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On Strongly J-Semiclean Rings

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Abstract We in this note introduce a new concept, so called strongly J-semiclean ring, that is a generalization of strongly J-clean rings. We first observe the basic properties of strongly J-semiclean rings, constructing typical examples. We next investigate conditions on a local ring R that imply that the upper triangular matrix ring $T_n(R)$ is a strongly J-semiclean ring. Also, the criteria on strong J-semicleanness of 2×2 matrices in terms of a quadratic equation are given. As a consequence, several known results relating to strongly J-clean rings are extended to a more general setting.

Keywords upper triangular matrix; local ring; strongly J-semiclean ring

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1. Introduction

Throughout this paper all rings R are associative with identity, and all modules are unitary R-modules. Let R be a ring. We use $\mathbb{N}$, $U(R)$, $J(R)$, $T_n(R)$ and $M_n(R)$ to represent the set of all natural numbers, the set of units of R, the Jacobson radical of R, the ring of all upper triangular matrices over R and the ring of all $n \times n$ matrices over R, respectively.

An element $a \in R$ is strongly clean provided that there exist an idempotent $e^2 = e \in R$ and a unit $u \in U(R)$ such that $a = e + u$ and $eu = ue$. A ring R is strongly clean in case every element in R is strongly clean. Strong cleanness over commutative rings was extensively studied by many authors from very different view points [1–8]. Replacing $U(R)$ by $J(R)$, in [9], Chen paralleled to introduce the concept of strong J-cleanness. An element $a \in R$ is strongly J-clean provided that there exist an idempotent $e \in R$ and an element $w \in J(R)$ such that $a = e + w$ and $ew = we$. A ring R is strongly J-clean in case every element in R is strongly J-clean. It was shown in [9] that every strongly J-clean element is strongly clean, but the converse is not true in general [9, Example 2.2]. For more details and properties of strongly J-clean rings [9, 10].

In this note we continue the study of strongly J-clean rings. As a generalization of strongly J-clean rings, we first introduce a notion of strongly J-semiclean rings and investigate its properties. We next provide some necessary and sufficient conditions for the upper triangular matrix ring $T_n(R)$ over a local ring R to be strongly J-semiclean. Also, a criterion in terms of solvability of a simple quadratic equation in R is obtained for $M_2(R)$ to be strongly J-semiclean.

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2. Examples

Let R be a ring. An element $a \in R$ is called a periodic element provided that there exist two distinct positive integers m and n such that $a^m = a^n$. Clearly, both nilpotent elements and idempotents are periodic elements. A ring R is called a periodic ring if every element of R is a periodic element.

Definition 2.1 *Let R be a ring. An element $a \in R$ is strongly J-semiclean provided that there exist a periodic element $\pi \in R$ and an element $w \in J(R)$ such that $a = \pi + w$ and $\pi w = w\pi$. A ring R is strongly J-semiclean in case every element in R is strongly J-semiclean.*

Clearly, every strongly J-clean ring is strongly J-semiclean. But the following example shows that the converse is not true. Hence a strongly J-semiclean ring is not a trivial extension of a strongly J-clean ring.

Example 2.2 Let $\mathbb{Z}_p$ denote the ring of integers modulo p, where p is a prime number. Consider the ring $\mathbb{Z}_p[x]/(x^{n+1})$, where (x^{n+1}) is the ideal generated by x^{n+1}. Denote $\overline{x}$ in $\mathbb{Z}_p[x]/(x^{n+1})$ by u. Then $\mathbb{Z}_p[x]/(x^{n+1}) = \mathbb{Z}_p[u] = \mathbb{Z}_p + \mathbb{Z}_p u + \mathbb{Z}_p u^2 + \cdots + \mathbb{Z}_p u^n$. Let $f = a_0 + a_1 u + \cdots + a_n u^n \in \mathbb{Z}_p[u]$. Since $a_0^p = a_0$, $a_1 u + a_2 u^2 + \cdots + a_n u^n \in J(\mathbb{Z}_p[u])$, we obtain that f is strongly J-semiclean. Therefore, $\mathbb{Z}_p[u] = \mathbb{Z}_p[x]/(x^{n+1})$ is a strongly J-semiclean ring.

Now we show that $\mathbb{Z}_p[u] = \mathbb{Z}_p[x]/(x^{n+1})$ is not a strongly J-clean ring when $p \neq 2$. Note that the idempotent in $\mathbb{Z}_p[u] = \mathbb{Z}_p[x]/(x^{n+1})$ is $\overline{0}$ or $\overline{1}$. So for any $g = b_0 + b_1 u + \cdots + b_n u^n \in \mathbb{Z}_p[u]$ with $b_0 \neq \overline{0}$ and $b_0 \neq \overline{1}$, we obtain that $b_0 + b_1 u + \cdots + b_n u^n$ and $b_0 - 1 + b_1 u + \cdots + b_n u^n$ are units of $\mathbb{Z}_p[u] = \mathbb{Z}_p[x]/(x^{n+1})$. So g is not a sum of an idempotent $e^2 = e \in \mathbb{Z}_p[u]$ and an element $w \in J(\mathbb{Z}_p[u])$. Hence g is not strongly J-clean. Therefore, $\mathbb{Z}_p[u] = \mathbb{Z}_p[x]/(x^{n+1})$ is not a strongly J-clean ring, as desired.

