aligned}$$

$$c_j^{\sigma,\lambda} = (\lambda^2 - \sigma^2)(2j + \sigma + \lambda + 1),$$
$$d_j^{\sigma,\lambda} = 2(j + \sigma)(j + \lambda)(2j + \sigma + \lambda + 2).$$

Set $\omega_T^{\sigma,-\lambda}(t) = (T - t)^{\sigma} t^{-\lambda}$, we introduce the other weighted Sobolev space

$$L^2_{\omega_T^{\sigma,-\lambda}}(0,T) = \{v \mid v \text{ is measurable on } (0,T) \text{ and } \|v\|_{\omega_T^{\sigma,-\lambda}} < \infty\}, \tag{2.24}$$

equipped with the following norm,

$$\|v\|_{\omega_T^{\sigma,-\lambda}} = \Big(\int_0^T v^2(t)\omega_T^{\sigma,-\lambda}(t)\mathrm{d}t\Big)^{\frac{1}{2}}. \tag{2.25}$$

Following the same line as in the derivation of [39, Remark 4.1], one can prove that $\{\mathcal{J}_{T,j}^{\sigma,-\lambda}(t), j = 0,1,\dots\}$ is dense in the Hilbert space and it forms a basis for $L^2_{\omega_T^{\sigma,-\lambda}}(0,T)$. Furthermore, the set of all the SGJFs $\mathcal{J}_{T,j}^{\sigma,-\lambda}(t)$ is a complete $L^2_{\omega_T^{\sigma,-\lambda}}(0,T)$-orthogonal system, namely,

$$\int_0^T \mathcal{J}_{T,j}^{\sigma,-\lambda}(t)\mathcal{J}_{T,k}^{\sigma,-\lambda}(t)\omega_T^{\sigma,-\lambda}(t)\mathrm{d}t = \gamma_{T,j}^{\sigma,-\lambda}\delta_{jk}, \tag{2.26}$$

where

$$\gamma_{T,j}^{\sigma,-\lambda} = \frac{T^{\sigma-\lambda+1}2^{2\lambda}\Gamma(j+\sigma+1)\Gamma(j+\lambda+1)}{(2j+\sigma+\lambda+1)j!\Gamma(j+\sigma+\lambda+1)}, \quad j = 0,1,\dots.$$

For any $v \in L^2_{\omega_T^{\sigma,-\lambda}(t)}(0,T)$, which may be expressed in terms of the SGJFs as

$$v(t) = \sum_{j=0}^{\infty} \tilde{c}_j \mathcal{J}_{T,j}^{\sigma,-\lambda}(t), \tag{2.27}$$

where the coefficients $\tilde{c}_j$ are given by

$$\tilde{c}_j = \frac{1}{\gamma_{T,j}^{\sigma,-\lambda}} \int_0^T v(x)\mathcal{J}_{T,j}^{\sigma,-\lambda}(t)\omega_T^{\sigma,-\lambda}(t)\mathrm{d}t, \quad j = 0,1,\dots. \tag{2.28}$$

Theorem 2.6 *Set*

$$v_N(t) = \sum_{j=0}^{N} \tilde{c}_j \mathcal{J}_{T,j}^{\sigma,-\lambda}(t), \ \textit{for } \sigma,\lambda > -1, \tag{2.29}$$

then for $\mu > 0$ *and* $\lambda > \max(\mu - 1, 0)$, *there holds*

$${}_0^C\mathcal{D}_t^{\mu} v_N(t) = (\frac{2}{T})^{\mu} \sum_{j=0}^{N} \tilde{c}_j \frac{\Gamma(j+\lambda+1)}{\Gamma(j+\lambda-\mu+1)} \mathcal{J}_{T,j}^{\sigma+\mu,-\lambda+\mu}(t). \tag{2.30}$$

