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Estimate and Fekete-Szegö Inequality for a Class of m-Fold Symmetric Bi-Univalent Function Defined by Subordination

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Abstract In this paper, we investigate the coefficient estimate and Fekete-Szegö inequality of a class of m-fold bi-univalent function defined by subordination. The results presented in this paper improve or generalize the recent works of other authors.

Keywords analytic functions; univalent functions; coefficient estimates; m-fold symmetric bi-univalent function; Fekete-Szegö inequality; subordination

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1. Introduction

Let A denote the class of functions of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disk $U=\{z:|z|<1\}$. Further, by S we denote the family of all functions in A which are univalent in U.

It is well known that every function $f\in S$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z))=z,\quad z\in U$$

and

$$f(f^{-1}(\omega))=\omega,\quad |\omega|<r_0(f),\ r_0(f)\geq\frac{1}{4}.$$

The inverse functions $g=f^{-1}$ is given by

$$f^{-1}(\omega)=\omega-a_2\omega^2+(2a_2^2-a_3)\omega^3-(5a_2^3-5a_2a_3+a_4)\omega^4+\cdots. \tag{1.2}$$

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A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of all bi-univalent functions in U given by (1.1). The class of bi-univalent functions was first introduced and studied by Lewin [1] and was showed that $|a_2| < 1.51$. Brannan and Clunie [2] improved Lewin's results to $|a_2| \leq \sqrt{2}$ and later Netanyahu [3] proved that max $|a_2| = 4/3$ if $f(z) \in \Sigma$. Recently, many authors investigated bounds for various subclasses of bi-univalent functions [4–10].

Let φ be an analytic and univalent function with positive real part in U such that $\varphi(0) = 1, \varphi'(0) > 0$ and $\varphi(U)$ is symmetric with respect to the real axis. The Taylor's series expansion of such function is of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots , \tag{1.3}$$

where all coefficients are real and $B_1 > 0$.

Recently, Tang and Orhan [11, 12] introduced the following subclass of bi-univalent function class Σ and obtained estimates on the first two coefficients $|a_2|$ and $|a_3|$ and the Fekete-Szegö inequality.

Definition 1.1 ([11]) *A function $f \in \Sigma$ given by (1.1) is said to be in the class $H^{\mu}_{\Sigma}(\lambda, \varphi)$ if it satisfies*

$$(1-\lambda)(\frac{f(z)}{z})^{\mu} + \lambda f'(z)(\frac{f(z)}{z})^{\mu-1} \prec \varphi(z), \quad \lambda \geq 1,\ \mu \geq 0,\ z \in U$$

and

$$(1-\lambda)(\frac{g(\omega)}{\omega})^{\mu} + \lambda g'(\omega)(\frac{g(\omega)}{\omega})^{\mu-1} \prec \varphi(\omega), \quad \lambda \geq 1,\ \mu \geq 0,\ \omega \in U,$$

where $g(\omega) = f^{-1}(\omega)$.

Theorem 1.2 ([11]) *Let the function f given by (1.1) be in the class $H^{\mu}_{\Sigma}(\lambda, \varphi)$, $\lambda \geq 1$ and $\mu \geq 0$. Then*

$$|a_2| \leq \min\{\frac{B_1}{\lambda+\mu}, \sqrt{\frac{2(B_1 + |B_2 - B_1|)}{(1+\mu)(2\lambda+\mu)}}\}$$

and

$$|a_3| \leq \begin{cases} \min\{\frac{B_1}{2\lambda+\mu} + \frac{B_1^2}{(\lambda+\mu)^2}, \frac{2(B_1+|B_2-B_1|)}{(1+\mu)(2\lambda+\mu)}\}, & 0 \leq \mu < 1, \\ \frac{B_1}{2\lambda+\mu} + \frac{2|B_2-B_1|}{(1+\mu)(2\lambda+\mu)}, & \mu \geq 1. \end{cases}$$

Theorem 1.3 ([12]) *Let the function f given by (1.1) be in the class $H^{\mu}_{\Sigma}(\lambda, \varphi)$, $\lambda \geq 1$ and $\mu \geq 0$. Then*

