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Point Spectra of the Operator Corresponding to the $M/M/1$ Queueing Model with Working Vacation and Vacation Interruption

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Abstract In this paper, we consider point spectra of the operator corresponding to the $M/M/1$ queueing model with working vacation and vacation interruption. We prove that the underlying operator has uncountable eigenvalues on the left real line and these results describe the point spectra of the operator. Then, we show that the essential growth bound of the C_0-semigroup generated by the operator is 0 and therefore it is not quasi compact, the essential spectral bound of the C_0-semigroup is equal to 1. Moreover, our results imply it is impossible that the time-dependent solution of the model exponentially converges to its steady-state solution.

Keywords $M/M/1$ queueing model; working vacation and vacation interruption; C_0-semigroup; eigenvalue; essential spectral bound

MR(2010) Subject Classification 47A10; 47A75; 60K25

1. Introduction

According to Zhang and Hou [1], the $M/M/1$ queueing system with working vacation and vacation interruption can be described by the following system of partial differential equations with integral boundary conditions:

$$\begin{aligned}
&\frac{dp_{0,0}(t)}{\mathrm{d}t} = -\lambda p_{0,0}(t) + \mu_0 \int_0^\infty p_{1,0}(x,t)\mathrm{d}x + \mu_1 \int_0^\infty p_{1,1}(x,t)\mathrm{d}x,\\
&\frac{\partial p_{1,0}(x,t)}{\partial t} + \frac{\partial p_{1,0}(x,t)}{\partial x} = -[\lambda+\theta+\mu_0]p_{1,0}(x,t),\\
&\frac{\partial p_{n,0}(x,t)}{\partial t} + \frac{\partial p_{n,0}(x,t)}{\partial x} = -[\lambda+\theta+\mu_0]p_{n,0}(x,t) + \lambda p_{n-1,0}(x,t),\ \forall n\ge 2,\\
&\frac{\partial p_{1,1}(x,t)}{\partial t} + \frac{\partial p_{1,1}(x,t)}{\partial x} = -[\lambda+\mu_1]p_{1,1}(x,t),\\
&\frac{\partial p_{n,1}(x,t)}{\partial t} + \frac{\partial p_{n,1}(x,t)}{\partial x} = -[\lambda+\mu_1]p_{n,1}(x,t) + \lambda p_{n-1,1}(x,t),\ \forall n\ge 2,
\end{aligned} \tag{1.1}$$

with boundary conditions:

$$\begin{aligned}
&p_{1,0}(0,t) = \lambda p_{0,0}(t),\\
&p_{n,0}(0,t) = 0,\quad \forall n\ge 2,
\end{aligned}$$

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$$p_{n,1}(0,t)=\theta\int_0^\infty p_{n,0}(x,t)\mathrm{d}x+\mu_0\int_0^\infty p_{n+1,0}(x,t)\mathrm{d}x+ \mu_1\int_0^\infty p_{n+1,1}(x,t)\mathrm{d}x,\ \forall n\geq 1, \tag{1.2}$$

and initial condition:

$$p_{0,0}(0)=\phi_0, p_{m,0}(x,0)=\phi_k(x),\ \forall k\geq 1;\ p_{m,1}(x,0)=\varphi_m(x),\ \forall m\geq 1, \tag{1.3}$$

where, $(x,t)\in[0,\infty)\times[0,\infty)$; $p_{0,0}(t)$ represents the probability that there is no customer in the system and the server is in a working vacation period at time t; $p_{n,0}(x,t)\mathrm{d}x$ $(n\geq 1)$ is the probability that at time t the server is in a working vacation period and there are n customers in the system with elapsed service time of the customer undergoing service lying in $(x,x+\mathrm{d}x]$; $p_{n,1}(x,t)\mathrm{d}x$ $(n\geq 1)$ is the probability that at time t the server is in a regular busy period and there are n customers in the system with elapsed service time of the customer undergoing service lying in $(x,x+\mathrm{d}x]$; λ is the mean arrival rate of customers; θ is the vacation duration rate of the server; μ_0 is the service rate of the server while the server is in a working vacation period. μ_1 is the service rate of the server while the server is in a regular busy period.

In classical vacation queueing models, the server completely stops service during the vacation period. However, there are numerous situations where the server remains active during the vacation period which is called working vacation [2]. In 2002, Servi and Finn [3] first studied the $M/M/1$ queueing system with multiple working vacation and obtained the transform formula for the distribution of the number of customers in the system and the sojourn time in a steady state. Moreover, Wu and Takagi [4] extended Servi and Finn's [3] $M/M/1$ queueing system to an $M/G/1$ queueing system with multiple working vocation. In 2007, Li and Tian [5] first introduced the vacation interruption policy in an $M/M/1$ queueing model. Since then, vacation interruption models have been studied by several researchers, see Ke et al. [6], Zhang and Hou [1], Gao et al. [7], Liu et al. [8], Lee and Kim [9] and the references given there.

In 2010, Zhang and Hou [1] considered the $M/G/1$ queueing system with working vacation and vacation interruption where the server enters into vacations when there are no customers and it can take service at a lower rate during the vacation period. If there are customers in the system at the instant of a service completion during the vacation period, the server will come back to the normal working level no matter whether or not the vacation has ended. Otherwise, it continues the vacation. Using supplementary variable technique they established the above model and gave the Laplace-Stieltjes transform of the stationary waiting time. Then, they presented the queue length distribution and service status at an arbitrary epoch in steady state condition. In 2016, by using the C_0-semigroups theory Kasim [10] did dynamic analysis and proved that the above model has a unique positive time-dependent solution which satisfies the probability condition. When the service completion rates are constant (in this case the $M/G/1$ queueing model is called $M/M/1$ queueing model), by studying spectral properties of the underlying operator corresponding to the model he obtained that the time-dependent solution of the model strongly converges to its steady-state solution. In 2017, Kasim and Gupur [11] studied the asymptotic property of the time-dependent solution of the general case of above model by using the boundary perturbation method which was developed by the Greiner [12]. So far, no other results have been found in the literature. In this paper, inspired from the queueing model studied by Gupur [13–15] we prove that if $\lambda < \mu_1$, then all points in

$$\left\{\gamma \in C \,\middle|\, \begin{array}{l} \Re\gamma + \theta + \mu_0 > 0,\ \Re\gamma + \mu_1 > 0, \\ \left|\gamma + \lambda + \mu_1 \pm \sqrt{(\gamma + \lambda + \mu_1)^2 - 4\lambda\mu_1}\right| < 2\mu_1 \end{array}\right\} \cup \{0\}$$

are eigenvalue of $A+U+E$ with geometric multiplicity one. In particular, the interval $(-\min\{\theta + \mu_0, \mu_1\}, 0]$ belongs to its point spectrum when $\lambda < \mu_1 < 2\sqrt{\lambda\mu_1} - \lambda - \theta - \mu_0$. Therefore, our results imply that the C_0-semigroup generated by the underlying operator is not compact, even not eventually compact. Moreover, by combining the results in this paper and the results in [10] together with [16, Corollary 2.11] we deduce that the essential growth bound of the C_0-semigroup is 0. So, it is not quasi-compact. In addition, we show that the spectral radius and the essential spectral radius of the C_0-semigroup are equal to 1, respectively. Altogether, we can conclude it is impossible that the time-dependent solution of the model exponentially converges to its steady-state solution.

