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The Structure of a Lie Algebra Attached to a Unit Form

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Abstract Let $n \geq 4$. The complex Lie algebra, which is attached to the unit form $\mathfrak{q}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n-1} x_i x_{i+1}) + x_1 x_n$ and defined by generators and generalized Serre relations, is proved to be a finite-dimensional simple Lie algebra of type $\mathbb{D}_n$, and realized by the Ringel-Hall Lie algebra of a Nakayama algebra. As its application of the realization, we give the roots and a Chevalley basis of the simple Lie algebra.

Keywords Nakayama algebras; finite dimensional simple Lie algebras; Ringel-Hall Lie algebras

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1. Introduction

There are a lot of papers studying the relationship between the representation theories with Kac-Moody algebras or elliptic algebras. An important tool used in the studying is Ringel-Hall Lie algebra. Here we recall some background on the Ringel-Hall Lie algebras. Let A be an associative algebra over a finite field and M, N and L finite A-modules. Let $F_{M,N}^{L}$ be the number of submodules V of L such that $V \simeq N$ and $L/V \simeq M$. By the definition in [1], the Ringel-Hall algebra of A is an associative ring with a $\mathbb{Z}$-basis, indexed by the isoclasses $[M]$ of all finite A-modules M, and the multiplication: $[M] \cdot [N] = \sum_{[L]} F_{M,N}^{L}[L]$. In case A is hereditary of finite type, Ringel [1–3] showed that the subring of the degenerate Ringel-Hall algebra with a $\mathbb{Z}$-basis indexed by isoclasses of all indecomposable A-modules is a Lie subalgebra under the Lie multiplication of commutators, and over complex numbers it is isomorphic to the positive part of the corresponding complex semisimple Lie algebras such that the isoclasses of all indecomposable A-modules correspond to a Chevalley basis. Such Lie subalgebra is called the Ringel-Hall Lie algebra. To realize the whole (not only the positive part) of a Kac-Moody Lie algebra, the Ringel-Hall Lie algebras of 2-period triangulated categories have been constructed in [4, 5]. Here the Ringel-Hall numbers are related to triangles instead of short exact sequences. Then any symmetrizable Kac-Moody Lie algebra can be realized by the Ringel-Hall Lie algebra of the root category of the corresponding hereditary algebra A (see [5]). Here the root category is the orbit category $\mathcal{R}(A) = \mathcal{D}^b(A)/T^2$, where $\mathcal{D}^b(A)$ is the derived category of A and T is the shift functor. Lin and Peng [6] proved that the elliptic Lie algebra of type $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$,

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$E_8^{(1,1)}$ is isomorphic to the Ringel-Hall Lie algebra of the root category of the tubular algebra with type $\mathbb{T}(2,2,2,2), \mathbb{T}(3,3,3), \mathbb{T}(4,4,2), \mathbb{T}(6,3,2)$.

Recently, the authors in [7] built a relationship between the representation theory of a Nakayama algebra and a Lie algebra attached to a unit form. The kind of Lie algebras are introduced by Barot, Kussin and Lenzing in [8], and they are defined by generators and generalized Serre relations. These generalized Serre relations are different from the Serre relations in the definitions of Kac-Moody algebras. The root systems of the Lie algebras in [8] associated with a root space decomposition are different to determine. In [7] the authors determined the root system and a Chevalley basis of a complex Lie algebra attached to the unit form $\mathfrak{q}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^n x_i^2 + \sum_{1\le i<j\le n}(-1)^{j-i}x_ix_j$ via the Ringel-Hall Lie algebra of root category of some Nakayama algebra. In this paper, in a similar way, we determine the root system and a Chevalley basis of the complex Lie algebra attached to the unit form $\mathfrak{q}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^n x_i^2 - (\sum_{i=1}^{n-1} x_ix_{i+1}) + x_1x_n$ via the representation theory of a different Nakayama algebra.

A square matrix $C = (C_{ij})_{n\times n}$ with integral coefficients is called a quasi-Cartan matrix if it is symmetrizable and $C_{ii} = 2$ for all i. A quasi-Cartan matrix C is called a Cartan matrix if it is positive definite and $C_{ij} \le 0$ for all $i \ne j$. Let $\mathbb{Z}^n$ be the set of the integral n-dimensional row vectors, and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ the canonical basis of $\mathbb{Z}^n$. A unit form is a quadratic form $\mathfrak{q} : \mathbb{Z}^n \to \mathbb{Z}$, $\mathfrak{q}(x) = \sum_{i=1}^n x_i^2 + \sum_{i<j} q_{ij}x_ix_j$, with integral coefficients $q_{ij} \in \mathbb{Z}$. Each unit form $\mathfrak{q} : \mathbb{Z}^n \to \mathbb{Z}$ has an associated symmetric quasi-Cartan matrix $C = C(\mathfrak{q})$ given by $C_{ij} = \mathfrak{q}(\epsilon_i + \epsilon_j) - \mathfrak{q}(\epsilon_i) - \mathfrak{q}(\epsilon_j)$. A unit form is called positive definite if $\mathfrak{q}(x) > 0$ for each $0 \ne x \in \mathbb{Z}^n$. We associate with any unit form $\mathfrak{q}$ a bigraph $B(\mathfrak{q})$, which has vertices $1, 2, \ldots, n$ and $|q_{ij}|$ solid (resp., dotted) lines between i and j if $q_{ij} < 0$ (resp., $q_{ij} > 0$). A unit form q is connected if so is $B(\mathfrak{q})$. If $C(\mathfrak{q})$ is a Cartan matrix, then $B(\mathfrak{q})$ is a graph (there are no broken edges) Γ, which by the Cartan-Killing classification is a disjoint union of Dynkin diagrams $\mathbb{A}_m$ $(m \ge 1)$, $\mathbb{D}_m$ $(m \ge 4)$ and $\mathbb{E}_m$ $(m = 6, 7, 8)$. In this case, we write $\mathfrak{q} = \mathfrak{q}_\Gamma$ and call Γ the Dynkin type of $\mathfrak{q}$.

Given a unit form $\mathfrak{q}$, set $C = C(\mathfrak{q})$ and let $\mathfrak{g}_4(\mathfrak{q})$ be the complex Lie algebra defined by the generators e_i, e_{-i}, h_i $(1 \le i \le n)$ which are homogeneous of degree ϵ_i, $-\epsilon_i$ and 0, respectively, and subject to the following relations:

(R_1) $[h_i, h_j] = 0$, for all i and j,

(R_2) $[h_i, e_{\delta j}] = \delta C_{ij} e_{\delta j}$, for all i, j and $\delta = \pm1$,

(R_3) $[e_{\delta i}, e_{-\delta i}] = \delta h_i$, for all i and $\delta = \pm1$,

(R_4) $(\mathrm{ad}\, e_{\delta i})^{1+m}(e_{\gamma j}) = 0$, where $m = \max\{0, -\delta\gamma C_{ij}\}$, for $\delta, \gamma \in \{1, -1\}$ and $1 \le i, j \le n$.

If $\mathfrak{q}$ is a positive definite unit form such that its quasi-Cartan matrix is a Cartan matrix, then by [9], $\mathfrak{g}_4(q)$ is a semisimple finite dimensional Lie algebra. Note that in general, when A is not necessarily a Cartan matrix, the relations (R_4) are a subset of the relations

(R_∞) $[e_{\delta_1 i_1}, e_{\delta_2 i_2}, \ldots, e_{\delta_t i_t}] = 0$, whenever $\mathfrak{q}(\sum_{j=1}^t \delta_j \epsilon_{i_j}) > 1$ for $\delta_j = \pm1$,
where the multibrackets are defined inductively by

$$[x_1, x_2, \ldots, x_t] = [x_1, [x_2, \ldots, x_t]].$$

Let $\mathfrak{g}_\infty(\mathfrak{q})$ be the Lie algebra defined by the generators e_i, e_{-i}, h_i $(1 \le i \le n)$ and by the relations (R_1), (R_2), (R_3), (R_∞). We recall that any positive definite unit form has a unique associated Dynkin type Γ such that $\mathfrak{q}$ is equivalent to $\mathfrak{q}_\Gamma$, that is $\mathfrak{q} = \mathfrak{q}_\Gamma \circ T$ for some $\mathbb{Z}$-invertible integral matrix T. In this case, it is denoted by $\mathfrak{q} \sim \mathfrak{q}_\Gamma$. By [8], if $\mathfrak{q}$ is positive definite of Dynkin type Γ, then $\mathfrak{g}_\infty(\mathfrak{q})$ is isomorphic to $\mathfrak{g}_4(\mathfrak{q}_\Gamma)$.

In this paper, we will study the complex Lie algebra $\mathfrak{g}_\infty(\mathfrak{q})$ attached to the following unit form

$$\mathfrak{q}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i^2 - \Big(\sum_{i=1}^{n-1} x_i x_{i+1}\Big) + x_1 x_n, \quad n \ge 4.$$

We will show that the above unit form $\mathfrak{q}$ is positive definite and connected, and so the attached Lie algebra $\mathfrak{g}_\infty(\mathfrak{q})$ is a finite dimensional simple Lie algebra. Next we determine the graph Γ satisfying that $\mathfrak{g}_\infty(\mathfrak{q}) \simeq \mathfrak{g}_4(\mathfrak{q}_\Gamma)$, and a Chevalley basis of $\mathfrak{g}_\infty(\mathfrak{q})$. To achieve this aim, we prove that the Lie algebra $\mathfrak{g}_\infty(\mathfrak{q})$ is isomorphic to the Ringel-Hall Lie algebra $\mathfrak{g}$ of a Nakayama algebra, and so $\mathfrak{g}_\infty(\mathfrak{q})$ is a simple Lie algebra of type $\mathbb{D}_n$. As its application of the realization, we give the roots, a root space decomposition and a Chevalley basis of the simple Lie algebra.

Let us give a brief view on the content of this article. In Section 2, we study some properties of the given unit form and the related Lie algebra defined by generators and generalized Serre relations. In Section 3, we show some properties of a class of Nakayama algebras, which will be applied to realize the above Lie algebra. In Section 4, we recall the Ringel-Hall Lie algebra of the above Nakayama algebra, and prove that it is isomorphic to the above simple Lie algebra. As an application of the isomorphism theorem, Section 5 shows that there is a bijection between the set of the indecomposable objects of the root category of the Nakayama algebra and the set of the roots of the positive definite unit form, and give the Dynkin type, a root space decomposition and a Chevalley basis of the above simple Lie algebra.

2. A positive definite unit form

The following lemma gives some properties of the unit form

$$\mathfrak{q}(x) = \sum_{i=1}^{n} x_i^2 - \Big(\sum_{i=1}^{n-1} x_i x_{i+1}\Big) + x_1 x_n.$$

Lemma 2.1 *Let $\Delta = \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n | \mathfrak{q}(x_1, x_2, \ldots, x_n) = 1\}$. Then*

(i) The associated Cartan matrix $C = C(\mathfrak{q})$ is

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

(ii) $\mathfrak{q}(x_1,x_2,\ldots,x_n)=\frac{1}{2}[(\sum_{i=1}^{n-1}(x_i-x_{i+1})^2+(x_1+x_n)^2]$, *and* $\mathfrak{q}$ *is connected positive definite.*

(iii) $\Delta=\{\pm\epsilon_i|1\le i\le n\}\cup\{\pm(\sum_{k=i}^{j}\epsilon_k)|1<i<j\le n\}\cup\{\pm(\sum_{k=1}^{i}\epsilon_k)|1<i<n\}\cup\{\pm(\sum_{k=1}^{i}\epsilon_k-\sum_{k=j+1}^{n}\epsilon_k)|1\le i<j\le n-1\}$, *and* $|\Delta|=2n^2-2n$.