Proof According to [39, Theorem 3.1], by using transformation $\eta = \frac{2t}{T} - 1$, we get that for $\mu > 0$ and $\lambda > \mu - 1$,

$${}_0^{RL}\mathcal{D}_t^{\mu}(\mathcal{J}_{T,j}^{\sigma,-\lambda}(t)) = (\frac{2}{T})^{\mu} \frac{\Gamma(j+\lambda+1)}{\Gamma(j+\lambda-\mu+1)} \mathcal{J}_{T,j}^{\sigma+\mu,-\lambda+\mu}(t). \tag{2.31}$$

On the other hand, (2.5) leads to that for $\lambda > \max(\mu - 1, 0)$,

$${}_0^C\mathcal{D}_t^{\mu}(\mathcal{J}_{T,j}^{\sigma,-\lambda}(t)) = {}_0^{RL}\mathcal{D}_t^{\mu}(\mathcal{J}_{T,j}^{\sigma,-\lambda}(t)). \tag{2.32}$$

Combining (2.31) and (2.32), we use (2.6) to derive the desired result (2.30) directly. □

Remark 2.7 It is important to point out that the use of generalized Jacobi polynomials as basis function for deriving efficient spectral algorithms for ordinary fractional differential equations can be found in [39]. The authors of [40] used shifted generalized Jacobi polynomials as basis function to propose a generalized Jacobi spectral-Galerkin method for the nonlinear Volterra integral equations with weakly singular kernels. The detailed approximation results on generalized Jacobi polynomials are also referred to [39,40] for theoretical evidences.

3. Mixed generalized Jacobi and Chebyshev collocation method

We are going to solve system (1.1)–(1.3) by using mixed generalized Jacobi and Chebyshev collocation scheme. In practice, we first split the exact solution $U(x,t)$ of the given problem as

$$U(x,t) = \widetilde{U}(x,t) + g(x). \tag{3.1}$$

By substituting (3.1) in system (1.1)–(1.3), we obtain that

$${}_0\mathcal{D}_t^{\alpha}\widetilde{U}(x,t) + a(x)\partial_x\widetilde{U}(x,t) + b(x)\partial_x^2\widetilde{U}(x,t) = \widetilde{f}(x,t), \quad x \in (0,L),\ t \in (0,T], \tag{3.2}$$

with homogeneous initial condition

$$\widetilde{U}(x,0) = 0, \quad x \in (0,L), \tag{3.3}$$

and inhomogeneous Dirichlet boundary conditions

$$\widetilde{U}(0,t) = h_1(t) - g(0), \quad \widetilde{U}(L,t) = h_2(t) - g(L),\ t \in (0,T] \tag{3.4}$$

where $\widetilde{f}(x,t) = f(x,t) - a(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x} - b(x)\frac{\mathrm{d}^2g(x)}{\mathrm{d}x^2}$.

In order to solve the new system (3.2)–(3.4) numerically, we approximate $\widetilde{U}(x,t)$ by $(M+1)$-terms shifted Chebyshev polynomials and $(N+1)$-terms SGJFs as

$$\widetilde{U}_{M,N}(x,t) = \sum_{i=0}^{M}\sum_{j=0}^{N} c_{i,j} T_{L,i}(x)\mathcal{J}_{T,j}^{\sigma,-\lambda}(t), \quad \lambda > 0. \tag{3.5}$$

Substituting (3.5) in (3.2), we obtain

$${}_0\mathcal{D}_t^{\alpha}\widetilde{U}_{M,N}(x,t) + a(x)\partial_x\widetilde{U}_{M,N}(x,t) + b(x)\partial_x^2\widetilde{U}_{M,N}(x,t) = \widetilde{f}(x,t). \tag{3.6}$$

We now collocate (3.6) in certain nodes. We take the Chebyshev-Gauss-Lobatto collocation nodes associated with interval $[0,L]$ for spatial collocation, that is

$$x_k = \frac{L}{2}\big(1 - \cos(\frac{k\pi}{M})\big), \quad 0 \le k \le M. \tag{3.7}$$