Our first aim is to rewrite the Eqs. (1.1)–(1.3) in the form of an abstract Cauchy problem on a suitable Banach space. For this purpose we select the following state space (we follow the notation of Gupur et al. [17], Kasim [10, 11]).

$$\begin{aligned} X &= \{(p_0, p_1) \,|\, p_0 \in Y_1, p_1 \in Y_2, \|(p_0, p_1)\| = \|p_0\|_{Y_1} + \|p_1\|_{Y_2} < \infty\}, \\ Y_1 &= \Big\{p_0 \in \mathbb{R} \times L^1[0,\infty) \times \cdots \,\Big|\, \|p_0\| = |p_{0,0}| + \sum_{n=1}^{\infty} \|p_{n,0}\|_{L^1[0,\infty)} < \infty\Big\}, \\ Y_2 &= \Big\{p_1 \in L^1[0,\infty) \times L^1[0,\infty) \cdots \,\Big|\, \|p_1\| = \sum_{n=1}^{\infty} \|p_{n,1}\|_{L^1[0,\infty)} < \infty\Big\}. \end{aligned}$$

It is obvious that X is a Banach space. For simplicity, we introduce

$$\Gamma_1 = \begin{pmatrix} e^{-x} & 0 & 0 & 0 & \cdots \\ \lambda e^{-x} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & \theta & 0 & 0 & \cdots \\ 0 & 0 & \theta & 0 & \cdots \\ 0 & 0 & 0 & \theta & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\Gamma_3 = \begin{pmatrix} 0 & 0 & \mu_0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \mu_0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \mu_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix} 0 & \mu_1 & 0 & 0 & \cdots \\ 0 & 0 & \mu_1 & 0 & \cdots \\ 0 & 0 & 0 & \mu_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now we define operators and their domain as follows.

$$A\left(\begin{pmatrix} p_{0,0} \\ p_{1,0}(x) \\ p_{2,0}(x) \\ p_{3,0}(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} p_{1,1}(x) \\ p_{2,1}(x) \\ p_{3,1}(x) \\ p_{4,1}(x) \\ \vdots \end{pmatrix}\right) = \left(\begin{pmatrix} -\lambda & 0 & 0 & 0 & \cdots \\ 0 & -\frac{\mathrm{d}}{\mathrm{d}x} & 0 & 0 & \cdots \\ 0 & 0 & -\frac{\mathrm{d}}{\mathrm{d}x} & 0 & \cdots \\ 0 & 0 & 0 & -\frac{\mathrm{d}}{\mathrm{d}x} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{0,0} \\ p_{1,0}(x) \\ p_{2,0}(x) \\ p_{3,0}(x) \\ \vdots \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}x} & 0 & 0 & \cdots \\ 0 & -\frac{\mathrm{d}}{\mathrm{d}x} & 0 & \cdots \\ 0 & 0 & -\frac{\mathrm{d}}{\mathrm{d}x} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{1,1}(x) \\ p_{2,1}(x) \\ p_{3,1}(x) \\ \vdots \end{pmatrix}\right),$$

$$D(A) = \left\{ (p_0, p_1) \in X \;\middle|\; \begin{array}{l} \frac{dp_{n,0}}{\mathrm{d}x} \in L^1[0,\infty),\ \frac{dp_{n,1}}{\mathrm{d}x} \in L^1[0,\infty), p_{n,0}(x) \text{ and } p_{n,1}(x) \\ (n \geq 1) \text{ are absolutely continuous and } p_0(0) = \Gamma_1 p_0; \\ p_1(0) = \int_0^\infty \Gamma_2 p_0 \mathrm{d}x + \int_0^\infty \Gamma_3 p_0 \mathrm{d}x + \int_0^\infty \Gamma_4 p_1 \mathrm{d}x \end{array} \right\}.$$

$$U\left(\begin{pmatrix} p_{0,0} \\ p_{1,0}(x) \\ p_{2,0}(x) \\ p_{3,0}(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} p_{1,1}(x) \\ p_{2,1}(x) \\ p_{3,1}(x) \\ p_{4,1}(x) \\ \vdots \end{pmatrix}\right) = \left(\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & \mathcal{D}_0 & 0 & 0 & \cdots \\ 0 & \lambda & \mathcal{D}_0 & 0 & \cdots \\ 0 & 0 & \lambda & \mathcal{D}_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{0,0} \\ p_{1,0}(x) \\ p_{2,0}(x) \\ p_{3,0}(x) \\ \vdots \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \mathcal{D}_1 & 0 & 0 & \cdots \\ \lambda & \mathcal{D}_1 & 0 & \cdots \\ 0 & \lambda & \mathcal{D}_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{1,1}(x) \\ p_{2,1}(x) \\ p_{3,1}(x) \\ \vdots \end{pmatrix}\right), \quad D(U) = X,$$

here

$$\mathcal{D}_0 = -(\lambda + \theta + \mu_0), \quad \mathcal{D}_1 = -(\lambda + \mu_1).$$

$$E\left(\begin{pmatrix} p_{0,0} \\ p_{1,0}(x) \\ p_{2,0}(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} p_{1,1}(x) \\ p_{2,1}(x) \\ p_{3,1}(x) \\ \vdots \end{pmatrix}\right)$$

$$= \left(\begin{pmatrix} \int_0^\infty \mu_0(x) p_{1,0}(x)\mathrm{d}x + \int_0^\infty \mu_1(x) p_{1,1}(x)\mathrm{d}x \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}\right), \quad D(E) = X.$$

Then the above system of equations (1.1)–(1.3) can be written as an abstract Cauchy problem in Banach space X.

$$\begin{cases} \frac{d(p_0,p_1)(t)}{\mathrm{d}t} = (A+U+E)(p_0,p_1)(t), \quad t \in (0,\infty), \\ (p_0,p_1)(0) = \left(\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix} \right). \end{cases} \tag{1.4}$$

Kasim [10] have obtained the following result.

Theorem 1.1 *$A+U+E$ generates a positive contraction C_0-semigroup $T(t)$. $T(t)$ is isometric for the initial value. Therefore, the system (1.4) has a unique positive time-dependent solution $(p_0,p_1)(x,t) = T(t)(p_0,p_1)(0)$ satisfying*

$$\|(p_0,p_1)(\cdot,t)\| = 1, \quad \forall t \in [0,\infty).$$

In addition, the time-dependent solution $(p_0,p_1)(x,t)$ of the system (1.4) strongly converges to its steady-state solution $(p_0,p_1)(x)$, i.e.,

$$\lim_{t\to\infty} \|(p_0,p_1)(\cdot,t) - \beta(p_0,p_1)(\cdot)\| = 0,$$

here β is decided by the eigenvector satisfying $(A+U+E)^*(q_0^*,q_1^*) = 0$ and initial value.