Proof (i) For $1\le i\le n$, $C_{ii}=\mathfrak{q}(2\epsilon_i)-2\mathfrak{q}(\epsilon_i)=4-2=2$. For $1\le i\le n-1$, $C_{i,i+1}=\mathfrak{q}(\epsilon_i+\epsilon_{i+1})-\mathfrak{q}(\epsilon_i)-\mathfrak{q}(\epsilon_{i+1})=-1$. For $1\le i\le n-2, 1<j<n$, $C_{i,i+j}=\mathfrak{q}(\epsilon_i+\epsilon_{i+j})-\mathfrak{q}(\epsilon_i)-\mathfrak{q}(\epsilon_{i+j})=0$. For $1<i\le n$, $C_{i-1,i}=\mathfrak{q}(\epsilon_{i-1}+\epsilon_i)-\mathfrak{q}(\epsilon_{i-1})-\mathfrak{q}(\epsilon_i)=-1$. $C_{1n}=C_{n1}=\mathfrak{q}(\epsilon_1+\epsilon_n)-\mathfrak{q}(\epsilon_1)-\mathfrak{q}(\epsilon_n)=1$. So (i) holds.

(ii) We prove it by induction on n. For $n=2$,

$$\mathfrak{q}=x_1^2+x_2^2-x_1x_2+x_1x_2=\frac{1}{2}[(x_1-x_2)^2+(x_1+x_2)^2].$$

Assume it holds for $n-1$, i.e.,

$$\mathfrak{q}(x_1,x_2,\ldots,x_{n-1})=\frac{1}{2}\Big[\sum_{i=1}^{n-2}(x_i-x_{i+1})^2+(x_1+x_{n-1})^2\Big].$$

Then

$$\begin{aligned}\mathfrak{q}(x_1,x_2,\ldots,x_n)&=\mathfrak{q}(x_1,x_2,\ldots,x_{n-1})+x_n^2-x_{n-1}x_n+x_1x_n-x_1x_{n-1}\\&=\frac{1}{2}\sum_{i=1}^{n-2}(x_i-x_{i+1})^2+\frac{1}{2}x_1^2+\frac{1}{2}x_{n-1}^2+x_1x_{n-1}+x_n^2-x_{n-1}x_n+x_1x_n-x_1x_{n-1}\\&=\frac{1}{2}\sum_{i=1}^{n-2}(x_i-x_{i+1})^2+\frac{1}{2}(x_1+x_n)^2+\frac{1}{2}(x_{n-1}-x_n)^2\\&=\frac{1}{2}\Big[\sum_{i=1}^{n-1}(x_i-x_{i+1})^2+(x_1+x_n)^2\Big]\ge 0.\end{aligned}$$

If $\mathfrak{q}(x_1,x_2,\ldots,x_n)=\frac{1}{2}[\sum_{i=1}^{n-1}(x_i-x_{i+1})^2+(x_1+x_n)^2]=0$, we have $x_1=x_2=\cdots=x_n=0$. So $\mathfrak{q}$ is positive definite. By the definition of the bigraph $B(\mathfrak{q})$, we have

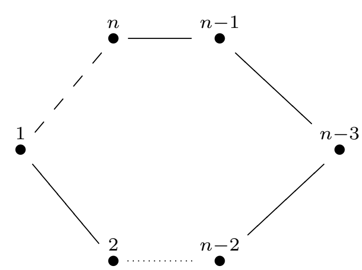


Figure 1 Bigraph $B(\mathfrak{q})$ of $\mathfrak{q}$

It is easy to see that the bigraph $B(\mathfrak{q})$ of $\mathfrak{q}$ is connected.

(iii) Let $(x_1,x_2,\ldots,x_n)\in\Delta$, where $x_i\in\mathbb{Z}$, $1\le i\le n$. Then $\sum_{i=1}^{n-1}(x_i-x_{i+1})^2+(x_1+x_n)^2=2$. Let $x_1-x_2=a_1, x_2-x_3=a_2,\ldots,x_{n-2}-x_{n-1}=a_{n-2}, x_{n-1}-x_n=a_{n-1}, x_1+x_n=a_n$. Then for $2\le j\le n-1$,

$$x_1=\frac{1}{2}\sum_{i=1}^{n}a_i, x_1+x_i=\sum_{j=i}^{n}a_j, x_1+x_n=a_n. \qquad (*)$$

In the set $\{a_1, a_2, a_3, \ldots, a_n\}$, there are only two elements with absolute value 1, and the others are zero. Assume that $|a_i| = |a_j| = 1$ for $1 \le i < j \le n$. We prove (iii) in the following cases.

Case 1 $i = 1$.

In this case $a_1 = \pm 1$.

Case 1.1 $a_1 = 1$, $a_j = \pm 1$, $j = 2, 3, \ldots, n$.

If $j = n, a_j = 1$, then $a_n = 1$, $a_l = 0$ for $2 \le l < n$. By the equalities $(*)$ we have $x_1 = 1, x_2 = x_3 = \cdots = x_n = 0$. Then $(x_1, x_2, \ldots, x_n) = \epsilon_1$. If $2 \le j \le n-1$, $a_j = 1$, then $a_2 = a_3 = \cdots = a_{j-1} = a_{j+1} = \cdots = a_n = 0$. By the equalities $(*)$ we have $x_1 = 1, x_2 = x_3 = \cdots = x_j = 0$, $x_{j+1} = \cdots = x_{n-1} = x_n = -1$. Then $(x_1, x_2, \ldots, x_n) = \epsilon_1 - \sum_{k=j+1}^n \epsilon_k$. If $j = 2$, $a_j = -1$, then $x_1 = 0, x_2 = -1, x_3 = x_4 = \cdots = x_n = 0$. So $(x_1, x_2, \ldots, x_n) = -\epsilon_2$. If $2 < j \le n$, $a_j = -1$, then $x_1 = 0, x_2 = x_3 = \cdots = x_j = -1, x_{j+1} \cdots = x_n = 0$. So $(x_1, x_2, \ldots, x_n) = -\sum_{k=2}^j \epsilon_k$.

Case 1.2 $a_1 = -1$, $a_j = \pm 1$, $j = 2, 3, \ldots, n$.

By computations similar to those in Case 1.1, if $j = n$, $a_j = -1$, then $(x_1, x_2, \ldots, x_n) = -\epsilon_1$. And if $2 \le j \le n-1$, $a_j = -1$, then $(x_1, x_2, \ldots, x_n) = -\epsilon_1 + \sum_{k=j+1}^n \epsilon_k$. And if $j = 2$, $a_j = 1$, then $(x_1, x_2, \ldots, x_n) = \epsilon_2$. And if $2 < j \le n$, $a_j = 1$, then $(x_1, x_2, \ldots, x_n) = \sum_{k=2}^j \epsilon_k$.

Case 2 $1 < i < n$.

In this case $a_i = \pm 1$.

Case 2.1 $a_i = 1$, $a_j = \pm 1$, $j = i+1, \ldots, n$.

If $j = n$, $a_j = 1$, and the others are zero, then $x_1 = x_2 = \cdots = x_i = 1$, $x_{i+1} = x_{i+2} = \cdots = x_n = 0$. So $(x_1, x_2, \ldots, x_n) = \sum_{k=1}^i \epsilon_k$. If $i+1 \le j \le n-1$, $a_j = 1$, and the others are zero, then $x_1 = x_2 = \cdots = x_i = 1, x_{i+1} = x_{i+2} = \cdots = x_j = 0, x_{j+1} = \cdots = x_n = -1$. Then $(x_1, x_2, \ldots, x_n) = \sum_{k=1}^i \epsilon_k - \sum_{k=j+1}^n \epsilon_k$. If $j = i+1$, $a_j = -1$, and the others are zero, then $x_1 = x_2 = \cdots = x_i = x_{i+2} = \cdots = x_n = 0$, $x_{i+1} = -1$. Therefore, $(x_1, x_2, \ldots, x_n) = -\epsilon_{i+1}$. If $i+1 < j \le n$, $a_j = -1$, then $x_1 = x_2 = \cdots = x_i = 0$, $x_{i+1} = x_{i+2} = \cdots = x_j = -1$, $x_{j+1} = x_{j+2} = \cdots = x_n = 0$. Therefore, $(x_1, x_2, \ldots, x_n) = -\sum_{k=i+1}^j \epsilon_k$.

Case 2.2 $a_i = -1$, $a_j = \pm 1$, $j = i+1, \ldots, n$.

If $j = n$, $a_j = -1$, and the others are zero, then $(x_1, x_2, \ldots, x_n) = -\sum_{k=1}^i \epsilon_k$. If $i+1 \le j \le n-1$, $a_j = -1$, and the others are zero, then $(x_1, x_2, \ldots, x_n) = -\sum_{k=1}^i \epsilon_k + \sum_{k=j+1}^n \epsilon_k$. If $j = i+1$, $a_j = 1$, then $(x_1, x_2, \ldots, x_n) = \epsilon_{i+1}$. If $i+1 < j \le n$, $a_j = 1$, then $(x_1, x_2, \ldots, x_n) = \sum_{k=i+1}^j \epsilon_k$.

In a word

$$\begin{aligned}\Delta =&\{\pm\epsilon_i | 1 \le i \le n\} \cup \{\pm(\sum_{k=i}^j \epsilon_k) | 1 < i < j \le n\} \cup \{\pm(\sum_{k=1}^i \epsilon_k) | 1 < i < n\} \cup \\ &\{\pm(\sum_{k=1}^i \epsilon_k - \sum_{k=j+1}^n \epsilon_k) | 1 \le i < j \le n-1\}.\end{aligned}$$

By simple computation, $|\Delta| = 2n^2 - 2n$. $\square$

Each element α in Δ is called a root of the unit form $\mathfrak{q}$. Let $\mathbb{N}$ be the set of the natural numbers. If a root $\alpha \in \mathbb{N}^n$, it is called a positive root of $\mathfrak{q}$. Since $\mathfrak{q}$ is a positive definite unit form, then by [8, Theorem 1.3], we have the following theorem.

Theorem 2.2 *The complex Lie algebra $\mathfrak{g}_\infty(\mathfrak{q})$ attached to the unit form $\mathfrak{q}$ is finite dimensional simple Lie algebra.*

Next we recall a $\mathbb{Z}^n$-graded structure on $\mathfrak{g}_\infty(\mathfrak{q})$ introduced in [8]. Let $H = \bigoplus_{i=1}^n \mathbb{C}h_i$, which is a commutative Lie subalgebra of $\mathfrak{g}_\infty(\mathfrak{q})$. For any $h = \sum_{i=1}^n \lambda_i h_i$, we define $r(h) = \sum_{i=1}^n \lambda_i \epsilon_i$ and $\langle h, \alpha\rangle = r(h)C\alpha^t$ for $\alpha \in \mathbb{C}^n$. For any $\alpha \in \mathbb{C}^n$, we set

$$(\mathfrak{g}_\infty(\mathfrak{q}))_\alpha = \{x \in \mathfrak{g}_\infty(\mathfrak{q}) | [h, x] = \langle h, \alpha\rangle x, \text{ for all } h \in H\}.$$

By [3, Lemma 2.1], the vector space $(\mathfrak{g}_\infty(\mathfrak{q}))_\alpha$ is generated by all expressions $[e_{\delta_1 i_1}, e_{\delta_2 i_2}, \ldots, e_{\delta_t i_t}]$ with $\sum_{j=1}^t \delta_j \epsilon_{i_j} = \alpha$. Then

$$\mathfrak{g}_\infty(\mathfrak{q}) = H \oplus \bigoplus_{\alpha\in\mathbb{Z}^n} (\mathfrak{g}_\infty(\mathfrak{q}))_\alpha,$$

where H is the Cartan subalgebra of $\mathfrak{g}_\infty(\mathfrak{q})$. If $0 \neq \alpha \in \mathbb{Z}^n$ such that $(\mathfrak{g}_\infty(\mathfrak{q}))_\alpha \neq 0$, then $\dim_{\mathbb{C}}(\mathfrak{g}_\infty(\mathfrak{q}))_\alpha = 1$, and α is called a root of the simple Lie algebra $\mathfrak{g}_\infty(\mathfrak{q})$. The set Δ' of the roots of $\mathfrak{g}_\infty(\mathfrak{q})$ is called a root system of the simple Lie algebra $\mathfrak{g}_\infty(\mathfrak{q})$. In fact, by the definition of $\mathfrak{g}_\infty(\mathfrak{q})$, $\Delta' \subseteq \Delta$. By the following Theorem 5.1, we will see that $\Delta' = \Delta$. A subset Π of Δ' is called *a base* if Π is a maximal linearly independent system of Δ', and each root $\beta \in \Delta'$ can be written as $\beta = \sum_{\alpha\in\Pi} k_\alpha \alpha$ with integral coefficients k_α all nonnegative or all nonpositive. The roots in Π are called simple roots. The cardinality of Π is n. So the Dynkin diagram Γ related to the simple roots is defined. By [8, Proposition 6.1], the Lie algebra $\mathfrak{g}_\infty(\mathfrak{q})$ is isomorphic to the simple Lie algebra $\mathfrak{g}_4(\mathfrak{q}_\Gamma)$ of the Dynkin type Γ. We are interested in the Dynkin type of Γ. To see this, we will first show that $\mathfrak{g}_\infty(\mathfrak{q})$ can be realized by the Ringel-Hall Lie algebra of a Nakayama algebra Λ, which is piecewise hereditary of type $\mathbb{D}_n$. Then Γ is proved to be of Dynkin type $\mathbb{D}_n$.