The mixed generalized Jacobi and Chebyshev collocation method in spatial is seeking $\widetilde{U}_{M,N}(x,t)$ such that

$$\widetilde{U}_{M,N}(0,t) = h_1(t) - g(0), \quad \widetilde{U}_{M,N}(L,t) = h_2(t) - g(L), \tag{3.8}$$

and that the equation holds at the interior collocation points $x_k,\ 1 \le k \le M-1$:

$${}_0\mathcal{D}_t^{\alpha}\widetilde{U}_{M,N}(x_k,t) + a(x_k)\partial_x\widetilde{U}_{M,N}(x_k,t) + b(x_k)\partial_x^2\widetilde{U}_{M,N}(x_k,t) = \widetilde{f}(x_k,t). \tag{3.9}$$

For suitable collocation points in time, we take the Jacobi-Gauss-Radau collocation nodes $t_l, 0 \le l \le N$ associated with interval $(0, T]$, which are the $N+1$ zeros of $(T-t)J_N^{(\sigma+1,\lambda)}(t)$. By using (2.16), (2.20), (2.30) and (3.9), we have that for $1 \le k \le M-1,\ 0 \le l \le N$

$$\begin{aligned}&\sum_{i=0}^{M}\sum_{j=0}^{N} c_{i,j} T_{L,i}(x_k)(\frac{2}{T})^{\alpha}\frac{\Gamma(j+\lambda+1)}{\Gamma(j+\lambda-\alpha+1)}\mathcal{J}_{T,j}^{\sigma+\alpha,-\lambda+\alpha}(t_l)+\\ &\quad\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{r=0}^{M} c_{i,j}\big(d_{i,r}^{(1)}a(x_k)+d_{i,r}^{(2)}b(x_k)\big)T_{L,r}(x_k)\mathcal{J}_{T,j}^{\sigma,-\lambda}(t_l)=\widetilde{f}(x_k,t_l). \qquad (3.10)\end{aligned}$$

Furthermore, collocating (3.8) in $N+1$ points t_l, and considering the properties of the shifted Chebyshev polynomials, we have that for $1 \le k \le M-1,\ 0 \le l \le N$

$$\sum_{i=0}^{M}\sum_{j=0}^{N} c_{i,j}(-1)^i\mathcal{J}_{T,j}^{\sigma,-\lambda}(t_l)=h_1(t_l)-g(0),\quad \sum_{i=0}^{M}\sum_{j=0}^{N} c_{i,j}\mathcal{J}_{T,j}^{\sigma,-\lambda}(t_l)=h_2(t_l)-g(L). \qquad (3.11)$$

The $(M+1)(N+1)$ unknown coefficients $c_{i,j}$ can be obtained from system (3.10) and (3.11). Consequently, $\widetilde{U}_{M,N}(x,t)$ given in (3.5) can be calculated.

To solve system (3.10) and (3.11) efficiently, we shall recast them in a more convenient form. Let us make the following notations of some matrices

$$\begin{aligned}&\mathbf{A}_{11}=(T_{L,i}(x_k))_{1\le k\le M-1,0\le i\le M},\\ &\mathbf{A}_{12}=(a(x_k)T_{L,r}(x_k))_{1\le k\le M-1,0\le r\le M},\\ &\mathbf{A}_{13}=(b(x_k)T_{L,r}(x_k))_{1\le k\le M-1,0\le r\le M},\\ &\mathbf{B}_{11}=(\frac{2}{T})^{\alpha}\big(\frac{\Gamma(j+\lambda+1)}{\Gamma(j+\lambda-\alpha+1)}\mathcal{J}_{T,j}^{\sigma+\alpha,-\lambda+\alpha}(t_l)\big)_{0\le l,j\le N},\\ &\mathbf{B}_{12}=\big(\mathcal{J}_{T,j}^{\sigma,-\lambda}(t_l)\big)_{0\le l,j\le N},\end{aligned} \qquad (3.12)$$