2. Main results

In this section, we describe the point spectra of $A+U+E$ and obtain the main results of this paper.

Theorem 2.1 *If $\lambda < \mu_1$, then all points in the set*

$$\left\{ \gamma \in C \;\middle|\; \begin{array}{l} \Re\gamma + \theta + \mu_0 > 0, \; \Re\gamma + \mu_1 > 0 \\ \left|\gamma\lambda + \mu_1 \pm \sqrt{(\gamma+\lambda+\mu_1)^2 - 4\lambda\mu_1}\right| < 2\mu_1 \end{array} \right\} \cup \{0\}$$

are eigenvalue of $A+U+E$ with geometric multiplicity one. In particular, the interval $(-\min\{\theta+\mu_0,\mu_1\},0]$ belongs to the point spectrum of $A+U+E$ when $\lambda < \mu_1 < 2\sqrt{\lambda\mu_1} - \lambda - \theta - \mu_0$.

Proof We consider the equation $[\gamma I - (A+U+E)](p_0,p_1) = 0$, which is equivalent to

$$(\gamma+\lambda)p_{0,0} = \mu_0 \int_0^\infty p_{1,0}(x)\mathrm{d}x + \mu_1 \int_0^\infty p_{1,1}(x)\mathrm{d}x, \tag{2.1}$$

$$\frac{dp_{1,0}(x)}{\mathrm{d}x} = -(\gamma+\lambda+\theta+\mu_0)p_{1,0}(x), \tag{2.2}$$

$$\frac{dp_{n,0}(x)}{\mathrm{d}x} = -(\gamma+\lambda+\theta+\mu_0)p_{n,0}(x) + \lambda p_{n-1,0}(x), \quad n \geq 2, \tag{2.3}$$

$$\frac{dp_{1,1}(x)}{\mathrm{d}x} = -(\gamma+\lambda+\mu_1)p_{1,1}(x), \tag{2.4}$$

$$\frac{dp_{n,1}(x)}{\mathrm{d}x} = -(\gamma+\lambda+\mu_1)p_{n,1}(x) + \lambda p_{n-1,1}(x), \quad n \geq 2, \tag{2.5}$$

$$p_{1,0}(0) = \lambda p_{0,0}, \tag{2.6}$$

$$p_{n,0}(0) = 0, \quad n \geq 2, \tag{2.7}$$

$$p_{n,1}(0) = \theta \int_0^\infty p_{n,0}(x)\mathrm{d}x + \mu_0 \int_0^\infty p_{n+1,0}(x)\mathrm{d}x + \mu_1 \int_0^\infty p_{n+1,1}(x)\mathrm{d}x, \quad n \geq 1. \tag{2.8}$$

Solving (2.2)–(2.5) we have

$$p_{1,0}(x) = p_{1,0}(0)e^{-(\gamma+\lambda+\theta+\mu_0)x}, \tag{2.9}$$

$$p_{n,0}(x) = p_{n,0}(0)e^{-(\gamma+\lambda+\theta+\mu_0)x} + \lambda e^{-(\gamma+\lambda+\theta+\mu_0)x} \int_0^x p_{n-1,0}(s)e^{(\gamma+\lambda+\theta+\mu_0)s}\mathrm{d}s, \quad n \geq 2, \tag{2.10}$$

$$p_{1,1}(x) = p_{1,1}(0)e^{-(\gamma+\lambda+\mu_1)x}, \tag{2.11}$$

$$p_{n,1}(x) = p_{n,1}(0)e^{-(\gamma+\lambda+\mu_1)x} + \lambda e^{-(\gamma+\lambda+\mu_1)x} \int_0^x p_{n-1,1}(s)e^{(\gamma+\lambda+\mu_1)s}\mathrm{d}s, \quad n \geq 2. \tag{2.12}$$

By combining (2.6) and (2.7) with (2.9) and (2.10) we deduce

$$p_{1,0}(x) = \lambda p_{0,0}\, e^{-(\gamma+\lambda+\theta+\mu_0)x}, \tag{2.13}$$

$$p_{2,0}(x) = \lambda e^{-(\gamma+\lambda+\theta+\mu_0)x} \int_0^x \lambda p_{0,0}\mathrm{d}s = \lambda p_{0,0}\frac{\lambda x}{1!}e^{-(\gamma+\lambda+\theta+\mu_0)x}, \tag{2.14}$$

$$p_{3,0}(x) = \lambda e^{-(\gamma+\lambda+\theta+\mu_0)x} \int_0^x \lambda p_{0,0}\frac{\lambda x}{1!}\mathrm{d}s = \lambda p_{0,0}\frac{(\lambda x)^2}{2!}e^{-(\gamma+\lambda+\theta+\mu_0)x}, \tag{2.15}$$

$$\cdots$$

$$\begin{aligned} p_{n,0}(x) &= \lambda e^{-(\gamma+\lambda+\theta+\mu_0)x} \int_0^x p_{n-1,0}(s)e^{(\gamma+\lambda+\theta+\mu_0)s}\mathrm{d}s \\ &= \lambda p_{0,0}\frac{(\lambda x)^{n-1}}{(n-1)!}e^{-(\gamma+\lambda+\theta+\mu_0)x}, \quad n \geq 1. \end{aligned} \tag{2.16}$$

By using (2.11) and (2.12) repeatedly we obtain

$$\begin{aligned} p_{2,1}(x) &= p_{2,1}(0)e^{-(\gamma+\lambda+\mu_1)x} + \lambda e^{-(\gamma+\lambda+\mu_1)x} \int_0^x p_{1,1}(s)e^{(\gamma+\lambda+\mu_1)s}\mathrm{d}s \\ &= p_{2,1}(0)e^{-(\gamma+\lambda+\mu_1)x} + \lambda e^{-(\gamma+\lambda+\mu_1)x} \int_0^x p_{1,1}(0)\mathrm{d}s \\ &= [p_{2,1}(0) + \frac{\lambda x}{1!}p_{1,1}(0)]e^{-(\gamma+\lambda+\mu_1)x} \\ &= e^{-(\gamma+\lambda+\mu_1)x}\sum_{k=1}^{2}\frac{(\lambda x)^{2-k}}{(2-k)!}p_{k,1}(0), \end{aligned} \tag{2.17}$$

$$\begin{aligned} p_{3,1}(x) &= p_{3,1}(0)e^{-(\gamma+\lambda+\mu_1)x} + \lambda e^{-(\gamma+\lambda+\mu_1)x} \int_0^x p_{2,1}(s)e^{(\gamma+\lambda+\mu_1)s}\mathrm{d}s \\ &= p_{3,1}(0)e^{-(\gamma+\lambda+\mu_1)x} + \lambda e^{-(\gamma+\lambda+\mu_1)x} \int_0^x [p_{2,1}(0) + \frac{\lambda s}{1!}p_{1,1}(0)]\mathrm{d}s \\ &= [p_{3,1}(0) + \frac{\lambda x}{1!}p_{2,1}(0) + \frac{(\lambda x)^2}{2!}p_{1,1}(0)]e^{-(\gamma+\lambda+\mu_1)x} \end{aligned}$$

$$= e^{-(\gamma+\lambda+\mu_1)x}\sum_{k=1}^{3}\frac{(\lambda x)^{2-k}}{(3-k)!}p_{k,1}(0), \tag{2.18}$$

$$\cdots$$

$$p_{n,1}(x) = [p_{n,1}(0) + \frac{\lambda x}{1!}p_{n-1,1}(0) + \frac{(\lambda x)^2}{2!}p_{n-2,1}(0) + \cdots + \frac{(\lambda x)^{n-1}}{(n-1)!}p_{1,1}(0)]e^{-(\gamma+\lambda+\mu_1)x}$$
$$= e^{-(\gamma+\lambda+\mu_1)x}\sum_{k=1}^{n}\frac{(\lambda x)^{n-k}}{(n-k)!}p_{k,1}(0), \quad n \geq 1. \tag{2.19}$$