3. Homological properties of a Nakayama algebra Λ

We consider a special class of Nakayama algebras obtained as follows. The linearly oriented quiver $\overrightarrow{\mathbb{A}}_n$ for $n \geq 3$ is

$$\overset{1}{\circ} \xrightarrow{\alpha_1} \overset{2}{\circ} \xrightarrow{\alpha_2} \overset{3}{\circ} \cdots \overset{n-1}{\circ} \xrightarrow{\alpha_{n-1}} \overset{n}{\circ}$$

Set $\Lambda = k\overrightarrow{\mathbb{A}}_n/I$, where k is a field, I is the ideal generated by the elements $\alpha_1\alpha_2\alpha_3\cdots\alpha_{n-1}$. Then Λ is a Nakayama algebra. By [10, Proposition 1.4], the global dimension g.l.dim $\Lambda = 2$. Let mod Λ be the finite-dimensional left Λ-modules. We denote by S_i the simple Λ-module associated with i, by P_i its projective cover and by I_i its injective envelope. Denote by $K_0(\Lambda)$ the Grothendieck group of an abelian category mod Λ, that is the free abelian group on isomorphism classes $[M]$ of objects in mod Λ modulo the relations $[M] = [N] + [L]$ for any exact sequence $0 \to N \to M \to L \to 0$. We identify $K_0(\Lambda)$ with $\mathbb{Z}^n$, and denote by p_j the image of P_j in the

Grothendieck group $K_0(\Lambda)$. The Cartan matrix $\bar{C}$ of Λ is an integral-valued matrix with entries $\bar{C}_{ij} = \dim_k \mathrm{Hom}_\Lambda(P_i, P_j)$, $1 \le i, j \le n$. Thus the j-th column vector of $\bar{C}$ is p_j^t, where t denotes the transpose of a matrix. So $\bar{C} = (\epsilon_1^t + \epsilon_2^t + \cdots + \epsilon_{n-1}^t, \epsilon_2^t + \epsilon_3^t + \cdots + \epsilon_n^t, \ldots, \epsilon_{n-1}^t + \epsilon_n^t, \epsilon_n^t)$, i.e,

$$\bar{C} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

Let $\mathcal{K}^b(\Lambda)$ be the homotopy category associated with the category of bounded complexes over mod Λ, and $\mathcal{D}^b(\Lambda)$ the bounded derived category of $\mathcal{K}^b(\Lambda)$ by localization with quasi-isomorphisms. Denote by T the shift functor of complexes in $\mathcal{D}^b(\Lambda)$. By [10, Proposition 2.1], Λ is piecewise hereditary of type $\mathbb{D}_n$, i.e., the derived category $\mathcal{D}^b(\Lambda)$ of Λ is isomorphic to the derived category of a hereditary algebra of type $\mathbb{D}_n$. Similarly to $K_0(\Lambda)$, $K_0(\mathcal{D}^b(\Lambda))$ is defined as the free abelian group on isomorphism classes $[X]$ of complexes in $\mathcal{D}^b(\Lambda)$ modulo the relations $[X] = [Y] + [Z]$ for any distinguished triangle $Y \to X \to Z \to TY$. It has been shown in [11, Lemma III.1.2], we can identify $K_0(\Lambda)$ and the Grothendieck group $K_0(\mathcal{D}^b(\Lambda))$. Given an object X in $\mathcal{D}^b(\Lambda)$, we denote by $\underline{\dim} X$ the corresponding element in $K_0(\mathcal{D}^b(\Lambda)) = K_0(\Lambda)$. Note that there is a canonical embedding of mod Λ into $\mathcal{D}^b(\Lambda)$ (as the full subcategory of complexes concentrated in degree zero), and the restriction of $\underline{\dim}$ to this full subcategory mod Λ coincides with the usual dimension vector function.

For any indecomposable object $X \in \mathcal{D}^b(\Lambda)$, there is an Auslander-Reiten triangle $\tau X \to Y \to X \to T\tau X$. So there is an autoequivalence τ on $\mathcal{D}^b(\Lambda)$ which induces an isomorphism denoted by Φ on the Grothendieck group $K_0(\Lambda)$, i.e., $\underline{\dim}\ \tau X = (\underline{\dim} X)\Phi$. Observe that $(\underline{\dim}\ P_i)\Phi = -\underline{\dim}\ I_i$. Thus $\Phi = -(\bar{C}^{-1})^t \bar{C}$. The following results give some properties of Φ.

Lemma 3.1 *Let Φ be an isomorphism on the Grothendieck group $K_0(\Lambda)$. Then*

$$\Phi = -(\epsilon_{n-1}^t, \epsilon_n^t, \epsilon_1^t - \epsilon_2^t + \epsilon_n^t, \epsilon_1^t - \epsilon_3^t + \epsilon_n^t, \ldots, \epsilon_1^t - \epsilon_{n-1}^t + \epsilon_n^t),$$

$$\Phi^i = \begin{cases} -(\epsilon_{n-i}^t, \ldots, \epsilon_n^t, \epsilon_1^t - \epsilon_2^t + \epsilon_n^t, \ldots, \epsilon_1^t - \epsilon_{n-i}^t + \epsilon_n^t), & i \text{ is odd and } 2 \le i \le n-1; \\ (\epsilon_1^t - \epsilon_{n-i}^t + \epsilon_n^t, \ldots, \epsilon_1^t - \epsilon_{n-1}^t + \epsilon_n^t, \epsilon_1^t, \ldots, \epsilon_{n-i}^t), & i \text{ is even and } 2 \le i \le n-1. \end{cases}$$

$$\Phi^n = \begin{cases} -(\epsilon_1^t - \epsilon_{n-1}^t + \epsilon_n^t, \epsilon_1^t, \epsilon_2^t, \ldots, \epsilon_{n-1}^t), & n \text{ is an odd integral;} \\ (\epsilon_{n-1}^t, \epsilon_n^t, \epsilon_1^t - \epsilon_2^t + \epsilon_n^t, \ldots, \epsilon_1^t - \epsilon_{n-1}^t + \epsilon_n^t,), & n \text{ is an even integral.} \end{cases}$$

$$\Phi^{2n-2} = E_n,$$

where E_n is the identity matrix of order n.

Proof By simple computation, $\bar{C}^{-1}=(\epsilon_1^t-\epsilon_2^t+\epsilon_n^t,\epsilon_2^t+\epsilon_3^t,\dots,\epsilon_i^t+\epsilon_{i+1}^t,\dots,\epsilon_n^t)$, i.e,

$$\bar{C}^{-1}=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0\\ -1 & 1 & 0 & \cdots & 0 & 0\\ 0 & -1 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 0\\ 1 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}.$$

So $(\bar{C}^{-1})^t=(\epsilon_1^t,-\epsilon_1^t+\epsilon_2^t,\dots,-\epsilon_{i-1}^t+\epsilon_i^t,\dots,\epsilon_1^t-\epsilon_{n-1}^t+\epsilon_n^t)$. Note that $\epsilon_s\epsilon_m^t=1$ for $s=m$, and $\epsilon_s\epsilon_m^t=0$ for any $s\neq m$. By simple computation,

$$\Phi=-(\bar{C}^{-1})^t\bar{C}=-\begin{pmatrix} 0 & 0 & 1 & 1 & \cdots & 1\\ 0 & 0 & -1 & 0 & \cdots & 0\\ 0 & 0 & 0 & -1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & 0 & 0 & \cdots & -1\\ 0 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$
$$=-(\epsilon_{n-1}^t,\epsilon_n^t,\epsilon_1^t-\epsilon_2^t+\epsilon_n^t,\epsilon_1^t-\epsilon_3^t+\epsilon_n^t,\dots,\epsilon_1^t-\epsilon_{n-1}^t+\epsilon_n^t).$$

Then

$$\Phi^2=(-1)^2\begin{pmatrix} 1 & 1 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 & 0\\ -1 & 0 & 0 & 0 & \cdots & 0 & 1\\ 0 & -1 & 0 & 0 & \cdots & 0 & 0\\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$
$$=(\epsilon_1^t-\epsilon_{n-2}^t+\epsilon_n^t,\epsilon_1^t-\epsilon_{n-1}^t+\epsilon_n^t,\epsilon_1^t,\epsilon_2^t,\dots,\epsilon_{n-2}^t).$$

And

$$\Phi^3=(-1)^3\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & \cdots & 1 & 1\\ 0 & 0 & 0 & 0 & -1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1\\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$
$$=-(\epsilon_{n-3}^t,\epsilon_{n-3+1}^t,\dots,\epsilon_n^t,\epsilon_1^t-\epsilon_2^t+\epsilon_n^t,\dots,\epsilon_1^t-\epsilon_{n-3}^t+\epsilon_n^t).$$

The others Φ^i, $4 \le i \le 2n-2$, are similarly computed. $\square$

The orbit category $\mathcal{R}(\Lambda)= \mathcal{D}^b(\Lambda)/T^2$ is called the root category of Λ, and the Galois covering functor $\mathcal{F} : \mathcal{D}^b(\Lambda) \to \mathcal{R}(\Lambda)$ is dense. Furthermore, since $D^b(\Lambda) \simeq D^b(\mathcal{H})$ for some hereditary abelian category $\mathcal{H}$, using a proof similar to [6, Theorem 3.3], we can prove that $\mathcal{R}(\Lambda)$ is a triangulated category. For convenience, we denote $\mathcal{R}(\Lambda)$ by $\mathcal{R}$ in the following. We consider the Grothendick group $K_0(\mathcal{R})$ as usual. Then $K_0(\mathcal{R}) = K_0(D^b(\Lambda))$, and this identification is made so that for any $X \in \mathcal{D}^b(\Lambda)$, $\underline{\dim} X$ in $K_0(\mathcal{D}^b(\Lambda))$ coincides with $\underline{\dim}\ \mathcal{F}X$ in $K_0(\mathcal{R})$.