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{B_1}{2\lambda+\mu}, & |1-\gamma| \leq \frac{\mu+1}{2}|1 + \frac{2(B_1-B_2)(\lambda+\mu)^2}{B_1^2(2\lambda+\mu)(1+\mu)}|, \\ \frac{2B_1^3|1-\gamma|}{|(2\lambda+\mu)(1+\mu)B_1^2+2(B_1-B_2)(\lambda+\mu)^2|}, & |1-\gamma| \geq \frac{\mu+1}{2}|1 + \frac{2(B_1-B_2)(\lambda+\mu)^2}{B_1^2(2\lambda+\mu)(1+\mu)}|. \end{cases}$$

For each functions $f \in S$, the function

$$h(z) = \sqrt[m]{f(z^m)}, \quad z \in U; m \in N$$

is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric [13, 14] if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad z \in U;\ m \in N. \tag{1.4}$$

Srivastava et al. [15] defined m-fold symmetric univalent functions in U, analogous to the concept of m-fold symmetric univalent functions. For the normalized form of f given by (1.4), they obtained the series expansion for f^{-1} as follows:

$$\begin{aligned} g(\omega) =& f^{-1}(\omega) = \omega - a_{m+1}\omega^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]\omega^{2m+1} - \\ & [\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}]\omega^{3m+1} + \cdots . \end{aligned} \tag{1.5}$$

We denote by Σ_m the class of m-fold symmetric bi-univalent function in U. For $m = 1$, the formula (1.5) coincides with the formula (1.2) of the class Σ.

Recently, many researchers [15–20] introduced and investigated a lot of interesting subclass of m-fold symmetric bi-univalent functions. Motivated by them, we investigate the estimates $|a_{m+1}|$ and $|a_{2m+1}|$ for functions belonging to the new general subclass $N^{\mu}_{\Sigma_m}(\lambda, \varphi)$ of Σ_m. A new subclass $N^{\mu}_{\Sigma_m}(\lambda, \varphi)$ is defined as follows:

Definition 1.4 *A function $f \in \Sigma_m$ given by (1.4) is said to be in the class $N^{\mu}_{\Sigma_m}(\lambda, \varphi)$ if it satisfies*

$$(1-\lambda)(\frac{f(z)}{z})^{\mu} + \lambda f'(z)(\frac{f(z)}{z})^{\mu-1} \prec \varphi(z), \quad \lambda \geq 1,\ \mu \geq 0,\ z \in U$$

and

$$(1-\lambda)(\frac{g(\omega)}{\omega})^{\mu} + \lambda g'(\omega)(\frac{g(\omega)}{\omega})^{\mu-1} \prec \varphi(\omega), \quad \lambda \geq 1,\ \mu \geq 0,\ \omega \in U,$$

where the function g is given by (1.5).

Remark 1.5 There are many choices of φ, λ, μ, and m which would provide interesting subclasses of class $N^{\mu}_{\Sigma_m}(\lambda, \varphi)$. For example

(1) For $\lambda = 1$, $\mu = 0$ and $m = 1$, $N^0_{\Sigma_1}(1, \varphi) = S^0_{\Sigma_1}(\varphi)$ introduced by Ma and Minda [4].

(2) For $\mu = 0$, $\lambda = 1$, $m = 1$ and $\varphi(z) = (\frac{1+z}{1-z})^{\alpha}$ $(0 < \alpha \leq 1)$, $N^0_{\Sigma_1}(1, (\frac{1+z}{1-z})^{\alpha}) = S^*_{\Sigma}[\alpha]$ studied by Brannan and Taha [7].

(3) For $\mu = 0$, $\lambda = 1$, $m = 1$ and $\varphi(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \leq \beta < 1)$, $N^0_{\Sigma_1}(1, \frac{1+(1-2\beta)z}{1-z}) = S^*_{\Sigma}(\beta)$ studied by Brannan and Taha [7].

(4) For $m = 1$ and $\varphi(z) = (\frac{1+z}{1-z})^{\alpha}$ $(0 < \alpha \leq 1)$, $N^{\mu}_{\Sigma_1}(\lambda, (\frac{1+z}{1-z})^{\alpha}) = N^{\mu}_{\Sigma}(\alpha, \lambda)$ introduced by Çağlar et al. [8].