Since $\int_0^\infty x^k e^{-\omega x}\mathrm{d}x = \dfrac{k!}{\omega^{k+1}}$ for $\omega > 0$ and $k \geq 1$, (2.16) and (2.19) imply, for $\Re\gamma+\lambda+\theta+\mu_0 > 0$, $\Re\gamma + \lambda + \mu_1 > 0$

$$\int_0^\infty p_{n,0}(x)\mathrm{d}x = \lambda p_{0,0}\frac{\lambda^{n-1}}{(n-1)!}\int_0^\infty x^{n-1}e^{-(\gamma+\lambda+\theta+\mu_0)x}\mathrm{d}x$$
$$= \frac{\lambda^{n-1}}{(\gamma+\lambda+\theta+\mu_0)^n}\lambda p_{0,0}, \quad n \geq 1, \tag{2.20}$$

$$\int_0^\infty p_{n,1}(x)\mathrm{d}x = \sum_{k=1}^{n}\frac{\lambda^{n-k}}{(n-k)!}p_{k,1}(0)\int_0^\infty x^{n-k}e^{-(\gamma+\lambda+\mu_1)x}\mathrm{d}x$$
$$= \sum_{k=1}^{n}\frac{\lambda^{n-k}}{(\gamma+\lambda+\mu_1)^{n+1-k}}p_{k,1}(0), \quad n \geq 1. \tag{2.21}$$

Combining (2.8) with (2.20) and (2.21) gives

$$p_{1,1}(0) = \frac{\mu_1}{\gamma+\lambda+\mu_1}p_{2,1}(0) + \frac{\mu_1\lambda}{(\gamma+\lambda+\mu_1)^2}p_{1,1}(0) + \frac{\lambda[\mu_0(\lambda+\theta)+\theta(\gamma+\lambda+\theta)]}{(\gamma+\lambda+\theta+\mu_0)^2}p_{0,0}, \tag{2.22}$$

$$p_{2,1}(0) = \frac{\mu_1}{\gamma+\lambda+\mu_1}p_{3,1}(0) + \frac{\mu_1\lambda}{(\gamma+\lambda+\mu_1)2}p_{2,1}(0) + \frac{\mu_1\lambda^2}{(\gamma+\lambda+\mu_1)^3}p_{1,1}(0) + \frac{\lambda^2[\mu_0(\lambda+\theta)+\theta(\gamma+\lambda+\theta)]}{(\gamma+\lambda+\theta+\mu_0)^3}p_{0,0}, \tag{2.23}$$

$$p_{3,1}(0) = \frac{\mu_1}{\gamma+\lambda+\mu_1}p_{4,1}(0) + \frac{\mu_1\lambda}{(\gamma+\lambda+\mu_1)^2}p_{3,1}(0) + \frac{\mu_1\lambda^2}{(\gamma+\lambda+\mu_1)^3}p_{2,1}(0) + \frac{\mu_1\lambda^3}{(\gamma+\lambda+\mu_1)^4}p_{1,1}(0) + \frac{\lambda^3[\mu_0(\lambda+\theta)+\theta(\gamma+\lambda+\theta)]}{(\gamma+\lambda+\theta+\mu_0)^4}p_{0,0}, \tag{2.24}$$

$$\cdots$$

$$p_{n,1}(0) = \frac{\mu_1}{\gamma+\lambda+\mu_1}p_{n+1,1}(0) + \frac{\mu_1\lambda}{(\gamma+\lambda+\mu_1)^2}p_{n,1}(0) + \frac{\mu_1\lambda^2}{(\gamma+\lambda+\mu_1)^3}p_{n-1,1}(0) + \cdots + \frac{\mu_1\lambda^{n-1}}{(\gamma+\lambda+\mu_1)^n}p_{2,1}(0) + \frac{\mu_1\lambda^n}{(\gamma+\lambda+\mu_1)^{n+1}}p_{1,1}(0) + \frac{\lambda^n[\mu_0(\lambda+\theta)+\theta(\gamma+\lambda+\theta)]}{(\gamma+\lambda+\theta+\mu_0)^{n+1}}p_{0,0}$$

$$= \sum_{k=1}^{n+1} \frac{\mu_1 \lambda^{n+1-k}}{(\gamma+\lambda+\mu_1)^{n+2-k}} p_{k,1}(0) + \frac{\lambda^n[\mu_0(\lambda+\theta)+\theta(\gamma+\lambda+\theta)]}{(\gamma+\lambda+\theta+\mu_0)^{n+1}} p_{0,0}, \quad n \geq 2, \tag{2.25}$$

$$\Longrightarrow$$

$$p_{n+1,1}(0) = \sum_{k=1}^{n+2} \frac{\mu_1 \lambda^{n+2-k}}{(\gamma+\lambda+\mu_1)^{n+3-k}} p_{k,1}(0) + \frac{\lambda^{n+1}[\mu_0(\lambda+\theta)+\theta(\gamma+\lambda+\theta)]}{(\gamma+\lambda+\theta+\mu_0)^{n+2}} p_{0,0}, \quad n \geq 1. \tag{2.26}$$

By (2.26) $-\frac{\lambda}{\gamma+\lambda+\mu_1}\times$ (2.25) it follows that

$$\begin{aligned} &p_{n+1,1}(0) - \frac{\lambda}{\gamma+\lambda+\mu_1} p_{n,1}(0) \\ &= \sum_{k=1}^{n+2} \frac{\mu_1 \lambda^{n+2-k}}{(\gamma+\lambda+\mu_1)^{n+3-k}} p_{k,1}(0) + \frac{\lambda^{n+1}[\mu_0(\lambda+\theta)+\theta(\gamma+\lambda+\theta)]}{(\gamma+\lambda+\theta+\mu_0)^{n+2}} p_{0,0} - \\ &\sum_{k=1}^{n+1} \frac{\mu_1 \lambda^{n+2-k}}{(\gamma+\lambda+\mu_1)^{n+3-k}} p_{k,1}(0) - \frac{\lambda^{n+1}[\mu_0(\lambda+\theta)+\theta(\gamma+\lambda+\theta)]}{(\gamma+\lambda+\mu_1)(\gamma+\lambda+\theta+\mu_0)^{n+1}} p_{0,0} \\ &= \frac{\mu_1}{\gamma+\lambda+\mu_1} p_{n+2,1}(0) + \frac{\lambda^{n+1}(\mu_1-\theta-\mu_0)[\mu_0(\lambda+\theta)+\theta(\gamma+\lambda+\theta)]}{(\gamma+\lambda+\mu_1)(\gamma+\lambda+\theta+\mu_0)^{n+2}} p_{0,0} \\ &\Longrightarrow \end{aligned}$$

$$p_{n+2,1}(0) = \frac{\gamma+\lambda+\mu_1}{\mu_1} p_{n+1,1}(0) - \frac{\lambda}{\mu_1} p_{n,1}(0) - \frac{\lambda^{n+1}(\mu_1-\theta-\mu_0)[\mu_0(\lambda+\theta)+\theta(\gamma+\lambda+\theta)]}{\mu_1(\gamma+\lambda+\theta+\mu_0)^{n+2}} p_{0,0}, \quad n \geq 2. \tag{2.27}$$