Lemma 3.2 *Let Λ be the Nakayama algebra over a field k and $\mathcal{R}$ the root category of Λ. Then for any indecomposable object X in $\mathcal{R}$, we have $\dim_k Hom_{\mathcal{R}}(X,X)$-$\dim_k Hom_{\mathcal{R}}(X,TX) = 1$.*

Proof Consider the covering functors $\mathcal{F} : D^b(\Lambda) \to \mathcal{R}$ and take an indecomposable object $\dot{X}$ in $D^b(\Lambda)$ such that $\mathcal{F}\dot{X} = X$. Since Λ is a piecewise hereditary algebra of type $\mathbb{D}_n$, from the structure of $D^b(\Lambda)$, $\mathrm{Hom}_{D^b(\Lambda)}(\dot{X}, T^i\dot{X}) = 0$ for $i \neq 0$, and $\dim_k \mathrm{Hom}_{D^b(\Lambda)}(\dot{X}, \dot{X}) = 1$. Note that $\mathcal{F}$ induces two isomorphisms $\mathrm{Hom}_{D^b(\Lambda)}(\dot{X}, \dot{X}) \simeq \mathrm{Hom}_{\mathcal{R}}(X,X)$ and $\mathrm{Hom}_{D^b(\Lambda)}(\dot{X}, T\dot{X}) \simeq \mathrm{Hom}_{\mathcal{R}}(X,TX)$. Thus $\dim_k \mathrm{Hom}_{\mathcal{R}}(X,X)$-$\dim_k \mathrm{Hom}_{\mathcal{R}}(X,TX) = 1$. $\square$

4. The isomorphism theorem

In this subsection, following [4], we recall the definition of the Ringel-Hall Lie algebras of the root category $\mathcal{R}$ of Λ, where Λ is the above Nakayama algebra over a finite field k with cardinality $|k| = q$. The definition is also seen in [6, Section 5].

Given $X, Y, L \in \mathcal{R}$, consider

$$W(X,Y;L) = \{(f,g,h) \in \mathrm{Hom}(X,L) \times \mathrm{Hom}(L,Y) \times \mathrm{Hom}(Y,TX) | \\ X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} TX \text{ is a triangle}\}.$$

Applying the action of $\mathrm{Aut}(X)\times\mathrm{Aut}(Y)$ on $W(X,Y;L)$ defined by

$$(a,c) \circ (f,g,h) = (af, gc^{-1}, ch(Ta)^{-1})$$

for $(a,c) \in \mathrm{Aut}(X)\times\mathrm{Aut}\ (Y)$, $(f,g,h) \in W(X,Y;L)$, we get the Ringel-Hall number $F^L_{YX} = |W(X,Y;L)/\mathrm{Aut}(X) \times \mathrm{Aut}(Y)|$.

For any $M \in \mathcal{R}$, we denote by $h_M := \underline{\dim}\ M$ the canonical image of $[M]$ in $K_0(\mathcal{R})$. Denote by $\mathrm{ind}\,\mathcal{R}$ the set of representatives of isoclasses of the indecomposable objects in $\mathcal{R}$. And denote by $\mathbf{h}'$ the subgroup of $K_0(\mathcal{R})\otimes_{\mathbb{Z}}\mathbb{C}$ generated by $\frac{h_M}{d(M)}$, $M \in \mathrm{ind}\,\mathcal{R}$, where $d(M) = \dim_k(\mathrm{End}M/\mathrm{Rad}(\mathrm{End}\ M))$.

We define a symmetric Euler bilinear function $I_{\mathcal{R}}(-,-)$ on $\mathbf{h}' \times \mathbf{h}'$ determined by

$$I_{\mathcal{R}}(h_X, h_Y) = \dim_k \mathrm{Hom}_{\mathcal{R}}(X,Y) - \dim_k \mathrm{Hom}_{\mathcal{R}}(X,TY) + \\ \dim_k \mathrm{Hom}_{\mathcal{R}}(Y,X) - \dim_k \mathrm{Hom}_{\mathcal{R}}(Y,TX),$$

for any $X, Y \in \mathcal{R}$.

Let $\mathbf{n}$ be the free abelian group with a basis $\{u_X | X \in \mathrm{ind}\ \mathcal{R}\}$. Let

$$g(\mathcal{R}) = \mathbf{h}' \oplus \mathbf{n},$$

be a direct sum of $\mathbb{Z}$-modules. We shall consider the quotient group

$$g(\mathcal{R})_{(q-1)} = g(\mathcal{R})/(q-1)g(\mathcal{R}).$$

We still use u_M, h_M to denote the corresponding residue classes for $M \in \mathcal{R}$.

Then by [4], $g(\mathcal{R})_{(q-1)}$ is a Lie algebra over $\mathbb{Z}/(q-1)\mathbb{Z}$ with the Lie operation $[-,-]$ as follows.

(1) For any two indecomposable objects $X, Y \in \mathcal{R}$,

$$[u_X, u_Y] = \begin{cases} \sum_{L \in \text{ind } \mathcal{R}} (F^L_{XY} - F^L_{YX}) u_L, & \text{if } Y \ncong TX, \\ -\frac{h_X}{d(X)}, & \text{if } Y \cong TX. \end{cases}$$

(2) For any objects $X, Y \in \mathcal{R}$ with Y indecomposable,

$$[h_X, u_Y] = I_{\mathcal{R}}(h_X, h_Y) u_Y, \text{ and } [u_Y, h_X] = -[h_X, u_Y].$$

(3) $[\mathbf{h}', \mathbf{h}'] = 0$.

Obviously, $g(\mathcal{R})_{(q-1)}$ has the canonical decomposition

$$g(\mathcal{R})_{(q-1)} = \mathbf{h}' \bigoplus \bigoplus_{\alpha = \underline{\dim} X,\ X \in \text{ind } \mathcal{R}} (g(\mathcal{R})_{(q-1)})_\alpha,$$

where $(g(\mathcal{R})_{(q-1)})_\alpha$ is the $\mathbb{Z}/(q-1)\mathbb{Z}$-submodule spanned by all u_X with $X \in \text{ind } \mathcal{R}$ and $\underline{\dim} X = \alpha$.

As in [4, Section 5.2], we define the direct product $\prod_{E \in \Omega} g(\mathcal{R}^E)_{(|E|-1)}$ of Lie algebras and let $\mathcal{LC}(\mathcal{R})_1$ be the Lie subalgebra of $\prod_{E \in \Omega} g(\mathcal{R}^E)_{(|E|-1)}$ generated by $u_{S_i} = (u_{S_i^E})_{E \in \Omega}$ and $u_{TS_i} = (u_{TS_i^E})_{E \in \Omega}$, $1 \le i \le n$. Write

$$\mathbf{g} = \mathcal{LC}(\mathcal{R})_1 \otimes_{\mathbb{Z}} \mathbb{C},$$

then $\mathbf{g}$ is a Lie algebra over $\mathbb{C}$, called the Ringel-Hall Lie algebra of the root category $\mathcal{R}$. In fact, we have the following proposition.

Proposition 4.1 *Let $\mathbf{g}$ be the Ringel-Hall Lie algebra of the root category $\mathcal{R}$, and $\prod_{E \in \Omega} g(\mathcal{R}^E)_{(|E|-1)}$ be the direct product of Lie algebras. Then*

$$\mathbf{g} = \prod_{E \in \Omega} g(\mathcal{R}^E)_{(|E|-1)} \otimes \mathbb{C}.$$

Proof We should prove that for any indecomposable object $X \in \text{ind } \mathcal{R}$, $u_X \in \mathbf{g}$. By the Auslander-Reiten quiver of $\mathcal{R}$, $\text{Hom}_{\mathcal{R}}(X, TX) = 0$. So X is an exceptional object. By a proof similar to that in [6, Proposition 8.2], X can be extended to a complete exceptional sequence $\mathcal{X} = (X_1, X_2, \ldots, X_n)$, where $X_1 = X$. Let $\mathcal{L}(\mathcal{X}, T\mathcal{X})$ be the Lie subalgebra of $\prod_{E \in \Omega} g(\mathcal{R}^E)_{(|E|-1)} \otimes \mathbb{C}$ generated by u_{X_i}, u_{TX_i}, $1 \le i \le n$. Similar to that in [6, Proposition 8.3], $\mathcal{L}(\mathcal{X}, T\mathcal{X}) = \mathbf{g}$. Since $u_X \in \mathcal{L}(\mathcal{X}, T\mathcal{X})$, we have $u_X \in \mathbf{g}$. □

Naturally $\mathbf{g}$ has the following grading

$$\mathbf{g} = \bigoplus_{\alpha \in K_0(\mathcal{R})} \mathbf{g}_\alpha$$

such that deg $(u_{S_i}) = \underline{\dim} S_i = \epsilon_i$ and deg $(u_{TS_i}) = \underline{\dim} TS_i = -\epsilon_i$, where $\mathbf{g_0}$ is just $\mathbf{h}'$.

Let $Q=\mathbb{Z}^n$. Define a functor $I_Q(-,-):Q\times Q\to\mathbb{Z}$ such that

$$I_Q(\alpha,\beta)=\mathfrak{q}(\alpha+\beta)-\mathfrak{q}(\alpha)-\mathfrak{q}(\beta),\quad \alpha,\beta\in Q.$$

Obviously, $I_Q(-,-)$ is a symmetric bilinear form on Q, and the matrix C in Lemma 2.1 is the Gram matrix of $I_Q(-,-)$. The relationship between $I_Q(-,-)$ and $I_{\mathcal{R}}(-,-)$ is as follows.

Lemma 4.2 *(1) $I_{\mathcal{R}}(h_{S_i},h_{S_i})=2$ for any $i=1,2,\dots,n$, and $I_{\mathcal{R}}(h_{S_i},h_{S_j})=-1$ for $|j-i|=1$, and $I_{\mathcal{R}}(h_{S_i},h_{S_j})=0$ for $1<|j-i|<n-1$, and $I_{\mathcal{R}}(h_{S_i},h_{S_j})=1$ for $|j-i|=n-1$.*

(2) There exists a group isomorphism $\xi:G(\mathcal{R})\to Q$ defined by $h_{S_i}\mapsto\epsilon_i$, $1\le i\le n$. And under ξ we have $I_{\mathcal{R}}(-,-)=I_Q(-,-)$.

(3) For any indecomposable object $X\in\mathcal{R}$, $\mathfrak{q}(\underline{\dim}X)=1$.

Proof (1) By the definition of root categories,

$$\mathrm{Hom}_{\mathcal{R}}(S_i,S_j)=\bigoplus_{m=1}^{\infty}\mathrm{Hom}_{D^b(\Lambda)}(S_i,T^{2m}S_j)=\bigoplus_{m=1}^{\infty}\mathrm{Ext}_{\Lambda}^{2m}(S_i,S_j),$$

$$\mathrm{Hom}_{\mathcal{R}}(S_i,TS_j)=\bigoplus_{m=1}^{\infty}\mathrm{Hom}_{D^b(\Lambda)}(S_i,T^{2m+1}S_j)=\bigoplus_{m=1}^{\infty}\mathrm{Ext}_{\Lambda}^{2m+1}(S_i,S_j).$$

It is easy to see that proj.dim$S_i=2$ for $i=1$, and proj.dim$S_i=1$ for $1<i<n$, proj.dim$S_n=0$. Similarly, inj.dim$S_j=2$ for $j=n$, inj.dim$S_j=1$ for $1<j<n$, inj.dim$S_j=0$ for $j=1$. Then $\mathrm{Ext}^3(S_i,S_j)=\mathrm{Ext}^4(S_i,S_j)=\cdots=\mathrm{Ext}^n(S_i,S_j)=0$ for $1\le i,j\le n$. If $i\ne1$, then proj.dim$S_i\le 1,\mathrm{Ext}^2(S_i,S_j)=0$. If $j\ne n$, then inj.dim$S_j\le1,\mathrm{Ext}^2(S_i,S_j)=0$. So $\mathrm{Ext}^2(S_i,S_j)=0$ for $i\ne1$ or $j\ne n$. There is a long exact sequence $0\to S_n\to P_2\to P_1\to S_1\to0$, and so $\mathrm{Ext}^2(S_1,S_n)=\mathrm{Hom}(S_n,S_n)=1$. Since $\mathrm{Ext}^1(S_i,S_j)\cong\mathrm{Hom}(P_{i+1},S_j)$ for $1\le i<n,1\le j\le n$, and $\mathrm{Ext}^1(S_n,S_j)=0$, we have

$$\mathrm{Ext}^1(S_i,S_j)=\begin{cases}1, & j-i=1;\\ 0, & j-i\ne1.\end{cases}$$

Therefore, $\dim_k\mathrm{Hom}_{\mathcal{R}}(S_j,S_i)=\dim_k\mathrm{Hom}_{\mathcal{R}}(S_j,TS_i)=0$ for any $1\le i<j\le n$, $\dim_k\mathrm{Hom}_{\mathcal{R}}(S_i,TS_j)=1$, $\dim_k\mathrm{Hom}_{\mathcal{R}}(S_i,S_j)=0$ for $j-i=1$, and $\dim_k\mathrm{Hom}_{\mathcal{R}}(S_1,S_n)=1$, $\dim_k\mathrm{Hom}_{\mathcal{R}}(S_1,TS_n)=0$. Then $I_{\mathcal{R}}(h_{S_1},h_{S_n})=\dim_k\mathrm{Hom}_{\mathcal{R}}(S_1,S_n)=1=C_{1n}$, and

$$I_{\mathcal{R}}(h_{S_i},h_{S_j})=\begin{cases}-\dim_k\mathrm{Hom}_{\mathcal{R}}(S_i,S_j)=-1, & \text{for } j-i=1;\\ -\dim_k\mathrm{Hom}_{\mathcal{R}}(S_j,S_i)=-1, & \text{for } i-j=1.\end{cases}$$

Then $I_{\mathcal{R}}(h_{S_i},h_{S_j})=-1$ for $|j-i|=1$. And $I_{\mathcal{R}}(h_{S_i},h_{S_j})=0$ for $1<|j-i|<n-1$. Then we have $I_{\mathcal{R}}(h_{S_i},h_{S_j})=C_{ij}$ for $i\ne j$. For any $i=1,2,\dots,n$, $I_{\mathcal{R}}(h_{S_i},h_{S_i})=2=C_{ii}$ by Lemma 3.2.