(5) For $m = 1$ and $\varphi(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \leq \beta < 1)$, $N^{\mu}_{\Sigma_1}(\lambda, \frac{1+(1-2\beta)z}{1-z}) = N^{\mu}_{\Sigma}(\beta, \lambda)$ introduced by Çağlar et al. [8].

(6) For $\mu = 1$, $m = 1$ and $\varphi(z) = (\frac{1+z}{1-z})^{\alpha}$ $(0 < \alpha \leq 1)$, $N^1_{\Sigma_1}(\lambda, (\frac{1+z}{1-z})^{\alpha}) = B_{\Sigma}(\alpha, \lambda)$ studied by Frasin and Aouf [9].

(7) For $\mu = 1$, $m = 1$ and $\varphi(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \leq \beta < 1)$, $N^1_{\Sigma_1}(\lambda, \frac{1+(1-2\beta)z}{1-z}) = B_{\Sigma}(\beta, \lambda)$ studied by Frasin and Aouf [9].

(8) For $\mu = 1$, $\lambda = 1$, $m = 1$ and $\varphi(z) = (\frac{1+z}{1-z})^{\alpha}$ $(0 < \alpha \leq 1)$, $N^1_{\Sigma_1}(1, (\frac{1+z}{1-z})^{\alpha}) = H^{\alpha}_{\Sigma}$ studied by Srivastava et al. [10].

(9) For $\mu=1$, $\lambda=1$, $m=1$ and $\varphi(z)=\frac{1+(1-2\beta)z}{1-z}$ $(0\le\beta<1)$, $N^1_{\Sigma_1}(1,\frac{1+(1-2\beta)z}{1-z})=H_\Sigma(\beta)$ studied by Srivastava et al. [10].

(10) For $m=1$, $N^\mu_{\Sigma_1}(\lambda,\varphi)=H^\mu_\sigma(\lambda,\varphi)$ studied by Tang et al. [11].

(11) For $\mu=1$, $\lambda=1$ and $\varphi(z)=(\frac{1+z}{1-z})^\alpha$ $(0<\alpha\le1)$, $N^1_{\Sigma_m}(1,(\frac{1+z}{1-z})^\alpha)=H^\alpha_{\Sigma,m}$ studied by Srivastava et al. [15].

(12) For $\mu=1,\lambda=1$ and $\varphi(z)=\frac{1+(1-2\beta)z}{1-z}$ $(0\le\beta<1)$, $N^1_{\Sigma_m}(1,\frac{1+(1-2\beta)z}{1-z})=H_{\Sigma,m}(\beta)$ studied by Srivastava et al. [15].

(13) For $\mu=1$ and $\lambda=1$, $N^1_{\Sigma_m}(1,\varphi)=H_{\sigma,m}(\varphi)$ studied by Çağlar and Gurusamy [17].

(14) For $\varphi(z)=(\frac{1+z}{1-z})^\alpha$ $(0<\alpha\le1)$, $N^\mu_{\Sigma_m}(\lambda,(\frac{1+z}{1-z})^\alpha)=N^\mu_{\Sigma,m}(\alpha,\lambda)$ studied by Bulut [18].

(15) For $\varphi(z)=\frac{1+(1-2\beta)z}{1-z}$ $(0\le\beta<1)$, $N^\mu_{\Sigma_m}(\lambda,\frac{1+(1-2\beta)z}{1-z})=N^\mu_{\Sigma,m}(\beta,\lambda)$ studied by Bulut [18].

(16) For $\mu=1$ and $\varphi(z)=(\frac{1+z}{1-z})^\alpha$ $(0<\alpha\le1)$ $N^1_{\Sigma_m}(\lambda,(\frac{1+z}{1-z})^\alpha)=A^{\alpha,\lambda}_{\Sigma,m}$ studied by Sümer [19].

(17) For $\mu=1$ and $\varphi(z)=\frac{1+(1-2\beta)z}{1-z}$ $(0\le\beta<1)$, $N^1_{\Sigma_m}(\lambda,\frac{1+(1-2\beta)z}{1-z})=A^\lambda_{\Sigma,m}(\beta)$ studied by Sümer [19].