If we set

$$\begin{aligned} &p_{n+2,1}(0) - \alpha p_{n+1,1}(0) - \pi_{n+2} = \beta(p_{n+1,1}(0) - \alpha p_{n,1}(0) - \pi_{n+1}) \Longleftrightarrow \\ &p_{n+2,1}(0) = (\alpha+\beta) p_{n+1,1}(0) - \alpha\beta p_{n,1}(0) - (\pi_{n+2} - \beta\pi_{n+1}), \quad n \geq 2, \end{aligned} \tag{2.28}$$

then comparison of (2.27) and (2.28) shows that

$$\alpha+\beta = \frac{\gamma+\lambda+\mu_1}{\mu_1}, \quad \alpha\beta = \frac{\lambda}{\mu_1}, \tag{2.29}$$

$$\pi_{n+2} - \beta\pi_{n+1} = \frac{\lambda^{n+1}(\mu_1-\theta-\mu_0)[\mu_0(\lambda+\theta)+\theta(\gamma+\lambda+\theta)]}{\mu_1(\gamma+\lambda+\theta+\mu_0)^{n+2}} p_{0,0}, \quad n \geq 2. \tag{2.30}$$

From (2.29) it follows that

$$\alpha = \frac{\gamma+\lambda+\mu_1+\sqrt{(\gamma+\lambda+\mu_1)^2-4\lambda\mu_1}}{2\mu_1}, \tag{2.31}$$

$$\beta = \frac{\gamma+\lambda+\mu_1-\sqrt{(\gamma+\lambda+\mu_1)^2-4\lambda\mu_1}}{2\mu_1}. \tag{2.32}$$

The equation (2.28) implies

$$p_{n+2,1}(0) - \alpha p_{n+1,1}(0) - \pi_{n+2} = \beta(p_{n+1,1}(0) - \alpha p_{n,1}(0) - \pi_{n+1})$$

$$
\begin{aligned}
&= \beta^2(p_{n,1}(0) - \alpha p_{n-1,1}(0) - \pi_n) = \beta^3(p_{n-1,1}(0) - \alpha p_{n-2,1}(0) - \pi_{n-1}) \\
&= \cdots \\
&= \beta^{n-1}(p_{3,1}(0) - \alpha p_{2,1}(0) - \pi_3), \quad n \geq 2. \qquad (2.33)
\end{aligned}
$$

Applying (2.33), we get

$$
\begin{aligned}
&p_{n+1,1}(0) - \alpha p_{n,1}(0) - \pi_{n+1} = \beta^{n-2}(p_{3,1}(0) - \alpha p_{2,1}(0) - \pi_3) \Longrightarrow \\
&\alpha p_{n+1,1}(0) - \alpha^2 p_{n,1}(0) - \alpha\pi_{n+1} = \alpha\beta^{n-2}(p_{3,1}(0) - \alpha p_{2,1}(0) - \pi_3), \qquad (2.34) \\
&p_{n,1}(0) - \alpha p_{n-1,1}(0) - \pi_n = \beta^{n-3}(p_{3,1}(0) - \alpha p_{2,1}(0) - \pi_3) \Longrightarrow \\
&\alpha^2 p_{n,1}(0) - \alpha^3 p_{n-1,1}(0) - \alpha^2\pi_n = \alpha^2\beta^{n-3}(p_{3,1}(0) - \alpha p_{2,1}(0) - \pi_3), \qquad (2.35) \\
&p_{n-1,1}(0) - \alpha p_{n-2,1}(0) - \pi_{n-1} = \beta^{n-4}(p_{3,1}(0) - \alpha p_{2,1}(0) - \pi_3) \Longrightarrow \\
&\alpha^3 p_{n-1,1}(0) - \alpha^4 p_{n-2,1}(0) - \alpha^3\pi_{n-1} = \alpha^3\beta^{n-4}(p_{3,1}(0) - \alpha p_{2,1}(0) - \pi_3), \qquad (2.36) \\
&\cdots \\
&p_{5,1}(0) - \alpha p_{4,1}(0) - \pi_5 = \beta^2(p_{3,1}(0) - \alpha p_{2,1}(0) - \pi_3) \Longrightarrow \\
&\alpha^{n-3} p_{5,1}(0) - \alpha^{n-2} p_{4,1}(0) - \alpha^{n-3}\pi_5 = \alpha^{n-3}\beta^2(p_{3,1}(0) - \alpha p_{2,1}(0) - \pi_3), \qquad (2.37) \\
&p_{4,1}(0) - \alpha p_{3,1}(0) - \pi_4 = \beta(p_{3,1}(0) - \alpha p_{2,1}(0) - \pi_3) \Longrightarrow \\
&\alpha^{n-2} p_{4,1}(0) - \alpha^{n-1} p_{3,1}(0) - \alpha^{n-2}\pi_4 = \alpha^{n-2}\beta(p_{3,1}(0) - \alpha p_{2,1}(0) - \pi_3). \qquad (2.38)
\end{aligned}
$$

Adding both side of (2.33) to (2.38) we have for $n \geq 2$

$$
\begin{aligned}
&p_{n+2,1}(0) - \alpha^{n-1} p_{3,1}(0) - \pi_{n+2} - \alpha\pi_{n+1} - \alpha^2\pi_n - \alpha^3\pi_{n-1} - \cdots - \alpha^{n-3}\pi_5 - \alpha^{n-2}\pi_4 \\
&= [\beta^{n-1} + \alpha\beta^{n-2} + \alpha^2\beta^{n-3} + \alpha^3\beta^{n-4} + \cdots + \alpha^{n-3}\beta^2 + \alpha^{n-2}\beta](p_{3,1}(0) - \alpha p_{2,1}(0) - \pi_3) \\
&\Longrightarrow \\
&p_{n+2,1}(0) = [\beta^{n-1} + \alpha\beta^{n-2} + \alpha^2\beta^{n-3} + \alpha^3\beta^{n-4} + \cdots + \alpha^{n-3}\beta^2 + \alpha^{n-2}\beta + \alpha^{n-1}]p_{3,1}(0) - \\
&\qquad [\beta^{n-1} + \alpha\beta^{n-2} + \alpha^2\beta^{n-3} + \alpha^3\beta^{n-4} + \cdots + \alpha^{n-3}\beta^2 + \alpha^{n-2}\beta]\alpha p_{2,1}(0) - \\
&\qquad [\beta^{n-1} + \alpha\beta^{n-2} + \alpha^2\beta^{n-3} + \alpha^3\beta^{n-4} + \cdots + \alpha^{n-3}\beta^2 + \alpha^{n-2}\beta]\pi_3 + \\
&\qquad \sum_{k=4}^{n+2} \alpha^{n+2-k}\pi_k.
\end{aligned}
$$