Proposition 2.3 *Every strongly J-semiclean element is strongly clean.*

Proof Let $x \in R$ be a strongly J-semiclean element. Then there exists a periodic element $\pi \in R$ and an element $w \in J(R)$ such that $x = \pi + w$ and $\pi w = w\pi$. Let k and l ($k > l$) be two distinct positive integers such that $\pi^k = \pi^l$. Then we have

$$\pi^l = \pi^k = \pi^l \pi^{k-l} = \pi^l \pi^{2(k-l)} = \cdots = \pi^l \cdot \pi^{l(k-l-1)} \cdot \pi^l.$$

Thus $\pi^l \cdot \pi^{l(k-l-1)} = \pi^{l(k-l)}$ and $1 - \pi^{l(k-l)}$ are idempotents. Write $x = \pi + w$ as

$$x = (1 - \pi^{l(k-l)}) + \pi - (1 - \pi^{l(k-l)}) + w.$$

Since

$$\begin{aligned}
&[\pi - (1 - \pi^{l(k-l)})][\pi^{l-1}\pi^{l(k-l-1)}\pi^{l(k-l)} - (\sum_{i=0}^{l-1} \pi^i)(1 - \pi^{l(k-l)})] \\
&= [\pi\pi^{l(k-l)} - (1-\pi)(1 - \pi^{l(k-l)})][\pi^{l-1}\pi^{l(k-l-1)}\pi^{l(k-l)} - \Big(\sum_{i=0}^{l-1} \pi^i\Big)(1 - \pi^{l(k-l)})] \\
&= \pi^l \pi^{l(k-l-1)} \pi^{l(k-l)} + (1 - \pi^l)(1 - \pi^{l(k-l)})
\end{aligned}$$

$$= \pi^{l(k-l)} + 1 - \pi^{l(k-l)} - \pi^l + \pi^l \pi^{l(k-l)} = 1$$

$$= [\pi^{l-1}\pi^{l(k-l-1)}\pi^{l(k-l)} - \Big(\sum_{i=0}^{l-1}\pi^i\Big)(1-\pi^{l(k-l)})][\pi - (1-\pi^{l(k-l)})],$$

we obtain that $\pi - (1-\pi^{l(k-l)})$ is a unit in R and so is $\pi - (1-\pi^{l(k-l)}) + w$. It follows from $\pi w = w\pi$ that

$$(1-\pi^{l(k-l)})[\pi - (1-\pi^{l(k-l)}) + w] = [\pi - (1-\pi^{l(k-l)}) + w](1-\pi^{l(k-l)}).$$

Hence x is a strongly clean element, as required. □

Corollary 2.4 ([9, Proposition 2.1]) *Every strongly J-clean element is strongly clean.*

Proof The result follows from Proposition 2.3. □

Corollary 2.5 *All strongly J-semiclean rings are strongly clean.*

Proof It follows from Proposition 2.3. □

The following example shows that the converse of Corollary 2.5 is not true.

Example 2.6 Let $\mathbb{Q}$ be the field of rational numbers. Then $\mathbb{Q}$ is strongly clean but not strongly J-semiclean.

Recall that an element $a \in R$ is strongly semiclean if $a = \pi + u$, where π is a periodic element in R and u is a unit in R such that $\pi u = u\pi$. A ring R is a strongly semiclean ring if every element in R is strongly semiclean [11].

Proposition 2.7 *All strongly J-semiclean rings are strongly semiclean.*

Proof For any $x \in R$, there exist a periodic element $\pi \in R$ and an element $w \in J(R)$ such that $x - 1 = \pi + w$ and $\pi w = w\pi$. Then $x = \pi + 1 + w$. Clearly, $1 + w$ is a unit in R and $\pi(1+w) = (1+w)\pi$. So x is a strongly semiclean element. Therefore, R is a strongly semiclean ring. □

From Proposition 2.7, one may suspect that that every strongly semiclean ring is strongly J-semiclean. But the following example eliminates the possibility.

Example 2.8 Let $R = \{\frac{m}{n} \mid m, n \in \mathbb{Z} \text{ and } 7 \nmid n\}$ and $G = \{g, g^2, g^3\}$ be a cyclic group of order 3. Then by [12, Theorem 3.1], the group ring RG is a strongly semiclean ring. But it is shown in [6] that RG is not clean and so RG is not strongly clean. So by Corollary 2.5, we obtain that RG is not strongly J-semiclean.