(18) For $\mu=0,\lambda=1$ and $\varphi(z)=\frac{1+(1-2\beta)z}{1-z}$ $(0\le\beta<1)$, $N^0_{\Sigma_m}(1,\frac{1+(1-2\beta)z}{1-z})$ introduced by Hamidi and Jahangiri [20].

(19) For $\mu=0,\lambda=1$ and $\varphi(z)=(\frac{1+z}{1-z})^\alpha$ $(0<\alpha\le1)$, $N^0_{\Sigma_m}(1,(\frac{1+z}{1-z})^\alpha)=S^\alpha_{\Sigma,m}$.

(20) For $\lambda=1$ and $\varphi(z)=(\frac{1+z}{1-z})^\alpha$ $(0<\alpha\le1)$, a new class $N^\mu_{\Sigma_m}(1,(\frac{1+z}{1-z})^\alpha)$ is obtained, which consists of m-fold symmetric bi-Bazilevič functions.

(21) For $\lambda=1$ and $\varphi(z)=\frac{1+(1-2\beta)z}{1-z}$ $(0\le\beta<1)$, a new class $N^\mu_{\Sigma_m}(1,\frac{1+(1-2\beta)z}{1-z})$ is obtained, which consists of m-fold symmetric bi-Bazilevič functions.

In order to derive our main results, we shall need the following lemma.

Lemma 1.6 ([21]) *If $p(z)\in P$, then $|c_n|\le2$ for each n, where P is the family of all functions p, analytic in U for which $R\{p(z)\}>0$, where*

$$p(z)=1+c_1z+c_2z^2+\cdots,\quad z\in U.$$

2. Coefficient estimates

Using Lemma 1.6, our first main results is given by Theorem 2.1 below:

Theorem 2.1 *Let $f(z)$ given by (1.4) be in the class $N^\mu_{\Sigma_m}(\lambda,\varphi)$. Then*

$$|a_{m+1}|\le\min\big\{\frac{B_1}{\mu+m\lambda},\sqrt{\frac{2(B_1+|B_2-B_1|)}{(m+\mu)(\mu+2m\lambda)}},\Omega_1\big\},\tag{2.1}$$

$$|a_{2m+1}| \leq \begin{cases} \frac{B_1}{\mu+2m\lambda}, & \frac{B_1-B_2}{B_1^2} \in (-\infty, -\frac{(1+2m+\mu)(\mu+2m\lambda)}{2(\mu+m\lambda)^2}] \cup [\frac{(1-\mu)(\mu+2m\lambda)}{2(\mu+m\lambda)^2}, +\infty), \\ \min\left\{\frac{(1+m)B_1^2}{2(\mu+m\lambda)^2} + \frac{B_1}{\mu+2m\lambda}, \Omega_2, \Omega_3\right\}, & \frac{B_1-B_2}{B_1^2} \in [-\frac{(1+2m+\mu)(\mu+2m\lambda)}{2(\mu+m\lambda)^2}, -\frac{(m+\mu)(\mu+2m\lambda)}{2(\mu+m\lambda)^2}) \cup (-\frac{(m+\mu)(\mu+2m\lambda)}{2(\mu+m\lambda)^2}), \frac{(1-\mu)(\mu+2m\lambda)}{2(\mu+m\lambda)^2}], \end{cases} \quad (2.2)$$

where

$$\Omega_1 = \frac{B_1\sqrt{2B_1}}{\sqrt{|(m+\mu)(\mu+2m\lambda)B_1^2 + 2(B_1-B_2)(\mu+m\lambda)^2|}},$$

$$\Omega_2 = \frac{(1+m)(B_1+|B_2-B_1|)}{(m+\mu)(\mu+2\lambda m)} + \frac{B_1}{\mu+2\lambda m},$$

$$\Omega_3 = \frac{(1+m)B_1^3}{|(m+\mu)(\mu+2\lambda m)B_1^2 + 2(B_1-B_2)(\mu+\lambda m)^2|}.$$

Proof Let $f \in N^{\mu}_{\Sigma_m}(\lambda, \varphi)$ and $g = f^{-1}$. Then there are analytic functions $u, v : U \to U$, with $u(0) = v(0) = 0$ satisfying

$$(1-\lambda)(\frac{f(z)}{z})^{\mu} + \lambda f'(z)(\frac{f(z)}{z})^{\mu-1} = \varphi(u(z)), \quad (2.3)$$

$$(1-\lambda)(\frac{g(\omega)}{\omega})^{\mu} + \lambda g'(\omega)(\frac{g(\omega)}{\omega})^{\mu-1} = \varphi(v(\omega)). \quad (2.4)$$