(2) It is clear that ξ is an additive group isomorphism. By (1), the Gram matrix for the symmetric Euler bilinear function $I_{\mathcal{R}}(-,-)$ is C. So $I_{\mathcal{R}}(-,-)=I_Q(-,-)$.

(3) $\mathfrak{q}(\underline{\dim}X)=\frac{1}{2}I_Q(\underline{\dim}X)=\frac{1}{2}I_{\mathcal{R}}(\underline{\dim}X)=\dim_k\mathrm{Hom}_{\mathcal{R}}(X,X)-\dim_k\mathrm{Hom}_{\mathcal{R}}(X,TX)$. By Lemma 3.2, $\mathfrak{q}(\underline{\dim}X)=1$. $\square$

Theorem 4.3 *Let* **g** *be the Ringel-Hall Lie algebra of the Nakayama algebra Λ in Section 3,*

$\mathfrak{g}_\infty(\mathfrak{q})$ *be the complex simple Lie algebra attached to the unit form* $\mathfrak{q}:\mathbb{Z}^n\to\mathbb{Z}$,

$$\mathfrak{q}(x)=\sum_{i=1}^{n}x_i^2-\Big(\sum_{i=1}^{n-1}x_ix_{i+1}\Big)+x_1x_n.$$

Then there is a Lie algebra isomorphism $\varphi:\mathfrak{g}_\infty(\mathfrak{q})\to\mathbf{g}$ *defined by*

$$h_i\mapsto h_{S_i},\quad 1\le i\le n,$$

$$e_i\mapsto u_{S_i},\quad 1\le i\le n,$$

$$e_{-i}\mapsto -u_{TS_i},\quad 1\le i\le n.$$

Proof We check the following equalities.

(1) $[h_{S_i},h_{S_j}]=0$ for all $1\le i,j\le n$.

(2) For $1\le i\le n$, $[h_{S_i},u_{S_i}]=I_{\mathcal{R}}(h_{S_i},h_{S_i})u_{S_i}=2u_{S_i}=C_{ii}u_{S_i}$ by Lemma 4.2. For $i\neq j$, $|j-i|=1$, $[h_{S_i},u_{S_j}]=I_{\mathcal{R}}(h_{S_i},h_{S_j})u_{S_j}=(-1)u_{S_j}=C_{ij}u_{S_j}$, $[h_{S_i},-u_{TS_j}]=-I_{\mathcal{R}}(h_{S_i},h_{TS_j})u_{TS_j}=I_{\mathcal{R}}(h_{S_i},h_{S_j})u_{TS_j}=u_{TS_j}=-C_{ij}(-u_{TS_j})$. For $1<|j-i|<n-1$, $[h_{S_i},u_{S_j}]=I_{\mathcal{R}}(h_{S_i},h_{S_j})u_{S_j}=0=0u_{S_j}$, $[h_{S_i},-u_{TS_j}]=-I_{\mathcal{R}}(h_{S_i},h_{TS_j})u_{TS_j}=0$. And $[h_{S_1},u_{S_n}]=I_{\mathcal{R}}(h_{S_1},h_{S_n})u_{S_n}=u_{S_n}=C_{1n}u_{S_n}$.

(3) For $1\le i\le n$, $[u_{S_i},-u_{TS_i}]=\frac{h_{S_i}}{d(S_i)}$. Since $d(S_i)=\dim_k\mathrm{Hom}_{\mathcal{R}}(S(i),S(i))=1$, we have $[u_{S_i},-u_{TS_i}]=h_{S_i}$. And $[-u_{TS_i},u_{S_i}]=-[u_{S_i},-u_{TS_i}]=-h_{S_i}$.

(4) If $\mathfrak{q}(\sum_{j=1}^{t}\delta_j\epsilon_{i_j})>1$ for $\delta_j=\pm1$, $1\le j\le t$, then by Lemma 4.2(4), there is no indecomposable object X in $\mathcal{R}$ with the vector dimension $\sum_{j=1}^{t}\delta_j\epsilon_{i_j}$. Set $u_{\epsilon_i}=u_{S_i}$, and $u_{-\epsilon_i}=-u_{TS_i}$, $1\le i\le n$. Then $[u_{\delta_1\epsilon_{i_1}},u_{\delta_1\epsilon_{i_2}},\dots,u_{\delta_1\epsilon_{i_t}}]=0$.

By the above (1)–(4), there exists a natural epimorphism φ from $\mathfrak{g}_\infty(\mathfrak{q})$ to $\mathbf{g}$ determined by $\varphi(h_i)=h_{S_i}$, $\varphi(e_i)=u_{S_i}$, $\varphi(e_{-i})=-u_{TS_i},1\le i\le n$. Since $\mathfrak{g}_\infty(\mathfrak{q})$ is a simple Lie algebra, we have $\ker\varphi=0$, and so φ must be an isomorphism. □

5. Applications of the isomorphism theorem

In this section, as an application of the isomorphism theorem, we prove that there is a bijection between the set of the indecomposable objects of the root category of the Nakayama algebra and the set of the roots of the positive definite unit form. moreover, we determine the Dynkin type, a root space decomposition and a Chevalley basis of the above simple Lie algebra.

Theorem 5.1 *(1) Let* Λ *be the Nakayama algebra defined in Section 3 over a field* k *and* $\mathcal{R}$ *the root category of* Λ. *Set* $\mathfrak{q}(x_1,x_2,\dots,x_n)=\sum_{i=1}^{n}x_i^2-(\sum_{i=1}^{n-1}x_ix_{i+1})+x_1x_n$,

$$\Delta=\{(x_1,x_2,\dots,x_n)\in\mathbb{Z}^n|\mathfrak{q}(x_1,x_2,\dots,x_n)=1\}.$$

There is a bijection ψ : *ind* $\mathcal{R}\to\Delta$ *defined by* $\psi(M)=\underline{\dim}\,M$. *In other words, there is a one-to-one correspondence between the indecomposable objects of the root category* $\mathcal{R}$ *and the roots of the unit form* $\mathfrak{q}$.

(2) The indecomposable object X *with the dimensional vector* $-\epsilon_1,-\epsilon_2,\dots,-\epsilon_{n-1}$, $\sum_{i=2}^{n}\epsilon_i$ *is mouth object in the Auslander-Reiten quiver of* $\mathcal{R}$.

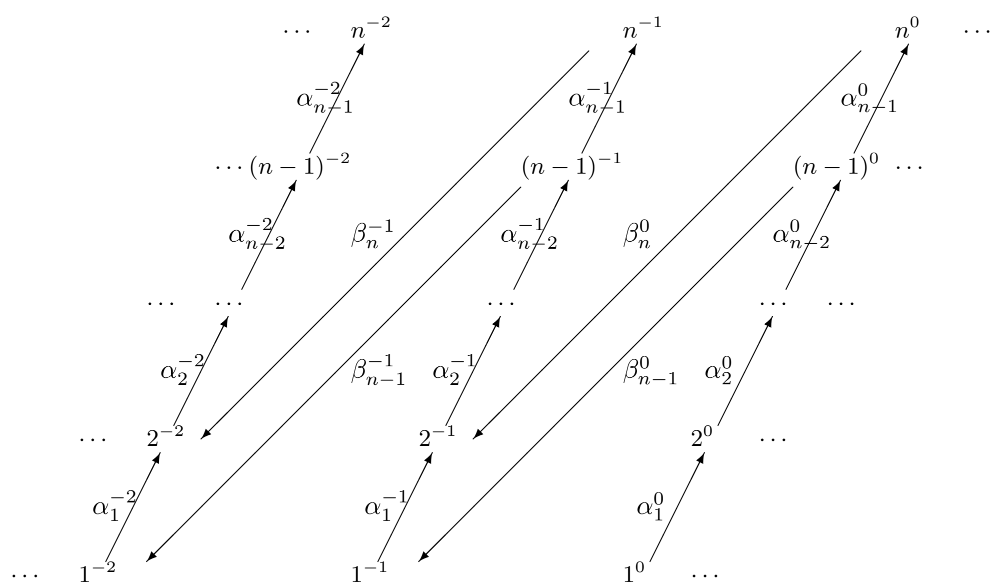
(3) $\mathfrak{g}_\infty(\mathfrak{q})$ *is a simple Lie algebra of type* $\mathbb{D}_n$*, and* $\mathfrak{g}_\infty(\mathfrak{q})$ *has a root space decomposition*

$$\mathfrak{g}_\infty(\mathfrak{q}) = H \oplus \bigoplus_{\alpha\in\Delta}(\mathfrak{g}_\infty(\mathfrak{q}))_\alpha,$$

where $H = \oplus_{i=1}^{n}\mathbb{C}h_i$*,* $\dim_{\mathbb{C}}(\mathfrak{g}_\infty(\mathfrak{q}))_\alpha = 1$ *for any* $\alpha \in \Delta$*.*

Proof (1) Let $X \in \operatorname{ind}\mathcal{R}$. By Lemma 4.2(3), $\underline{\dim}\, X \neq 0$, and $\underline{\dim}\, X \in \Delta$. Thus the map ψ is well-defined. Let $\alpha \in \mathbb{Z}^n = G(\mathcal{R})$ with $\mathfrak{q}(\alpha) = 1$. If X is an indecomposable object with $\underline{\dim} X = \alpha$ in $\mathcal{R}$, then by Proposition 4.1, $u_X \in \mathfrak{g}$. In fact, $u_X \in \mathfrak{g}_\alpha$. By Theorem 4.3, $\varphi^{-1}(u_X) \in \mathfrak{g}_\infty(\mathfrak{q}))_\alpha$. In the simple Lie algebra $\mathfrak{g}_\infty(\mathfrak{q})$, $\dim_{\mathbb{C}}(\mathfrak{g}_\infty(\mathfrak{q}))_\alpha \leq 1$ for any $0 \neq \alpha \in \mathbb{Z}^n$. Then the indecomposable object X in $\mathcal{R}$ such that $\underline{\dim}\, X = \alpha$ is unique. Thus ψ is injective. Note that the cardinality $|\operatorname{ind}\mathcal{R}|$ is $2n^2 - 2n$, which is equal to the cardinality $|\Delta|$ by Lemma 2.1(3). Then ψ is bijective.