Hence, when $\alpha = \beta$ by the Cauchy products we calculate

$$
\begin{aligned}
p_{n+2,1}(0) &= n\alpha^{n-1} p_{3,1}(0) - (n-1)\alpha^n p_{2,1}(0) - (n-1)\alpha^{n-1}\pi_3 + \\
&\qquad \sum_{k=4}^{n+2} \alpha^{n+2-k}\pi_k, \quad n \geq 2 \\
&\Longrightarrow
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=2}^{\infty} |p_{n+2,1}(0)| &\leq |p_{3,1}(0)| \sum_{n=2}^{\infty} n|\alpha|^{n-1} + |p_{2,1}(0)| \sum_{n=2}^{\infty} (n-1)|\alpha|^n + \\
&\qquad |\pi_3| \sum_{n=2}^{\infty} (n-1)|\alpha|^{n-1} + \sum_{n=2}^{\infty} |\pi_{n+2}| \sum_{k=1}^{\infty} |\alpha|^{k-1}. \qquad (2.39)
\end{aligned}
$$

When $\alpha \neq \beta$ also by the Cauchy products we derive

$$
\begin{aligned}
p_{n+2,1}(0) = & \frac{\alpha^n - \beta^n}{\alpha - \beta} p_{3,1}(0) - \alpha\left(\frac{\alpha^n - \beta^n}{\alpha - \beta} - \alpha^{n-1}\right) p_{2,1}(0) - \\
& \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} - \alpha^{n-1}\right)\pi_3 + \sum_{k=4}^{n+2} \alpha^{n+2-k}\pi_k, \quad n \geq 2 \\
& \Longrightarrow
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=2}^{\infty} |p_{n+2,1}(0)| \leq & \frac{|p_{3,1}(0)|}{|\alpha - \beta|}\Big(\sum_{n=2}^{\infty} |\alpha|^n + \sum_{n=2}^{\infty} |\beta|^n\Big) + \\
& \alpha |p_{2,1}(0)| \Big[\frac{1}{|\alpha - \beta|}\Big(\sum_{n=2}^{\infty} |\alpha|^n + \sum_{n=2}^{\infty} |\beta|^n\Big) + \sum_{n=2}^{\infty} |\alpha|^{n-1}\Big] + \\
& |\pi_3| \Big[\frac{1}{|\alpha - \beta|}\Big(\sum_{n=2}^{\infty} |\alpha|^n + \sum_{n=2}^{\infty} |\beta|^n\Big) + \sum_{n=2}^{\infty} |\alpha|^{n-1}\Big] + \\
& \sum_{n=2}^{\infty} |\pi_{n+2}| \sum_{k=1}^{\infty} |\alpha|^{k-1}. \qquad (2.40)
\end{aligned}
$$

From (2.30) we deduce

$$
\begin{aligned}
\pi_{n+2} - \beta\pi_{n+1} &= \frac{\lambda^{n+1}(\mu_1 - \theta - \mu_0)[\mu_0(\lambda + \theta) + \theta(\gamma + \lambda + \theta)]}{\mu_1(\gamma + \lambda + \theta + \mu_0)^{n+2}} p_{0,0}, \\
\beta\pi_{n+1} - \beta^2\pi_n &= \beta \frac{\lambda^{n}(\mu_1 - \theta - \mu_0)[\mu_0(\lambda + \theta) + \theta(\gamma + \lambda + \theta)]}{\mu_1(\gamma + \lambda + \theta + \mu_0)^{n+1}} p_{0,0}, \\
\beta^2\pi_n - \beta^3\pi_{n-1} &= \beta^2 \frac{\lambda^{n-1}(\mu_1 - \theta - \mu_0)[\mu_0(\lambda + \theta) + \theta(\gamma + \lambda + \theta)]}{\mu_1(\gamma + \lambda + \theta + \mu_0)^{n}} p_{0,0}, \\
& \cdots \\
\beta^{n-4}\pi_6 - \beta^{n-3}\pi_5 &= \beta^{n-4} \frac{\lambda^{5}(\mu_1 - \theta - \mu_0)[\mu_0(\lambda + \theta) + \theta(\gamma + \lambda + \theta)]}{\mu_1(\gamma + \lambda + \theta + \mu_0)^{6}} p_{0,0}, \\
\beta^{n-3}\pi_5 - \beta^{n-2}\pi_4 &= \beta^{n-3} \frac{\lambda^{4}(\mu_1 - \theta - \mu_0)[\mu_0(\lambda + \theta) + \theta(\gamma + \lambda + \theta)]}{\mu_1(\gamma + \lambda + \theta + \mu_0)^{5}} p_{0,0}.
\end{aligned}
$$

Adding both sides of the above equation and using the Cauchy products, we obtain

$$
\begin{aligned}
\pi_{n+2} = & \beta^{n-2}\pi_4 + \sum_{k=0}^{n-3} \beta^k \frac{\lambda^{n+1-k}(\mu_1 - \theta - \mu_0)[\mu_0(\lambda + \theta) + \theta(\gamma + \lambda + \theta)]}{\mu_1(\gamma + \lambda + \theta + \mu_0)^{n+2-k}} p_{0,0}, \\
& \Longrightarrow
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=2}^{\infty} \pi_{n+2} \leq & |\pi_4| + |\pi_4| \sum_{n=3}^{\infty} |\beta|^{n-2} + \Big|\frac{(\mu_1 - \theta - \mu_0)[\mu_0(\lambda + \theta) + \theta(\gamma + \lambda + \theta)]}{\mu_1}\Big| |p_{0,0}| \times \\
& \sum_{n=3}^{\infty} \sum_{k=0}^{n-3} \Big|\frac{\lambda^{n+1-k}}{(\gamma + \lambda + \theta + \mu_0)^{n+2-k}}\Big| |\beta|^k \\
= & |\pi_4| \sum_{n=2}^{\infty} |\beta|^{n-2} + \Big|\frac{(\mu_1 - \theta - \mu_0)[\mu_0(\lambda + \theta) + \theta(\gamma + \lambda + \theta)]}{\mu_1(\gamma + \lambda + \theta + \mu_0)}\Big| |p_{0,0}| \times
\end{aligned}
$$

$$\sum_{n=3}^{\infty}|\frac{\lambda}{\gamma+\lambda+\theta+\mu_0}|^n\sum_{k=0}^{\infty}|\beta|^k. \tag{2.41}$$