Define the functions $p_1(z)$ and $p_2(z)$ by

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_m z^m + c_{2m} z^{2m} + c_{3m} z^{3m} + \cdots$$

and

$$p_2(z) = \frac{1+v(z)}{1-v(z)} = 1 + b_m z^m + b_{2m} z^{2m} + b_{3m} z^{3m} + \cdots$$

or equivalently,

$$u(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}c_m z^m + (\frac{1}{2}c_{2m} - \frac{c_m^2}{4})z^{2m} + \cdots \quad (2.5)$$

and

$$v(z) = \frac{p_2(z)-1}{p_2(z)+1} = \frac{1}{2}b_m z^m + (\frac{1}{2}b_{2m} - \frac{b_m^2}{4})z^{2m} + \cdots. \quad (2.6)$$

From (2.3)–(2.6), we have

$$(1-\lambda)(\frac{f(z)}{z})^{\mu} + \lambda f'(z)(\frac{f(z)}{z})^{\mu-1} = \varphi(\frac{p_1(z)-1}{p_1(z)+1}) \quad (2.7)$$

and

$$(1-\lambda)(\frac{g(\omega)}{\omega})^{\mu} + \lambda g'(\omega)(\frac{g(\omega)}{\omega})^{\mu-1} = \varphi(\frac{p_2(\omega)-1}{p_2(\omega)+1}). \quad (2.8)$$

Using (2.5) and (2.6), together with (1.3) we get

$$\varphi(\frac{p_1(z)-1}{p_1(z)+1}) = 1 + \frac{1}{2}B_1 c_m z^m + [\frac{1}{2}B_1 c_{2m} + \frac{(B_2-B_1)c_m^2}{4}]z^{2m} + \cdots, \quad (2.9)$$

$$\varphi(\frac{p_2(\omega)-1}{p_\omega(z)+1}) = 1 + \frac{1}{2}B_1 b_m \omega^m + [\frac{1}{2}B_1 b_{2m} + \frac{(B_2-B_1)b_m^2}{4}]\omega^{2m} + \cdots. \quad (2.10)$$

Since

$$(1-\lambda)(\frac{f(z)}{z})^{\mu}+\lambda f'(z)(\frac{f(z)}{z})^{\mu-1}=1+(\mu+m\lambda)a_{m+1}z^m$$
$$(\mu+2m\lambda)[\frac{\mu-1}{2}a_{m+1}^2+a_{2m+1}]z^{2m}+\cdots$$

and

$$(1-\lambda)(\frac{g(\omega)}{\omega})^{\mu}+\lambda g'(\omega)(\frac{g(\omega)}{\omega})^{\mu-1}=1-(\mu+m\lambda)a_{m+1}\omega^m+$$
$$(\mu+2m\lambda)[(m+\frac{\mu+1}{2})a_{m+1}^2-a_{2m+1}]\omega^{2m}+\cdots$$

it follows frrom (2.7)–(2.10) that

$$(\mu+m\lambda)a_{m+1}=\frac{1}{2}B_1c_m, \tag{2.11}$$

$$(\mu+2m\lambda)[\frac{\mu-1}{2}a_{m+1}^2+a_{2m+1}]=\frac{1}{2}B_1c_{2m}+\frac{(B_2-B_1)c_m^2}{4}, \tag{2.12}$$

$$-(\mu+m\lambda)a_{m+1}=\frac{1}{2}B_1b_m, \tag{2.13}$$

$$(\mu+2m\lambda)[(m+\frac{\mu+1}{2})a_{m+1}^2-a_{2m+1}]=\frac{1}{2}B_1b_{2m}+\frac{(B_2-B_1)b_m^2}{4}. \tag{2.14}$$