(2) At first we show that the indecomposable object X with the dimensional vector $-\epsilon_1$ is a mouth object in the Auslander-Reiten quiver of $D^b(\Lambda)$, others can be similarly proven. Let $\hat{\Lambda}$ be the repetitive algebra of Λ. Then $\hat{\Lambda}$ is a self-injective algebra. Since the global dimension of Λ is finite, then its stable module category $\underline{\operatorname{mod}}\hat{\Lambda}$ as a triangulated category is triangle-equivalent to $D^b(\Lambda)$. So from now on we think $\underline{\operatorname{mod}}\hat{\Lambda} = D^b(\Lambda)$. The vertices of ordinary quiver $\hat{Q}$ of $\hat{\Lambda}$ can be denoted by i^j, $1 \leq i \leq n$, $j \in \mathbb{Z}$, such that for each j the full subquiver of $\hat{Q}$ consisting of $\{1^j, 2^j, \ldots, n^j\}$ coincides with the ordinary quiver of Λ with the same numbering vertices as in Section 3. Note that the subalgebra determined by the full subquiver consisting of $\{1^0, 2^0, \ldots, n^0\}$ is Λ. $\hat{Q}$ is described as follows:



where the diagram is commutative, and any composition of $n-1$ arrows with a same direction is zero, i.e., $\alpha_1^i\alpha_2^i\cdots\alpha_{n-1}^i = 0$, for $i \in \mathbb{Z}$.

Note that the indecomposable module $\begin{array}{c} S_{1^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{1^{-1}} \end{array}$, which denotes the indecomposable object with the simple composition factors $S_{1^0}, S_{2^0}, \ldots, S_{(n-1)^0}, S_{1^{-1}}$, is the projective cover of S_{1^0}, and also the injective envelope of $S_{1^{-1}}$. In the Auslander-Reiten quiver of $\hat{\Lambda}$, the inclusion

$$\begin{array}{c} S_{2^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{1^{-1}} \end{array} \hookrightarrow \begin{array}{c} S_{1^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{1^{-1}} \end{array}$$

is right almost split and is an irreducible morphism. Similarly, the canonical epimorphism

$$\begin{array}{c} S_{1^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{1^{-1}} \end{array} \twoheadrightarrow \begin{array}{c} S_{1^0} \\ \vdots \\ S_{(n-1)^0} \end{array}$$

is left almost split and is an irreducible morphism in the Auslander-Reiten quiver of $\hat{\Lambda}$. It is easy to see that there is a full subquiver

$$\begin{tikzcd}
 & \begin{array}{c} S_{1^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{1^{-1}} \end{array} \arrow[dr] & \\
\begin{array}{c} S_{2^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{1^{-1}} \end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c} S_{1^0} \\ \vdots \\ S_{(n-1)^0} \end{array} \\
 & \begin{array}{c} S_{2^0} \\ \vdots \\ S_{(n-1)^0} \end{array} \arrow[ur] &
\end{tikzcd}$$

in the Auslander-Reiten quiver of $\hat{\Lambda}$, where $\tau\left(\begin{array}{c} S_{1^0} \\ \vdots \\ S_{(n-1)^0} \end{array}\right) = \begin{array}{c} S_{2^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{1^{-1}} \end{array}$. Therefore, the image of $\begin{array}{c} S_{2^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{1^{-1}} \end{array}$ in $\underline{\mathrm{mod}}\hat{\Lambda}$ is a mouth object of the Auslander-Reiten quiver of $D^b(\Lambda) = \underline{\mathrm{mod}}\ \hat{\Lambda}$. There is an exact sequence,

$$0 \to \begin{matrix} S_{2^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{1^{-1}} \end{matrix} \to \begin{matrix} S_{1^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{1^{-1}} \end{matrix} \to S_{1^0} \to 0,$$

in the mod $\hat{\Lambda}$, then the image of $\begin{matrix} S_{2^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{1^{-1}} \end{matrix}$ in $\underline{\mathrm{mod}}\hat{\Lambda}$ is of the dimensional vector $-\epsilon_1$. Similarly, we can prove that the object with the dimensional vector $-\epsilon_2$ is a mouth object. Similarly, there is a full subquiver

$$\begin{tikzcd}
 & \begin{matrix} S_{2^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{n^o}S_{1^{-1}} \\ S_{2^{-1}} \end{matrix} \arrow[dr] & \\
\begin{matrix} S_{3^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{n^0}S_{1^{-1}} \\ S_{2^{-1}} \end{matrix} \arrow[ur] \arrow[dr] & & \begin{matrix} S_{2^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{n^o}S_{1^{-1}} \end{matrix} \\
 & \begin{matrix} S_{3^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{n^o}S_{1^{-1}} \end{matrix} \arrow[ur] &
\end{tikzcd}$$

in the Auslander-Reiten quiver of $\hat{\Lambda}$, where $\tau(\begin{matrix} S_{2^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{n^o}S_{1^{-1}} \end{matrix}) = \begin{matrix} S_{3^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{n^0}S_{1^{-1}} \\ S_{2^{-1}} \end{matrix}$. Therefore, the image of $\begin{matrix} S_{3^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{n^0}S_{1^{-1}} \\ S_{2^{-1}} \end{matrix}$ in $\underline{\mathrm{mod}}\hat{\Lambda}$ is a mouth object of the Auslander-Reiten quiver of $D^b(\Lambda) = \underline{\mathrm{mod}}\ \hat{\Lambda}$. And there is an exact sequence

$$0 \to \begin{matrix} S_{3^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{n^0}S_{1^{-1}} \\ S_{2^{-1}} \end{matrix} \to \begin{matrix} S_{2^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{n^0}S_{1^{-1}} \\ S_{2^{-1}} \end{matrix} \to S_{2^0} \to 0$$

in the $\underline{\bmod}\,\hat{\Lambda}$, then the image of $\begin{array}{c} S_{3^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{n^0} S_{1^{-1}} \\ S_{2^{-1}} \end{array}$ in $\underline{\bmod}\hat{\Lambda}$ is of the dimensional vector $-\epsilon_2$. Let M be the image of $\begin{array}{c} S_{3^0} \\ \vdots \\ S_{(n-1)^0} \\ S_{n^0} S_{1^{-1}} \\ S_{2^{-1}} \end{array}$. Then $\tau^i M, 0 \le i \le n-3$, are mouth objects in $\underline{\bmod}\,\hat{\Lambda}$. Because $\underline{\dim}\tau^i M = -\epsilon_{2+i}$, $0 \le i \le n-3$, the indecomposable objects X with the dimensional vector $-\epsilon_1, \ldots, -\epsilon_{n-1}, \sum_{i=2}^{n} \epsilon_i$ are mouth objects in the Auslander-Retein quiver of $\mathcal{R}$.

(3) Let M_1 be the unique indecomposable object X with the dimensional vector $-\epsilon_{n-1}$ in $\mathcal{R}$, $\underline{\dim} M_{i+1} = \underline{\dim}\tau^{-i} M_1, 1 \le i \le n-3$, M_{n-1} with the dimensional vector $-\epsilon_1$, and M_n with the dimensional vector $\epsilon_2 + \epsilon_3 + \cdots + \epsilon_n$. From (2), we know that $M_i, 1 \le i \le n$, are mouth objects of in $\bmod\hat{\Lambda}$. We describe them in the Auslander-Reiten quiver of $\mathcal{R}$ as follows.

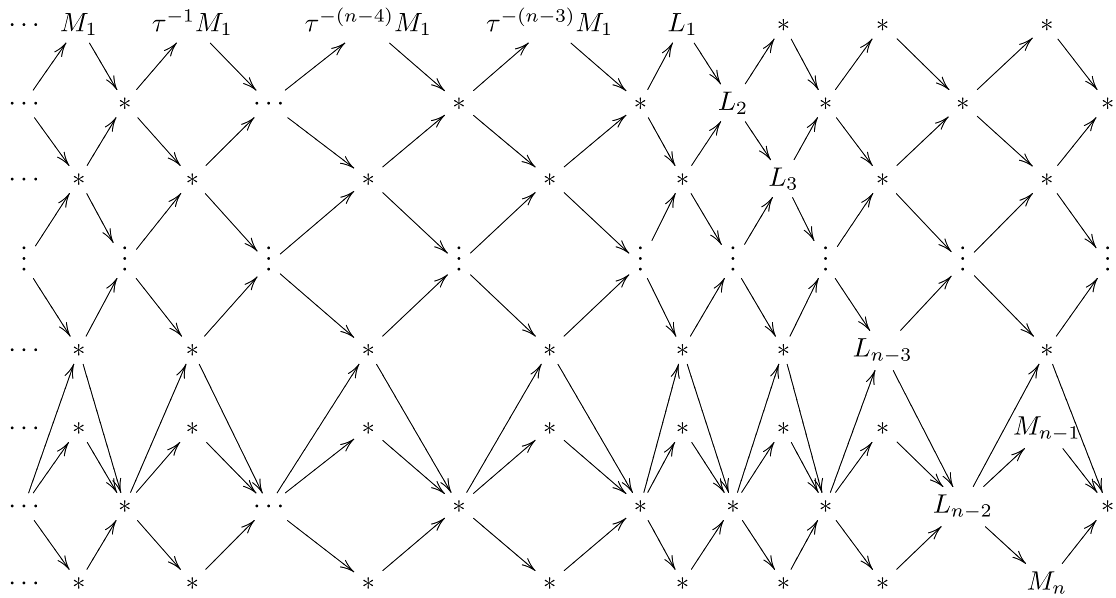


Figure 2 AR-quiver of $\mathcal{R}$

There is a slice $\begin{array}{ccccccccccccc} & & & & & & & & & & M_n & & \\ & & & & & & & & & & \uparrow & & \\ L_1 & \longrightarrow & L_2 & \longrightarrow & L_3 & \longrightarrow & \cdots & \longrightarrow & L_{n-3} & \longrightarrow & L_{n-2} & \longrightarrow & M_{n-1}. \end{array}$
So $\Lambda(n, n-1)$ is a tilted algebra of type $\mathbb{D}_n$, and so a piecewise hereditary algebra of type $\mathbb{D}_n$. As in Section 2, the simple Lie algebra $\mathfrak{g}_\infty(\mathfrak{q})$ has a root space decomposition

$$\mathfrak{g}_\infty(\mathfrak{q}) = H \oplus \bigoplus_{\alpha \in \Delta'} (\mathfrak{g}_\infty(\mathfrak{q}))_\alpha,$$