and denote some vectors by

$$\begin{aligned}&\vec{\mathbf{a}}_1=[(-1)^0,(-1)^1,\ldots,(-1)^M],\quad \vec{\mathbf{a}}_2=[1^0,1^1,\ldots,1^M],\\ &\vec{\mathbf{c}}=[c_{0,0},c_{0,1},\ldots,c_{0,N},\ldots,c_{M,0},c_{M,1},\ldots,c_{M,N}]^T,\\ &\vec{\mathbf{f}}=[\widetilde{f}_{1,0},\widetilde{f}_{1,1},\ldots,\widetilde{f}_{1,N},\ldots,\widetilde{f}_{M-1,0},\widetilde{f}_{M-1,1},\ldots,\widetilde{f}_{M-1,N}]^T,\\ &\vec{\mathbf{h}}_1=[h_{1,0},h_{1,1},\ldots,h_{1,N}]^T,\quad \vec{\mathbf{h}}_2=[h_{2,0},h_{2,1},\ldots,h_{2,N}]^T,\end{aligned} \qquad (3.13)$$

where $\widetilde{f}_{k,l}=\widetilde{f}(x_k,t_l)$, $1\le k\le M-1, 0\le l\le N$, $h_{1,l}=h_1(t_l)-g(0)$ and $h_{2,l}=h_2(t_l)-g(L)$, $0\le l\le N$.

Then we can make use of the Kronecker product (represented by $\otimes$) to express system (3.10) and (3.11) as the following matrix system

$$\begin{pmatrix}\mathbf{A}_{11}\otimes\mathbf{B}_{11}+\Big(\mathbf{A}_{12}\big(\mathbf{D}^{(1)}\big)^T+\mathbf{A}_{13}\big(\mathbf{D}^{(2)}\big)^T\Big)\otimes\mathbf{B}_{12}\\ \vec{\mathbf{a}}_1\otimes\mathbf{B}_{12}\\ \vec{\mathbf{a}}_2\otimes\mathbf{B}_{12}\end{pmatrix}\vec{\mathbf{c}}=\begin{pmatrix}\vec{\mathbf{f}}\\ \vec{\mathbf{h}}_1\\ \vec{\mathbf{h}}_2\end{pmatrix}. \qquad (3.14)$$

Remark 3.1 Under the following Neumann boundary conditions

$$\partial_x U(0,t)=h_1(t),\quad \partial_x U(L,t)=h_2(t),\ t\in(0,T], \qquad (3.15)$$

instead of Dirichlet boundary conditions (1.3), a similar matrix system will be solved as

$$\begin{pmatrix} \mathbf{A}_{11}\otimes\mathbf{B}_{11}+\Big(\mathbf{A}_{12}\big(\mathbf{D}^{(1)}\big)^T+\mathbf{A}_{13}\big(\mathbf{D}^{(2)}\big)^T\Big)\otimes\mathbf{B}_{12} \\ \Big(\vec{\mathbf{a}}_1\big(\mathbf{D}^{(1)}\big)^T\Big)\otimes\mathbf{B}_{12} \\ \Big(\vec{\mathbf{a}}_2\big(\mathbf{D}^{(1)}\big)^T\Big)\otimes\mathbf{B}_{12} \end{pmatrix}\vec{\mathbf{c}}=\begin{pmatrix} \vec{\mathbf{f}} \\ \vec{\mathbf{h}}_1 \\ \vec{\mathbf{h}}_2 \end{pmatrix}, \tag{3.16}$$

where new vectors $\vec{\mathbf{h}}_1$ and $\vec{\mathbf{h}}_2$ are denoted by

$$\vec{\mathbf{h}}_1=[\tilde{h}_{1,0},\tilde{h}_{1,1},\dots,\tilde{h}_{1,N}]^T,\quad \vec{\mathbf{h}}_2=[\tilde{h}_{2,0},\tilde{h}_{2,1},\dots,\tilde{h}_{2,N}]^T, \tag{3.17}$$

with $\tilde{h}_{1,l}=h_1(t_l)-\frac{\mathrm{d}g(0)}{\mathrm{d}x}$ and $\tilde{h}_{2,l}=h_2(t_l)-\frac{\mathrm{d}g(L)}{\mathrm{d}x}$, $0\le l\le N$.