From (2.11) and (2.13), we get

$$c_m=-b_m, \tag{2.15}$$

$$a_{m+1}^2=\frac{B_1^2(c_m^2+b_m^2)}{8(\mu+m\lambda)^2}. \tag{2.16}$$

Applying Lemma 1.6 for the coefficients c_m and b_m, we have

$$|a_m|\le\frac{B_1}{\mu+m\lambda}. \tag{2.17}$$

Adding (2.12) and (2.14), we have

$$(m+\mu)(\mu+2m\lambda)a_{m+1}^2=\frac{B_1(c_{2m}+b_{2m})}{2}+\frac{B_2-B_1}{4}(c_m^2+b_m^2). \tag{2.18}$$

Applying Lemma 1.6 for the coefficients c_m, c_{2m}, b_m and b_{2m}, we have

$$|a_{m+1}|\le\sqrt{\frac{2(B_1+|B_2-B_1|)}{(m+\mu)(2\lambda m+\mu)}}. \tag{2.19}$$

Substituting (2.15) and (2.16) into (2.18), we get

$$c_m^2=\frac{2B_1(\mu+m\lambda)^2(c_{2m}+b_{2m})}{(m+\mu)(\mu+2m\lambda)B_1^2+2(B_1-B_2)(\mu+m\lambda)^2}. \tag{2.20}$$

From (2.15), (2.20) and (2.16), we get

$$a_{m+1}^2=\frac{B_1^3(c_{2m}+b_{2m})}{2(m+\mu)(\mu+2m\lambda)B_1^2+4(B_1-B_2)(\mu+m\lambda)^2}. \tag{2.21}$$

Then, in view of Lemma 1.6, we have

$$|a_{m+1}|\le\frac{B_1\sqrt{2B_1}}{\sqrt{|(m+\mu)(\mu+2\lambda m)B_1^2+2(B_1-B_2)(\mu+\lambda m)^2|}}. \tag{2.22}$$

Now, from (2.17), (2.19) and (2.22), we get

$$|a_{m+1}| \leq \min\{\frac{B_1}{\lambda m+\mu}, \sqrt{\frac{2(B_1+|B_2-B_1|)}{(m+\mu)(2\lambda m+\mu)}}, \frac{B_1\sqrt{2B_1}}{\sqrt{|(m+\mu)(\mu+2\lambda m)B_1^2+2(B_1-B_2)(\mu+\lambda m)^2|}}\}.$$

By subtracting (2.14) from (2.12), we obtain

$$a_{2m+1} = \frac{1+m}{2}a_{m+1}^2 + \frac{B_1}{4(\mu+2m\lambda)}(c_{2m}-b_{2m}). \tag{2.23}$$

Substituting (2.11) into (2.23) and using Lemma 1.6, we get

$$|a_{2m+1}| \leq \frac{(1+m)B_1^2}{2(\mu+\lambda m)^2} + \frac{B_1}{\mu+2\lambda m}. \tag{2.24}$$

Substituting (2.18) into (2.23) and using Lemma 1.6, we get

$$|a_{2m+1}| \leq \frac{(1+m)(B_1+|B_2-B_1|)+(m+\mu)B_1}{(m+\mu)(\mu+2\lambda m)}. \tag{2.25}$$

From (2.11) and (2.23) it follows that

$$a_{2m+1} = B_1\{[h(\frac{B_1-B_2}{B_1^2}) + \frac{1}{4(\mu+2m\lambda)}]c_{2m} + [h(\frac{B_1-B_2}{B_1^2}) - \frac{1}{4(\mu+2m\lambda)}]b_{2m}\},$$

where

$$h(\frac{B_1-B_2}{B_1^2}) = \frac{(1+m)}{4(m+\mu)(\mu+2m\lambda)+8\frac{B_1-B_2}{B_1^2}(\mu+m\lambda)^2}.$$

Since all B_i are real and $B_1>0$, we conclude that

$$|a_3| \leq \begin{cases} 4B_1|h(\frac{B_1-B_2}{B_1^2})|, & |h(\frac{B_1-B_2}{B_1^2}) \geq \frac{1}{4(\mu+2m\lambda)}, \\ \frac{B_1}{\mu+2m\lambda}, & 0 \leq |h(\frac{B_1-B_2}{B_1^2})| \leq \frac{1}{4(\mu+2m\lambda)}. \end{cases}$$