where $H = \oplus_{i=1}^{n}\mathbb{C}h_i$ is the Cartan subalgebra of $\mathfrak{g}_\infty(\mathfrak{q})$, Δ' is the root system of $\mathfrak{g}_\infty(\mathfrak{q})$. From the Auslander-Reiten quiver we know $\underline{\dim}M_1 = -\epsilon_{n-1}$, $\underline{\dim}L_1 = \underline{\dim}\tau^{-(n-2)}M_1 = (-\epsilon_{n-1})\Phi^{-(n-2)} = -\epsilon_1 + \epsilon_n$. By computation, $\underline{\dim}L_1 = \underline{\dim}M_1 + \underline{\dim}M_2 + \cdots + \underline{\dim}M_{n-1} + \underline{\dim}M_n$. Since there is a distinguished triangle $L_1 \to M \to S_{n-1} \to TL_1$ in $\mathcal{R}$, we have $\underline{\dim}M = -\epsilon_1 + \epsilon_{n-1} + \epsilon_n$. Then $\underline{\dim}M = \underline{\dim}M_2 + \underline{\dim}M_3 + \cdots + \underline{\dim}M_{n-1} + \underline{\dim}M_n$. So if $X = -\sum_{k=1}^{i}\epsilon_k + \sum_{k=j+1}^{n}\epsilon_k$, $\underline{\dim}X$ can be written as $\underline{\dim}X = \sum_{i=1}^{n} k_\alpha \underline{\dim}M_i$ with integral coefficients k_α all nonnegative. Similarly, $\underline{\dim}TL_1 = \epsilon_1 - \epsilon_n$, and $\underline{\dim}TL_1 = \underline{\dim}TM_1 + \underline{\dim}TM_2 + \cdots + \underline{\dim}TM_{n-1} + \underline{\dim}TM_n = -\sum_{i=1}^{n}\underline{\dim}M_i$. By computations, if $Z = \sum_{k=1}^{i}\epsilon_k - \sum_{k=j+1}^{n}\epsilon_k$, $\underline{\dim}Z$ can be written as $\underline{\dim}Z = \sum_{i=1}^{n} k_\alpha^{'}\underline{\dim}M_i$ with integral coefficients $k_\alpha^{'}$ all nonpositive. By computations, each root X can be written as $\underline{\dim}X = \sum_{i=1}^{n} k_\alpha\underline{\dim}M_i$ with integral coefficients k_α all nonnegative or all nonpositive. So $\{M_i | 1 \le i \le n\}$ is a minimal generating subcategory of $\mathcal{R}$. By (1) we know $\{\psi(M_i) | 1 \le i \le n\}$ is a base of the root system Δ', and $\psi(M_i)$, $1 \le i \le n$, are seemed as simple roots of Δ'. Set $\pi_i = \psi(M_i), 1 \le i \le n$. In the Dynkin diagram Γ of Δ', for $i \ne j$, π_i is joined to π_j, if and only if there is a distinguished triangle $M_i \to N \to M_j \to TM_i$ or $M_j \to N' \to M_i \to TM_j$ for $N, N' \in \mathcal{R}$, if and only if $|i-j| = 1, 1 \le i < j \le n-1$, or $\begin{cases} i = n, & j = n-2; \\ i = n-2, & j = n. \end{cases}$ Thus the root system Δ' is of Dynkin type $\mathbb{D}_n$. Therefore the simple Lie algebra $\mathfrak{g}_\infty(\mathfrak{q})$ is of Dynkin type $\mathbb{D}_n$, and $\mathfrak{g}_\infty(\mathfrak{q})$ is a simple Lie algebra of type $\mathbb{D}_n$, $\dim_{\mathbb{C}}\mathfrak{g}_\infty(\mathfrak{q}) = 2n^2 - n$. So $|\Delta'| = 2n^2 - 2n = |\Delta|$. Since the root system $\Delta' \subseteq \Delta$, we have $\Delta' = \Delta$. Thus (3) holds. □

Corollary 5.2 *There is a one-to-one correspondence between the objects of indΛ and the positive roots of Δ.*

Proof Let Δ^+ be the set of the positive roots of Δ. By Lemma 2.1(3) $\Delta^+ = \{\epsilon_i | 1 \le i \le n\} \cup \{\sum_{k=i}^{j}\epsilon_k | 1 < i < j \le n\} \cup \{\sum_{k=1}^{i}\epsilon_k | 1 < i < n\}$, $|\Delta^+| = \frac{n^2+n-2}{2}$. From Theorem 5.1, there is an injective map $\psi : \text{ind}\,\Lambda \to \Delta^+$ defined by $\psi(M) = \underline{\dim}\ M$. So $\psi(\text{ind}\Lambda) \subseteq \Delta^+$. Since $|\psi(\text{ind}\Lambda)| = |\text{ind}\Lambda| = \frac{n^2+n-2}{2} = |\Delta^+|$, we have $\psi(\text{ind}\Lambda) = \Delta^+$. Therefore, $\psi : \text{ind}\Lambda \to \Delta^+$ is bijective. □

Lemma 5.3 *(i) Assume that X is an indecomposable object of $\mathcal{R}$ with dimension vector $\alpha = \delta(-\sum_{k=1}^{i}\epsilon_k + \sum_{k=j+1}^{n}\epsilon_k)$, where $1 \le i < j \le n$. Then in the Ringel-Hall Lie algebra $\mathbf{g}$,*

$$u_X = \begin{cases} (-1)^i[u_{TS_i}, u_{TS_{i-1}}, \ldots, u_{TS_2}, u_{S_{j+1}}, \ldots, u_{S_{n-1}}, u_{S_n}, u_{TS_1}], & \text{if } \delta = 1; \\ (-1)^i[u_{S_i}, u_{S_{i-1}}, \ldots, u_{S_2}, u_{TS_{j+1}}, \ldots, u_{TS_{n-1}}, u_{TS_n}, u_{S_1}], & \text{if } \delta = -1. \end{cases}$$

(ii) Assume that X' is an indecomposable object of $\mathcal{R}$ with dimension vector $\alpha = \delta(\sum_{k=1}^{i}\epsilon_k)$, where $1 \le i < n$. Then in the Ringel-Hall Lie algebra $\mathbf{g}$, if $\delta = 1$, $u_{X'} = [u_{S_1}, u_{S_2}, u_{S_3}, \ldots, u_{S_{i-1}}, u_{S_i}]$; if $\delta = -1$, $u_{X'} = [u_{TS_1}, u_{TS_2}, u_{TS_3}, \ldots, u_{TS_{i-1}}, u_{TS_i}]$.

(iii) Assume that X'' is an indecomposable object of $\mathcal{R}$ with dimension vector $\alpha = \delta(\sum_{k=i}^{j}\epsilon_k)$, where $1 < i < j \le n$. Then in the Ringel-Hall Lie algebra $\mathbf{g}$, $u_{X''} = [u_{TS_i}, u_{TS_{i+1}}, \ldots, u_{TS_{j-1}}, u_{TS_j}]$ for $\delta = -1$; $u_{X''} = [u_{S_i}, u_{S_{i+1}}, \ldots, u_{S_{j-1}}, u_{S_j}]$ for $\delta = 1$.

Proof (i) Let $L_{n-1} = M_{n-1}, L_n = M_n$. As shown in Figure 2, there is a slice

$$\begin{array}{ccccccccccccc} & & & & & & & & & & L_n & & \\ & & & & & & & & & & \uparrow & & \\ L_1 & \longrightarrow & L_2 & \longrightarrow & L_3 & \longrightarrow & \cdots & \longrightarrow & L_{n-3} & \longrightarrow & L_{n-2} & \longrightarrow & L_{n-1} \end{array}$$

in the Auslander-Reiten quiver of $\mathcal{R}$. By the structure of Auslander-Reiten quiver of $\mathcal{R}$, we know that $\underline{\dim} L_1 = -\epsilon_1 + \epsilon_n$, and there is a distinguished triangle $S_n \to L_1 \to TS_1 \to TS_n$ in $\mathcal{R}$. By the structure of Auslander-Reiten quiver of $\mathcal{R}$, $u_{L_1} = -[u_{S_n}, u_{TS_1}]$. We can get an exact sequence $0 \to L_1 \to L_2 \to S_{n-1} \to 0$, then $u_{L_2} = [u_{S_{n-1}}, u_{L_1}] = -[u_{S_{n-1}}, u_{S_n}, u_{TS_1}]$. Similarly, since $0 \to L_i \to L_{i+1} \to S_{n-i} \to 0$ is exact, we have

$$u_{L_i} = -[u_{S_{n-i+1}}, u_{S_{n-i+2}}, \dots, u_{S_n}, u_{TS_1}] \qquad (**)$$

for $1 \le i \le n-2$. For any indecomposable object N with dimension vector $\delta(-\sum_{k=1}^{i} \epsilon_k + \sum_{k=j+1}^{n} \epsilon_k)$ or $\delta(-\sum_{k=1}^{i} \epsilon_k)$ or $\delta(-\sum_{k=i}^{j} \epsilon_k), \delta \in \{1, -1\}$, there is some $m \in \{1, 2, \dots, n\}$ such that $\mathrm{Hom}_{\mathcal{R}}(N, L_m) \neq 0$ or $\mathrm{Hom}_{\mathcal{R}}(TN, L_m) \neq 0$ or $\mathrm{Hom}(L_m, N) \neq 0$ or $\mathrm{Hom}(L_m, TN) \neq 0$, we only prove (i) for the case $\mathrm{Hom}_{\mathcal{R}}(N, L_m) \neq 0$. Let $\delta = 1$, i.e., $\underline{\dim} N = -\sum_{k=1}^{i} \epsilon_k + \sum_{k=j+1}^{n} \epsilon_k$, $1 \le i < j \le n$. Choose $0 \neq f \in \mathrm{Hom}_{\mathcal{R}}(N, L_m)$. We can construct a distinguished triangle $N' \to N \to L_m \to TN'$ in $\mathcal{R}$. Since $\mathrm{Hom}_k(L_m, TN) = 0$ by the structure of $\mathcal{R}$, we have by [11, Corollary 1.4], N' is indecomposable, and so $\underline{\dim} N' \in \Delta$. On the other hand, $\underline{\dim} N' = \underline{\dim} N - \underline{\dim} L_m$. Since $\mathrm{Hom}(N, L_m) \neq 0$, then $\underline{\dim} L_m = -\epsilon_1 + \sum_{k=j+1}^{n} \epsilon_k$ for $2 < j < n$. In fact, $L_m = L_{n-j}$, then $\underline{\dim} N' = -\sum_{k=2}^{i} \epsilon_k$ for $2 \le i < n-1$. Otherwise, $\underline{\dim} N' \notin \Delta$, a contradiction. If $\underline{\dim} N' = -\epsilon_2$, then $\underline{\dim} L_m = -\epsilon_1 + \sum_{k=j+1}^{n} \epsilon_k$, $2 \le j < n$. We can get an exact sequence $0 \to N' \to N \to L_m \to 0$. By the structure of Auslander-Reiten quiver of $\mathcal{R}$, then $u_N = [u_{L_m}, u_{N'}]$. By Theorem 5.1, $N' \simeq TS_2$, $u_N = [u_{L_m}, u_{TS_2}]$. By $(**)$, $u_N = (-1)^2 [u_{TS_2}, u_{S_{j+1}}, \dots, u_{S_n}, u_{TS_1}]$. If $\underline{\dim} N' = -\epsilon_2 - \epsilon_3$, there is an exact sequence $0 \to TS_3 \to N' \to TS_2 \to 0$, then $u_{N'} = [u_{TS_2}, u_{TS_3}]$. Therefore,

$$\begin{aligned} u_N =& [u_{L_m}, u_{N'}] = [u_{L_m}, [u_{TS_2}, u_{TS_3}]] \\ =& -[u_{TS_2}, [u_{TS_3}, u_{L_m}]] - [u_{TS_3}, [u_{L_m}, u_{TS_2}]] \\ =& -[u_{TS_3}, [u_{L_m}, u_{TS_2}]] = (-1)^2 [u_{TS_3}, [u_{TS_2}, u_{L_m}]] \\ =& (-1)^3 [u_{TS_3}, u_{TS_2}, u_{S_{j+1}}, \dots, u_{S_n}, u_{TS_1}]. \end{aligned}$$

If $\underline{\dim} N' = -\sum_{k=2}^{i} \epsilon_k$, then we know $u_N = (-1)^i [u_{TS_i}, u_{TS_{i-1}}, \dots, u_{TS_2}, u_{S_{j+1}}, \dots, u_{S_n}, u_{TS_1}]$ for $\underline{\dim} N = -\sum_{k=1}^{i} \epsilon_k + \sum_{k=j+1}^{n} \epsilon_k$. Next, for the case $\delta = -1$, i.e., $\underline{\dim} N = \sum_{k=1}^{i} \epsilon_k - \sum_{k=j+1}^{n} \epsilon_k$, $1 \le i < j \le n$, we can similarly obtain that $u_N = (-1)^i [u_{S_i}, u_{S_{i-1}}, \dots, u_{S_2}, u_{TS_{j+1}}, \dots, u_{TS_n}, u_{S_1}]$. So (i) holds.