4. Numerical results

In this section, we describe the implementations for the mixed generalized Jacobi and Chebyshev collocation scheme (3.10) and (3.11). For measuring numerical errors, let $\omega_k^{M_1}$ and $\omega_l^{M_2}$ denote the corresponding weights of Chebyshev-Gauss-Lobatto and Jacobi-Gauss-Radau quadrature with respect to variables $x_k\in[0,L]$ and $t_l\in(0,T]$, respectively. We denote the following discrete weighted L^2-norm error as

$$E_{L^2}=\Big(\frac{L}{2}\frac{T}{2}\sum_{k=0}^{M_1}\sum_{l=0}^{M_2}(\widetilde{U}(x_k,t_l)-\widetilde{U}_{M,N}(x_k,t_l))^2\omega_k^{M_1}\omega_l^{M_2}\Big)^{\frac{1}{2}}, \tag{4.1}$$

and the numerical absolute error in maximum norm is denoted by

$$E_{\mathrm{Max}}=\max_{0\le k\le M_1,0\le l\le M_2}|\widetilde{U}(x_k,t_l)-\widetilde{U}_{M,N}(x_k,t_l)|. \tag{4.2}$$

We take $M_1=M+10, M_2=N+10$ in the following numerical implementations.

Example 4.1 We first consider the following time-fractional diffusion equation of order $\alpha\in(0,1]$, with homogeneous initial and Dirichlet boundary conditions

$$\begin{cases} {}_0\mathcal{D}_t^{\alpha}U(x,t)-\frac{\partial^2U(x,t)}{\partial x^2}=f(x,t), & x\in(0,L),\ t\in(0,T], \\ U(x,0)=0, & x\in(0,L), \\ U(0,t)=0,\quad U(L,t)=0, & t\in(0,T], \end{cases} \tag{4.3}$$

where $f(x,t)=(\frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)}t^{\mu-\alpha}+4\pi^2t^{\mu})\sin(2\pi x)$, $\mu\ge\alpha$. The exact solution of problem (4.3) is given by [20]

$$U(x,t)=t^{\mu}\sin(2\pi x). \tag{4.4}$$

In Figure 1, we plot $\log_{10}E_{L^2}$ and $\log_{10}E_{\mathrm{Max}}$ versus M with fixed $N=0$, $\lambda=0.9$ and arbitrary $\sigma>-1$ of problem (4.3) for $\mu=0.9$ and $L=T=1$ with $\alpha=0.3,\ 0.5$ and 0.8. All of them decay rapidly as M increases. Meanwhile, comparing these results with the absolute error e_n vs n obtained in [42], one can see that our method derives more accurate numerical solution for the same larger mode in space.