This completes the proof of Theorem 2.1. □

Theorem 2.2 *Let $f(z)$ given by (1.4) be in the class $N^{\mu}_{\Sigma_m}(\lambda,\varphi)$. Then*

$$|a_3-\gamma a_2^2| \leq \begin{cases} \frac{B_1}{\mu+2m\lambda}, & 0 \leq |h(\gamma)| < \frac{1}{4(\mu+2m\lambda)}, \\ \frac{|1+m-2\gamma|B_1^3}{|(m+\mu)(\mu+2m\lambda)B_1^2+2(B_1-B_2)(\mu+m\lambda)^2|}, & |h(\gamma)| \geq \frac{1}{4(\mu+2m\lambda)}. \end{cases} \tag{2.26}$$

Proof By using the equalities (2.21) and (2.23), we have

$$a_{2m+1}-\gamma a_{m+1}^2 = B_1[(h(\gamma)+\frac{1}{4(\mu+2m\lambda)})c_{2m} + (h(\gamma)-\frac{1}{4(\mu+2m\lambda)})b_{2m}],$$

where

$$h(\gamma) = \frac{(1+m-2\gamma)B_1^2}{4(m+\mu)(\mu+2m\lambda)B_1^2+8(B_1-B_2)(\mu+m\lambda)^2}.$$

Since all B_i are real and $B_1>0$, we conclude that

$$|a_3-\gamma a_2^2| \leq \begin{cases} \frac{B_1}{\mu+2m\lambda}, & 0 \leq |h(\gamma)| < \frac{1}{4(\mu+2m\lambda)}, \\ \frac{|1+m-2\gamma|B_1^3}{|(m+\mu)(\mu+2m\lambda)B_1^2+2(B_1-B_2)(\mu+m\lambda)^2|}, & |h(\gamma)| \geq \frac{1}{4(\mu+2m\lambda)}, \end{cases}$$

which completes the proof. □

3. Corollaries and consequences

Setting $m=1$ in Theorem 2.1, we have the following corollary.

Corollary 3.1 *Let $f(z)$ given by (1.4) be in the class $N^{\mu}_{\Sigma_1}(\lambda,\varphi)$. Then*

$$|a_2|\le \min\Big\{\frac{B_1}{\lambda+\mu},\sqrt{\frac{2(B_1+|B_2-B_1|)}{(1+\mu)(2\lambda+\mu)}},\Omega_1\Big\},$$

$$|a_{2m+1}|\le \begin{cases} \frac{B_1}{\mu+2\lambda}, & \frac{B_1-B_2}{B_1^2}\in(-\infty,-\frac{(3+\mu)(\mu+2\lambda)}{2(\mu+\lambda)^2}]\cup [\frac{(1-\mu)(\mu+2\lambda)}{2(\mu+\lambda)^2},+\infty),\\ \min\big\{\frac{B_1^2}{(\mu+\lambda)^2}+\frac{B_1}{\mu+2\lambda},\Omega_2,\Omega_3\big\}, & \frac{B_1-B_2}{B_1^2}\in[-\frac{(3+\mu)(\mu+2\lambda)}{2(\mu+\lambda)^2},-\frac{(1+\mu)(\mu+2\lambda)}{2(\mu+\lambda)^2})\cup(-\frac{(1+\mu)(\mu+2\lambda)}{2(\mu+\lambda)^2}),\frac{(1-\mu)(\mu+2\lambda)}{2(\mu+\lambda)^2}]. \end{cases}$$

where

$$\Omega_1=\frac{B_1\sqrt{2B_1}}{\sqrt{|(1+\mu)(\mu+2\lambda)B_1^2+2(B_1-B_2)(\mu+\lambda)^2|}},$$

$$\Omega_2=\frac{2(B_1+|B_2-B_1|)}{(1+\mu)(\mu+2\lambda)}+\frac{B_1}{\mu+2\lambda},$$

$$\Omega_3=\frac{2B_1^3}{|(1+\mu)(\mu+2\lambda)B_1^2+2(B_1-B_2)(\mu+\lambda)^2|}.$$

Remark 3.2 The estimates of the coefficients $|a_2|$ and $|a_3|$ of Corollary 3.1 are the improvements of the estimates obtained in [11, Theorem 2.1].

Setting $\varphi(z)=(\frac{1+z}{1-z})^{\alpha}$ $(0<\alpha\le 1)$ in Theorem 2.1, we have the following corollary.