From (2.21) and the Cauchy products of series we estimate, when $\Re\gamma+\mu_1>0$

$$\begin{aligned}\|p_{n,1}\|_{L^1[0,\infty)} &\le \sum_{k=1}^{n}\frac{\lambda^{n-k}}{(\Re\gamma+\lambda+\mu_1)^{n+1-k}}|p_{k,1}(0)|, \quad n\ge 1,\\ &\Longrightarrow\\ \sum_{n=1}^{\infty}\|p_{n,1}\|_{L^1[0,\infty)} &\le \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\lambda^{n-k}}{(\Re\gamma+\lambda+\mu_1)^{n+1-k}}|p_{k,1}(0)|\\ &= \sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(\Re\gamma+\lambda+\mu_1)^{n}}\sum_{n=1}^{\infty}|p_{n,1}(0)|\\ &= \frac{1}{\Re\gamma+\mu_1}\sum_{n=1}^{\infty}|p_{n,1}(0)|. \end{aligned}\tag{2.42}$$

From (2.20) and the Cauchy products of series we estimate, when $\Re\gamma+\theta+\mu_0>0$

$$\begin{aligned}\|p_{n,0}\|_{L^1[0,\infty)} &\le \frac{\lambda^{n-1}}{(\Re\gamma+\lambda+\theta+\mu_0)^{n}}\lambda|p_{0,0}|,\\ &\Longrightarrow\\ \sum_{n=1}^{\infty}\|p_{n,0}\|_{L^1[0,\infty)} &\le \sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(\Re\gamma+\lambda+\theta+\mu_0)^{n}}\lambda|p_{0,0}|\\ &\le \frac{\lambda}{\Re\gamma+\theta+\mu_0}|p_{0,0}|. \end{aligned}\tag{2.43}$$

For simplicity, let

$$\Omega := \left\{\gamma\in\mathcal{C}\;\middle|\; \begin{array}{l}\Re\gamma+\theta+\mu_0>0, \quad \Re\gamma+\mu_1>0\\ \left|\gamma+\lambda+\mu_1\pm\sqrt{(\gamma+\lambda+\mu_1)^2-4\lambda\mu_1}\right|<2\mu_1\end{array}\right\}.$$

It is easy to see that

$$\gamma\in\Omega \Longleftrightarrow \Re\gamma+\theta+\mu_0>0, \;\; \Re\gamma+\mu_1>0, \; |\alpha|<1, \; |\beta|<1.$$

Consequently, together with (2.39) and (2.40) we know that $\sum_{n=2}^{\infty}|p_{n+2,1}(0)|<\infty$ when $\gamma\in\Omega$. Substituting (2.9) and (2.11) into (2.7) and noting $\Re\gamma+\theta+\mu_0>0$, $\Re\gamma+\mu_1>0$ yields

$$\begin{aligned}(\gamma+\lambda)p_{0,0} &= \mu_0\int_0^{\infty}p_{1,0}(x)\mathrm{d}x+\mu_1\int_0^{\infty}p_{1,1}(x)\mathrm{d}x\\ &= \mu_0\int_0^{\infty}p_{1,0}(0)e^{-(\gamma+\lambda+\theta+\mu_0)x}\mathrm{d}x+\mu_1\int_0^{\infty}p_{1,1}(0)e^{-(\gamma+\lambda+\mu_1)x}\mathrm{d}x\\ &= \frac{\mu_1}{\gamma+\lambda+\mu_1}p_{1,1}(0)+\frac{\lambda\mu_1}{\gamma+\lambda+\theta+\mu_0}p_{0,0}\\ &\Longrightarrow\\ p_{1,1}(0) &= \frac{(\gamma+\lambda+\mu_1)[\mu_0\gamma+(\gamma+\lambda)(\gamma+\lambda+\theta)]}{\mu_1(\gamma+\lambda+\theta+\mu_0)}p_{0,0}. \end{aligned}\tag{2.44}$$

Combining (2.44) with (2.22) and (2.23), we can get that $|p_{1,1}(0)|$, $|p_{2,1}(0)|$ and $|p_{3,1}(0)|$ are finite for $\gamma \in \Omega$. Therefore

$$\sum_{n=1}^{\infty} |p_{n,1}(0)| = |p_{1,1}(0)| + |p_{2,1}(0)| + |p_{3,1}(0)| + \sum_{n=2}^{\infty} |p_{n+2,1}(0)| < \infty. \tag{2.45}$$

Together with (2.42) and (2.43) we conclude that

$$\|(p_0, p_1)\| = |p_{0,0}| + \sum_{n=1}^{\infty} \|p_{n,0}\|_{L^1[0,\infty)} + \sum_{n=1}^{\infty} \|p_{n,1}\|_{L^1[0,\infty)} < \infty,$$

which shows that all $\gamma \in \Omega$ are eigenvalues of $A + U + E$. Moreover, from (2.16), (2.19) and (2.43) it is evident that the eigenvectors corresponding to each γ span 1 dimensional linear space, i.e., their geometric multiplicity is one.

In the following, we consider the case that γ is a real number and obtain explicit results, which includes the following three cases:

Case 1 If $(\gamma+\lambda+\mu_1)^2 > 4\lambda\mu_1 \Rightarrow \gamma+\lambda+\mu_1 > 2\sqrt{\lambda\mu_1} \Rightarrow \gamma > 2\sqrt{\lambda\mu_1} - \lambda - \mu_1$, then by noting $\gamma + \mu_1 > 0,\ \gamma + \theta + \mu_0 > 0$, an easy computation shows that

$$0 < (\gamma+\lambda+\mu_1)^2 - 4\lambda\mu_1 < (\gamma+\lambda-\mu_1)^2 \Rightarrow |\alpha| < 1, |\beta| < 1,$$

which together with $\lambda < \mu_1 < 2\sqrt{\lambda\mu_1} - \lambda + \theta + \mu_0$ implies $\gamma \in \Omega$, that is, all points in $(2\sqrt{\lambda\mu_1} - \lambda - \mu_1, 0)$ are eigenvalues of $A + U + E$.

Case 2 If $(\gamma+\lambda+\mu_1)^2 = 4\lambda\mu_1 \Rightarrow \gamma = 2\sqrt{\lambda\mu_1} - \lambda - \mu_1$, then the condition $\lambda < \mu_1$ implies

$$\alpha = \beta = \frac{\gamma+\lambda+\mu_1}{2\mu_1} \Rightarrow |\alpha| = |\beta| = \frac{2\sqrt{\lambda\mu_1}}{2\mu_1} = \sqrt{\frac{\lambda}{\mu_1}} < 1$$
$$\Longrightarrow \gamma \in \Omega.$$

Therefore, $\gamma = 2\sqrt{\lambda\mu_1} - \lambda - \mu_1$ is eigenvalue of $A + U + E$.