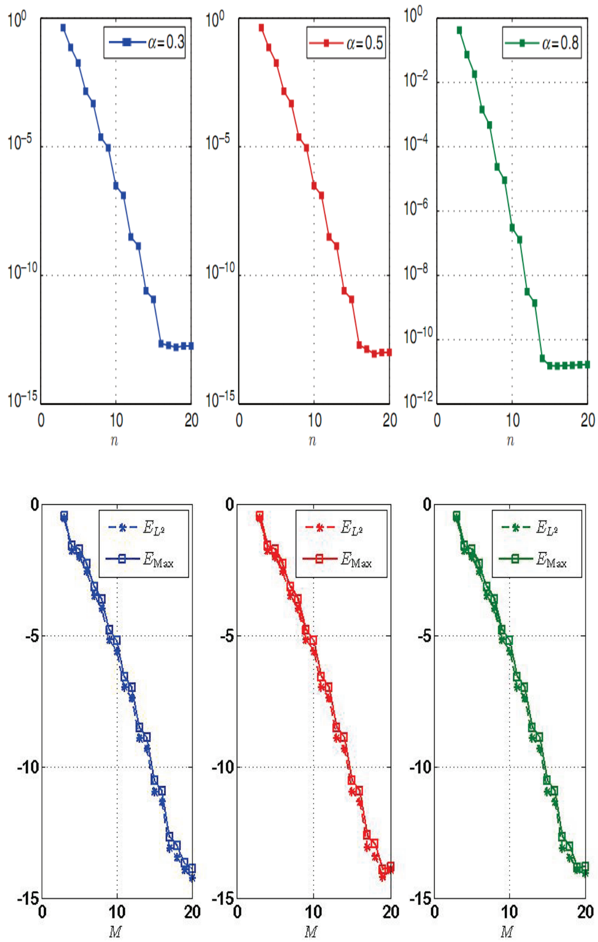

Figure 1 Absolute error e_n vs n given in [42] (top), $\log_{10}$ errors vs M of the scheme (3.10) and (3.11)(bottom) for problem (4.3) with $\alpha = 0.3, 0.5$ and 0.8 (from left to right)

$\alpha = 0.1$	$M = 11, N = 2$		$\alpha = 0.1$	$M = 11, N = 2$	
(σ, λ)	E_{Max}	E_{L^2}	(σ, λ)	E_{Max}	E_{L^2}
$(2, 1)$	2.63e-07	4.54e-08	$(2, 2)$	2.63e-07	5.24e-08
$(1, 1)$	2.63e-07	6.42e-08	$(1, 2)$	2.63e-07	7.87e-08
$(\frac{1}{2}, 1)$	2.63e-07	8.34e-08	$(\frac{1}{2}, 2)$	2.63e-07	1.06e-08
$(0, 1)$	2.63e-07	1.20e-07	$(0, 2)$	2.63e-07	1.57e-07
$(-\frac{1}{2}, 1)$	2.63e-07	2.13e-07	$(-\frac{1}{2}, 2)$	2.63e-07	2.13e-07

Table 1 Discrete errors of problem (4.3) with $\alpha = 0.1$ and taking different σ and λ in (3.5)

One can see from Table 1, accurate numerical solutions are achieved by using $N = 2$ and not larger M. Authors of [32] have applied the Sinc-Legendre collocation method for solving problem (4.3) with $\alpha = 0.1$, $\mu = 2$ and $L = T = 1$ by using $N = 8$ and $M = 10, 15, 20$ and $M = 25$. For $N = 8$ and $M = 25$, they obtained 10^{-5} accuracy as the maximum value of absolute error of the numerical solution. And by the Gegenbauer spectral method proposed in [36], the absolute error of the numerical solution reached to 10^{-7} accuracy with $N = 2$ and $M = 11$ for problem (4.3) with the same α, μ, L and T as above. Comparing these error results to that in Table 1, one may clearly find that, our method can derive more accurate numerical solution even by using smaller N, and also can afford much more choices on base functions of numerical solutions. Furthermore, we can observe that the discrete weighted L^2-norm error decreases slightly as σ increases for the same M, N and λ.

Example 4.2 We next consider the initial boundary values problem of FPDE of order $\alpha \in (0, 1]$, with inhomogeneous initial and Dirichlet boundary conditions

$$\begin{cases} {}_0\mathcal{D}_t^\alpha U(x,t) + x\frac{\partial U(x,t)}{\partial x} + \frac{\partial^2 U(x,t)}{\partial x^2} = f(x,t), & x \in (0,L),\ t \in (0,T], \\ U(x,0) = x^2, & x \in (0,L), \\ U(0,t) = 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} t^{2\alpha}, & t \in (0,T], \\ U(L,t) = L^2 + 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} t^{2\alpha} \mathrm{e}^{\mu L}, & t \in (0,T], \end{cases} \tag{4.5}$$

where $f(x,t) = 2(t^\alpha \mathrm{e}^{\mu x} + x^2 + 1) + 2\mu(x+\mu)\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} t^{2\alpha} \mathrm{e}^{\mu x}$, $\mu \in \mathbb{R}$. The exact solution of problem (4.5) is

$$U(x,t) = x^2 + 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} t^{2\alpha} \mathrm{e}^{\mu x}. \tag{4.6}$$