(ii) By the structure of Auslander-Reiten quiver of $\mathcal{R}$, we know that if $\underline{\dim} G = \epsilon_i$, $\underline{\dim} H = \epsilon_{i-1}$, $\underline{\dim} Q = \epsilon_i + \epsilon_{i-1}$, and there is an exact sequence $0 \to G \to Q \to H \to 0$ in $\mathcal{R}$, then $u_Q = [u_H, u_G] = [u_{S_{i-1}}, u_{S_i}]$. And if $\underline{\dim} I = \epsilon_{i-2}, \underline{\dim} F = \epsilon_i + \epsilon_{i-1} + \epsilon_{i-2}$, we can get an exact sequence $0 \to Q \to F \to I \to 0$, then $u_F = [u_I, u_Q] = [u_{S_{i-2}}, u_{S_{i-1}}, u_{S_i}]$. By computation, $u_{X'} = [u_{S_1}, u_{S_2}, u_{S_3}, \dots, u_{S_{i-1}}, u_{S_i}]$ for $\underline{\dim} X' = \sum_{k=1}^{i} \epsilon_k$. Similarly, $u_{X'} = [u_{TS_1}, u_{TS_2}, \dots, u_{TS_{i-1}}, u_{TS_i}]$ for $\underline{\dim} X' = -\sum_{k=1}^{i} \epsilon_k$.

(iii) Similarly to the (ii), $u_{X''} = [u_{TS_i}, u_{TS_{i+1}}, \dots, u_{TS_{j-1}}, u_{TS_j}]$ for $\underline{\dim}X'' = -\sum_{k=i}^{j}\epsilon_k$, $u_{X''} = [u_{S_i}, u_{S_{i+1}}, \dots, u_{S_{j-1}}, u_{S_j}]$ for $\underline{\dim}X'' = \sum_{k=i}^{j}\epsilon_k$. □

Theorem 5.4 *The Lie algebra $\mathfrak{g}_\infty(\mathfrak{q})$ attached to the unit form*

$$\mathfrak{q}(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} x_i^2 - \Big(\sum_{i=1}^{n-1} x_i x_{i+1}\Big) + x_1 x_n$$

has a basis

$$\begin{aligned}\mathcal{B} =&\{h_i, e_{\pm i} | 1 \le i \le n\} \cup \{[e_{-i}, e_{-(i-1)}, \dots, e_{-2}, e_{j+1}, \dots, e_{n-1}, e_n, e_{-1}],\\ &[e_i, e_{i-1}, \dots, e_2, e_{-(j+1)}, \dots, e_{-(n-1)}, e_{-n}, e_1] | 1 \le i < j \le n-1\} \cup\\ &\{[e_{-i}, e_{-(i+1)}, \dots, e_{-(j-1)}, e_{-j}], [e_i, e_{i+1}, \dots, e_{j-1}, e_j] | 1 < i < j \le n\} \cup\\ &\{[e_1, e_2, \dots, e_{i-1}, e_i], [e_{-1}, e_{-2}, \dots, e_{-(i-1)}, e_{-i}] | 1 < i < n\}.\end{aligned}$$

Proof By the construction of $\mathbf{g}$, $\mathbf{g}$ has a Chevalley basis $\bar{\mathcal{B}} = \{h_{S_i} | 1 \le i \le n\} \cup \{u_M | M \in \mathrm{ind}\,\mathcal{R}\}$. By Theorem 4.3, $\mathfrak{g}_\infty(\mathfrak{q})$ has a Chevalley basis $\varphi^{-1}(\bar{\mathcal{B}}) = \{h_i | 1 \le i \le n\} \cup \{\varphi^{-1}(u_M) | M \in \mathrm{ind}\,\mathcal{R}\}$. If $\underline{\dim}M = \pm\epsilon_i, 1 \le i \le n$, then $\varphi^{-1}(u_M) = \pm e_{\pm i}$. Let $\underline{\dim}M \ne \pm\epsilon_i, 1 \le i \le n$. Next we prove in the following cases that for any $M \in \mathrm{ind}\mathcal{R}$, $\varphi^{-1}(u_M)$ is a scalar multiplication of some element in $\mathcal{B}$.

Case 1 $\underline{\dim}M = \delta(-\sum_{k=1}^{i}\epsilon_k + \sum_{k=j+1}^{n}\epsilon_k)$.

If $\delta = 1$ and i is odd, then by Lemma 5.2, $u_M = (-1)[u_{TS_i}, u_{TS_{i-1}}, \dots, u_{TS_2}, u_{S_{j+1}}, \dots, u_{S_{n-1}}, u_{S_n}, u_{TS_1}]$. So $\varphi^{-1}(u_M) = -[-e_{-i}, -e_{-(i-1)}, \dots, -e_{-2}, e_{j+1}, \dots, e_{n-1}, e_n, -e_{-1}] = (-1)^{i+1}[e_{-i}, e_{-(i-1)}, \dots, e_{-2}, e_{j+1}, \dots, e_{n-1}, e_n, e_{-1}]$. If $\delta = 1$ and i is even, then $\varphi^{-1}(u_M) = (-1)^i[e_{-i}, e_{-(i-1)}, \dots, e_{-2}, e_{j+1}, \dots, e_{n-1}, e_n, e_{-1}]$.

If $\delta = -1$, then $u_M = (-1)^i[u_{S_i}, u_{S_{i-1}}, \dots, u_{S_2}, u_{TS_{j+1}}, \dots, u_{TS_{n-1}}, u_{TS_n}, u_{S_1}]$ by Lemma 5.2, and so $\varphi^{-1}(u_M) = (-1)^i[e_i, e_{i-1}, \dots, e_2, -e_{-(j+1)}, \dots, -e_{-(n-1)}, -e_{-n}, e_1] = (-1)^{n-j+i}[e_i, e_{i-1}, \dots, e_2, e_{-(j+1)}, \dots, e_{-(n-1)}, e_{-n}, e_1]$. So if $n-j$ is even, then $\varphi^{-1}(u_M) = (-1)^i[e_i, e_{i-1}, \dots, e_2, e_{-(j+1)}, \dots, e_{-(n-1)}, e_{-n}, e_1]$, and if $n-j$ is odd, then $\varphi^{-1}(u_M) = (-1)^{i+1}[e_i, e_{i-1}, \dots, e_2, e_{-(j+1)}, \dots, e_{-(n-1)}, e_{-n}, e_1]$.

Case 2 $\underline{\dim}W = \delta(\sum_{k=1}^{i}\epsilon_k)$.

If $\delta = 1$, then $u_W = [u_{S_1}, u_{S_2}, \dots, u_{S_{i-1}}, u_{S_i}]$, so $\varphi^{-1}(u_W) = [e_1, e_2, \dots, e_{i-1}, e_i]$. If $\delta = -1$, then $u_W = [u_{TS_1}, u_{TS_2}, \dots, u_{TS_{i-1}}, u_{TS_i}]$, so $\varphi^{-1}(u_M) = [-e_{-1}, -e_{-2}, \dots, -e_{-(i-1)}, -e_{-i}] = (-1)^i[e_{-1}, e_{-2}, \dots, e_{-(i-1)}, e_{-i}]$. If i is even, then $\varphi^{-1}(u_M) = [e_{-1}, e_{-2}, \dots, e_{-(i-1)}, e_{-i}]$. If i is odd, then $\varphi^{-1}(u_M) = -[e_{-1}, e_{-2}, \dots, e_{-(i-1)}, e_{-i}]$.

Case 3 $\underline{\dim}X = \delta(\sum_{k=i}^{j}\epsilon_k)$.

If $\delta = 1$, then $u_X = [u_{S_i}, u_{S_{i+1}}, \dots, u_{S_{j-1}}, u_{S_j}]$, and $\varphi^{-1}(u_X) = [e_i, e_{i+1}, \dots, e_{j-1}, e_j]$. If $\delta = -1$, then $u_X = [u_{TS_i}, u_{TS_{i+1}}, \dots, u_{TS_{j-1}}, u_{TS_j}]$, so $\varphi^{-1}(u_X) = [-e_{-i}, -e_{-(i+1)}, \dots, -e_{-(j-1)}, -e_{-j}] = (-1)^{j-i+1}[e_{-i}, e_{-(i+1)}, \dots, e_{-(j-1)}, e_{-j}]$. If $j-i+1$ is odd, then $\varphi^{-1}(u_X) = -[e_{-i}, e_{-(i+1)}, \dots, e_{-(j-1)}, e_{-j}]$. If $j-i+1$ is even, then $\varphi^{-1}(u_X) = [e_{-i}, e_{-(i+1)}, \dots, e_{-(j-1)}, e_{-j}]$.

So $\mathcal{B}$ is also a Chevalley basis of $\mathfrak{g}_\infty(\mathfrak{q})$. □

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References

[1] C. M. RINGEL. *Hall Algebras.*Topics in Algebra, 1990, **26**(1): 433–447.

[2] C. M. RINGEL. *Hall polynominals for the representation-finite hereditary algebras*. Adv. Math., 1990, **84**(2): 137–178.

[3] C. M. RINGEL. *Lie Algebras Arising in Representation Theory*. Cambridge Univ. Press, Cambridge, 1992.

[4] Liangang PENG, Jie XIAO. *Root categories and simple Lie algebras*. J. Algebra, 1997, **198**(1): 19–56.

[5] Liangang PENG, Jie XIAO. *Triangulated categories and Kac-Moody algebras*. Invent. Math., 2000, **140**(3): 563–603.

[6] Ya-nan LIN, Liangang PENG. *Elliptic Lie algebras and tubular algebras*. Adv. Math., 2005, **196**(2): 487–530.

[7] Jinjing CHEN, Zhengxin CHEN. *A class of simple Lie algebras attached to unit forms*. Front. Math. China, 2017, **12**(4): 787–803.

[8] M. BAROT. D. KUSSIN. H. LENZING. *The Lie algebra associated to a unit form*. J. Algebra, 2006, **296**(1): 1C-17.

[9] J. P. SERRE. *Complex Semisimple Lie Algebras*. Springer-Verlag, New York, 1987.

[10] D. HAPPEL. U. SEIDEL. *Piecewise hereditary Nakayama algebras*. Algebr. Represent. Theory, 2010, **13**(6): 693–704.

[11] D. HAPPEL. *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras*. Cambridge University Press, Cambridge, 1988.

[12] H. ASASHIBA. *Realization of simple Lie algebras via Hall algebras of tame hereditary algebras*. J. Math. Soc. Japan, 2004, **56**(3): 889–905.

[13] H. ASASHIBA. *Domestic canonical algebras and simple Lie algebras*. Math. Z., 2008, **259**(4): 713–754.

[14] Zhengxin CHEN, Ya-nan LIN. *Tubular algebras and affine Kac-Moody algebras*. Sci. China Ser. A, 2007, **50**(4): 521–532.

[15] Zhengxin CHEN, Ya-nan LIN. *A realization of elliptic Lie algebras of type $F_4^{(2,2)}$ by the Ringel-Hall approach*. J. Algebra, 2012, **350**: 108–131.

[16] Bangming DENG, Jie DU. *Folding derived categories with Frobenius functors*. J. Pure Appl. Algebra, 2007, **208**(3): 1023–1050.

[17] Ming DING, Jie XIAO, Fan XU. *Realizing enveloping algebras via varieties of modules*. Acta Math. Sin. (Engl. Ser.), 2010, **26**(1): 29–48.

[18] Changjian FU. *On root categories of finite-dimensional algebras*. J. Algebra, 2012, **370**: 233–265.

[19] D. HAPPEL, D. ZACHARIA. *A homological characterization of piecewise hereditary algebras*. Math. Z., 2008, **260**(1): 177–185.

[20] Liangang PENG. *Intersection Matrix Lie Algebras and Ringel-Hall Lie Algebras of Tilted Algebras*. Beijing Norm. Univ. Press, Beijing, 2002.

[21] Liangang PENG, Mang XU. *Symmetrizable intersection matrices and their root systems*. arXiv: 0912.1024.

[22] C. RIEDTMANN. *Lie algebras generated by indecomposables*. J. Algebra, 1997, **198**: 19–56.

[23] C. M. RINGEL. *Tame algebras and integral quadratic forms*. Springer-Verlag, Berlin, 1984.

[24] K. SAITO, D. YOSHII. *Extended affine root system. IV. Simply-laced elliptic Lie algebras*. Publ. Res. Inst. Math. Sci., 2000, **36**(3): 385–421.