Case 3 If $(\gamma+\lambda+\mu_1)^2 < 4\lambda\mu_1 \Rightarrow \gamma+\lambda+\mu_1 < 2\sqrt{\lambda\mu_1} \Rightarrow \gamma < 2\sqrt{\lambda\mu_1} - \lambda - \mu_1$, then the condition $\lambda < \mu_1$ gives

$$\begin{aligned}
\alpha &= \frac{\gamma+\lambda+\mu_1 + \sqrt{(\gamma+\lambda+\mu_1)^2 - 4\lambda\mu_1}}{2\mu_1} \\
&= \frac{\gamma+\lambda+\mu_1 + i\sqrt{4\lambda\mu_1 - (\gamma+\lambda+\mu_1)^2}}{2\mu_1} \\
\beta &= \frac{\gamma+\lambda+\mu_1 - i\sqrt{4\lambda\mu_1 - (\gamma+\lambda+\mu_1)^2}}{2\mu_1} \\
&\Longrightarrow
\end{aligned}$$

$$|\alpha| = |\beta| = \frac{\sqrt{(\gamma+\lambda+\mu_1)^2 - 4\lambda\mu_1 - (\gamma+\lambda+\mu_1)^2}}{2\mu_1} = \frac{2\sqrt{\lambda\mu_1}}{2\mu_1} = \sqrt{\frac{\lambda}{\mu_1}} < 1.$$

If $\lambda < \mu_1 < 2\sqrt{\lambda\mu_1} - \lambda + \theta + \mu_0$, then from $\gamma + \mu_1 > 0,\ \gamma + \theta + \mu_0 > 0$ we know that all points in $(-\min\{\mu_0 + \theta, \mu_1\}, 2\sqrt{\lambda\mu_1} - \lambda - \mu_1)$ are eigenvalues of $A + U + E$. Moreover, Kasim [10] proved that 0 is an eigenvalue of $A + U + E$ when $\lambda < \mu_1$.

To sum up, all points in $\Omega \cup \{0\}$ are eigenvalues of $A + U + E$ with geometric multiplicity one. In particular, the interval $(-\min\{\mu_0 + \theta, \mu_1\}, 0]$ belongs to the point spectrum of $A+U+E$ when $\lambda < \mu_1 < 2\sqrt{\lambda\mu_1} - \lambda - \theta - \mu_0$. $\square$

3. Conclusion and discussion

Let $\sigma_p(T(t))$ and $\sigma_p(A + U + E)$ be the point spectrum of $T(t)$ and $A + U + E$ respectively. From Theorem 2.1 and the spectral mapping theorem for the point spectrum [16, p. 277]

$$\sigma_p(T(t)) = e^{t\sigma_p(A+U+E)} \cup \{0\}$$

we know that $T(t)$ has uncountable eigenvalues and therefore it is not compact, even not eventually compact [16, P. 330].

Corollary 2.11 in Engel and Nagel [16, P. 258] states that if $T(t)$ is a C_0-semigroup on the Banach space X with generator $A + U + E$, then

(1) $\omega_0 = \max\{\omega_{ess}, s(A+U+E)\}$, where ω_0 is the growth bound of $T(t)$, ω_{ess} is the essential growth bound of $T(t)$, $s(A + U + E)$ is the spectral bound of $A + U + E$.

(2) $\sigma(A + U + E) \cup \{\gamma \in C|\Re\gamma \geq w\}$ is finite for each $\omega > \omega_{ess}$. Here, $\sigma(A + U + E)$ is the spectrum of $A + U + E$.

From the Theorem 1.1 we know that $\omega_0 = 0$ and $s(A + U + E) = 0$. These together with the statements (1) and (2) above yield $\omega_{ess} = 0$. From this and Proposition 3.5 in [16, P. 332], we conclude that $T(t)$ in not quasi-compact. Hence, this implies the essential difference between this model and the reliability models that are described by a finite number of partial differential equations [18], population equations [19–21].

Since $\omega_0 = 0$ and $\omega_{ess} = 0$, from Nagel [22, P. 74] it follows that

$$r(T(t)) = r_{ess}(T(t)) = e^{\omega_{ess}t} = e^0 = 1,$$

where $r(T(t))$ and $r_{ess}(T(t))$ are the spectral radius and essential spectral radius of $T(t)$, respectively.

Theorem 1.1 and Proposition 4.3.14 in Arendt et al. [23, P. 268] give the decomposition

$$(p_0, p_1)(x) = (p_0, p_1)_0(x) + (\phi_0, \phi_1)(x), \tag{2.46}$$

here $(p_0, p_1)(x) \in X$; $(p_0, p_1)_0(x)$ is the eigenvector with respect to 0, i.e., $(A+U+E)(p_0, p_1)_0(x) = 0$; $(\phi_0, \phi_1) \in \overline{\text{Range}(A + U + E)}$ and $\lim_{t\to\infty} \|T(t)(\phi_0, \phi_1)\| = 0$ by ABLV theorem [23, P. 374]. Let $(p_0, p_1)_\zeta(x)$ be eigenvector with respect to $-\min\{\mu_0 + \theta, \mu_1\}\zeta$ for $\zeta \in (0, 1)$ in Theorem 2.1. Then by using $(A + U + E)(p_0, p_1)_\zeta(x) = -\min\{\mu_0 + \theta, \mu_1\}\zeta(p_0, p_1)_\zeta(x)$ we have

$$\begin{aligned}
&T(t)((p_0, p_1)_0(x) + (A + U + E)(p_0, p_1)_\zeta(x)) \\
&\quad = (p_0, p_1)_0(x) + T(t)(A + U + E)(p_0, p_1)_\zeta(x) \\
&\quad = (p_0, p_1)_0(x) + T(t)[-\min\{\mu_0 + \theta, \mu_1\}\zeta(p_0, p_1)_\zeta(x)] \\
&\quad = (p_0, p_1)_0(x) - \min\{\mu_0 + \theta, \mu_1\}\zeta\, T(t)(p_0, p_1)_\zeta(x) \\
&\quad = (p_0, p_1)_0(x) - \min\{\mu_0 + \theta, \mu_1\}\zeta\, e^{-\min\{\mu_0+\theta,\mu_1\}\zeta t}(p_0, p_1)_\zeta(x) \\
&\quad \Longrightarrow \|T(t)((p_0, p_1)_0(\cdot) + (A + U + E)(p_0, p_1)_\zeta(\cdot)) - (p_0, p_1)_0(\cdot)\|
\end{aligned}$$

$$= \min\{\mu_0+\theta,\mu_1\}\zeta e^{-\min\{\mu_0+\theta,\mu_1\}\zeta t}\|(p_0,p_1)_\zeta\|, \quad \forall t\geq 0.$$

This shows that there are no positive constants $\nu>0$ and $\varrho>0$ such that

$$\begin{aligned}&\|T(t)\Big((p_0,p_1)_0(\cdot)+(A+U+E)(p_0,p_1)(\cdot)\Big)-(p_0,p_1)_0(\cdot)\|\\ &\leq \nu e^{\varrho t}\|(p_0,p_1)\|, \quad \forall t\geq 0,\ \forall (p_0,p_1)\in D(A).\end{aligned}$$

That is, it is impossible that the time-dependent solution of the system (1.4) exponentially converges to its steady-state solution. i.e., the convergence result given in Theorem 1.1 is optimal.

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