$\alpha = 0.2$	$M = 11, N = 1$		$\alpha = 0.5$	$M = 11, N = 1$	
(σ, λ)	E_{Max}	E_{L^2}	(σ, λ)	E_{Max}	E_{L^2}
$(2, 0.4)$	$2.66e$-15	$7.10e$-16	$(2, 1)$	$3.55e$-15	$7.94e$-16
$(1, 0.4)$	$1.80e$-15	$6.16e$-16	$(1, 1)$	$2.66e$-15	$1.53e$-15
$(\frac{1}{2}, 0.4)$	$1.80e$-15	$7.35e$-16	$(\frac{1}{2}, 1)$	$2.66e$-15	$1.87e$-15
$(0, 0.4)$	$1.80e$-15	$1.21e$-15	$(0, 1)$	$1.78e$-15	$9.11e$-16
$(-\frac{1}{2}, 0.4)$	$1.78e$-15	$1.32e$-15	$(-\frac{1}{2}, 1)$	$2.66e$-15	$3.07e$-15
$\alpha = 0.6$	$M = 11, N = 1$		$\alpha = 0.6$	$M = 11, N = 1$	
(σ, λ)	E_{Max}	E_{L^2}	(σ, λ)	E_{Max}	E_{L^2}
$(2, 0.2)$	$1.78e$-15	$8.81e$-15	$(2, 1.2)$	$2.66e$-15	$1.31e$-15
$(1, 0.2)$	$3.55e$-15	$2.12e$-16	$(1, 1.2)$	$8.88e$-16	$7.85e$-16
$(\frac{1}{2}, 0.2)$	$2.66e$-15	$1.87e$-15	$(\frac{1}{2}, 1.2)$	$4.44e$-15	$2.87e$-15
$(0, 0.2)$	$2.66e$-15	$1.61e$-15	$(0, 1.2)$	$1.33e$-15	$1.15e$-15
$(-\frac{1}{2}, 0.2)$	$3.55e$-15	$1.09e$-15	$(-\frac{1}{2}, 1.2)$	$2.66e$-15	$3.81e$-15

Table 2 Discrete errors of problem (4.5) with $\alpha = 0.2, 0.5$ and 0.6

From Table 2, we list errors E_{Max} and E_{L^2} of the mixed generalized Jacobi and Chebyshev collocation method for problem (4.4) with $\mu = 1$ and $L = T = 1$, by taking different σ and

λ in the numerical solution. One can clearly see that, no matter the value of given $\alpha \in (0,1)$ and $\sigma > -1$, once taking the parameter $\lambda = 2\alpha - \lfloor 2\alpha \rfloor, 2\alpha - \lfloor 2\alpha \rfloor + 1, \dots, 2\alpha$ in the expansions of numerical solution, our method may derive more accurate numerical solution of the given problem, for very small discretization in time, and not bigger discretization in space. We also observe that, for the smooth ($\alpha = 0.5$) or not smooth ($\alpha = 0.2, 0.6$) solution in time of given equation, one can take proper parameter λ in numerical expansion to improve its approximation accuracy. These results illustrate the generalized Jacobi functions are very efficient in deriving numerical solutions of fractional equations.

5. Concluding remarks

In this paper, we apply the mixed generalized Jacobi and Chebyshev collocation method to solve the time-fractional convection-diffusion equation with Dirichlet boundary conditions. This method utilizes an easy procedure to implement and yields higher accuracy. The basis functions have the following properties: easy computation, rapid convergence, and much more flexibility. At last, accuracy and efficiency of the proposed method are illustrated by two model problems.

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