Corollary 3.3 *Let $f(z)$ given by (1.4) be in the class $N^{\mu}_{\Sigma_m}(\lambda,(\frac{1+z}{1-z})^{\alpha})=N^{\mu}_{\Sigma,m}(\alpha,\lambda)$. Then*

$$|a_{m+1}|\le \min\Big\{\frac{2\alpha}{\lambda m+\mu},\sqrt{\frac{8\alpha-4\alpha^2}{(m+\mu)(2\lambda m+\mu)}},\frac{2\alpha}{\sqrt{(\mu+m\lambda)^2+m\alpha(\mu+2m\lambda-m\lambda^2)}}\Big\},$$

$$|a_{2m+1}|\le \begin{cases} \frac{2\alpha}{\mu+2\lambda}, & 0<\alpha\le\frac{(\mu+m\lambda)^2}{\mu+2m\lambda+m^2\lambda^2},\\ \min\big\{\frac{2(1+m)\alpha^2}{(\mu+\lambda m)^2}+\frac{2\alpha}{\mu+2\lambda m},\Omega_1,\Omega_2\big\}, & \frac{(\mu+m\lambda)^2}{\mu+2m\lambda+m^2\lambda^2}<\alpha\le 1, \end{cases}$$

where $\Omega_1=\frac{(1+m)(4\alpha-2\alpha^2)}{(m+\mu)(\mu+2\lambda m)}+\frac{2\alpha}{\mu+2\lambda m}$, $\Omega_2=\frac{2(1+m)\alpha^2}{(\mu+\lambda m)^2+m\alpha(\mu+2m\lambda-m\lambda^2)}$.

Remark 3.4 The estimates of the coefficients $|a_{m+1}|$ and a_{2m+1} of Corollary 3.2 are the improvement of the estimates obtained in [18, Theorem 4].

Setting $\varphi(z)=\frac{1+(1-2\beta)z}{1-z}$ $(o\le\beta<1)$ in Theorem 2.1, we have the following corollary.

Corollary 3.5 *Let $f(z)$ given by (1.4) be in the class $N^{\mu}_{\Sigma_m}(\lambda,\frac{1+(1-2\beta)z}{1-z})=N^{\mu}_{\Sigma,m}(\beta,\lambda)$. Then*

$$|a_{m+1}|\le \begin{cases} \sqrt{\frac{4(1-\beta)}{(m+\mu)(2\lambda m+\mu)}}, & 0\le\beta<\frac{m(1+2m\lambda-m\lambda^2)}{(1+2m\lambda)(1+m)},\\ \frac{2(1-\beta)}{\lambda m+\mu}, & \frac{m(1+2m\lambda-m\lambda^2)}{(1+2m\lambda)(1+m)}\le\beta<1. \end{cases}$$

$$|a_{2m+1}|\le \begin{cases} \frac{2(1-\beta)}{\mu+2m\lambda}, & \mu\ge 1,\\ \min\{\frac{2(1+m)(1-\beta)^2}{(\mu+m\lambda)^2}+\frac{2(1-\beta)}{\mu+2m\lambda},\frac{2(1+m)(1-\beta)}{(m+\mu)(\mu+2\lambda m)}\}, & 0\le\mu<1. \end{cases}$$

Remark 3.6 The estimate of the coefficients $|a_{2m+1}|$ of Corollary 3.3 is the improvement of the estimate obtained in [18, Theorem 15]. Setting $m=1$ in Theorem 2.2, we have the following corollary.

Corollary 3.7 ([12, Theorem 2.1]) *Let $f(z)$ given by (1.4) be in the class $N^{\mu}_{\Sigma_1}(\lambda,\varphi)$. Then*

$$|a_3-\gamma a_2^2|\le\begin{cases}\frac{B_1}{\mu+2\lambda}, & 0\le|h(\gamma)|<\frac{1}{4(\mu+2\lambda)},\\ \frac{2|1-\gamma|B_1^3}{|(1+\mu)(\mu+2\lambda)B_1^2+2(B_1-B_2)(\mu+\lambda)^2|}, & |h(\gamma)|\ge\frac{1}{4(\mu+2\lambda)}.\end{cases}$$

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