Example 2.9 Here are some examples of strongly J-semiclean rings:

(1) Any factor ring of a strongly J-semiclean ring is strongly J-semiclean.

(2) A direct sum $R = \oplus_{i=1}^n R_i$ of rings $\{R_i\}$ is strongly J-semiclean if and only if so is each R_i, $1 \le i \le n$.

(3) $R[x]$ is certainly never a strongly J-semiclean ring.

(4) Let R be a commutative ring. Then $R[x]/(x^n)$ is strongly J-semiclean if and only if so

is R.

(5) Every power series ring over a strongly J-semiclean commutative ring is strongly J-semiclean.

Proof (1) It is directly verified.

(2) It suffices to show that if R_1 and R_2 are strongly J-semiclean, then so is $R_1 \oplus R_2$. Let $x = (x_1, x_2) \in R_1 \oplus R_2$, where $x_1 \in R_1$ and $x_2 \in R_2$. For each i $(1 \le i \le 2)$, write $x_i = \pi_i + w_i$, where π_i is a periodic element in R_i, $w_i \in J(R_i)$ and $\pi_i w_i = w_i \pi_i$. Then $x = (x_1, x_2) = (\pi_1, \pi_2) + (w_1, w_2)$. Let k_i and l_i ($k_i > l_i$ for each $i = 1, 2$) be positive integers such that $\pi_1^{k_1} = \pi_1^{l_1}$ and $\pi_2^{k_2} = \pi_2^{l_2}$. Then

$$\pi_1^{l_1} = \pi_1^{k_1} = \pi_1^{l_1}\pi_1^{k_1-l_1} = \pi_1^{l_1}\pi_1^{2(k_1-l_1)} = \cdots = \pi_1^{l_1}\pi_1^{(k_2-l_2)(k_1-l_1)},$$

and

$$\pi_2^{l_2} = \pi_2^{k_2} = \pi_2^{l_2}\pi_2^{k_2-l_2} = \pi_2^{l_2}\pi_2^{2(k_2-l_2)} = \cdots = \pi_2^{l_2}\pi_2^{(k_1-l_1)(k_2-l_2)}.$$

Without loss of generality, we may assume that $l_1 < l_2$. Then

$$\pi_1^{l_2} = \pi_1^{l_1}\pi_1^{l_2-l_1} = \pi_1^{l_1}\pi_1^{(k_1-l_1)(k_2-l_2)}\pi_1^{l_2-l_1} = \pi_1^{l_2}\pi_1^{(k_1-l_1)(k_2-l_2)},$$

and so $(\pi_1, \pi_2)^{l_2} = (\pi_1, \pi_2)^{(k_1-l_1)(k_2-l_2)+l_2}$. Thus (π_1, π_2) is a periodic element in $R_1 \oplus R_2$. Clearly, $(\pi_1, \pi_2)(w_1, w_2) = (w_1, w_2)(\pi_1, \pi_2)$ and $(w_1, w_2) \in J(R_1 \oplus R_2)$. Hence $x = (x_1, x_2)$ is strongly J-semiclean in $R_1 \oplus R_2$. Therefore, $R_1 \oplus R_2$ is a strongly J-semiclean ring, as desired.

(3) It follows from the fact that $R[x]$ is not strongly clean [9, Example 2.5].

(4) It is trivial.

(5) It is trivial. □

Corollary 2.10 *Let $R = \oplus_{i=1}^n R_i$. Then $a = (x_1, x_2, \ldots, x_n) \in R$ is a periodic element in R if and only if each x_i is a periodic element in R_i $(1 \le i \le n)$.*

3. Elementary properties

Let I be an ideal of R. An element $a \in R$ is said to lift strongly modulo I, if, whenever $a^m - a^n \in I$, there exists $b^s = b^t \in R$ such that $b - a \in I$ and $ab = ba$, where m, n, s, $t \in \mathbb{N}$ and $m \ne n$, $s \ne t$. A ring R is said to be a J-ring if every element in R lifts strongly modulo $J(R)$. Clearly, if $a \in R$ is strongly J-semiclean, then a lifts strongly modulo $J(R)$.

Proposition 3.1 *Let $R = \oplus_{i=1}^n R_i$. Then R is a J-ring if and only if so is each R_i $(1 \le i \le n)$.*

Proof It suffices to show that $R = R_1 \oplus R_2$ is a J-ring if and only if R_1 and R_2 are J-rings. Suppose that $R_1 \oplus R_2$ is a J-ring. Let $x_1 \in R_1$ be such that $x_1^s - x_1^t \in J(R_1)$ where $s \ne t \in \mathbb{N}$. Then $a = (x_1, 0) \in R_1 \oplus R_2$ and $a^s - a^t = (x_1^s - x_1^t, 0) \in J(R_1 \oplus R_2)$. So there exists a periodic element $\pi = (y_1, y_2) \in R_1 \oplus R_2$ such that $a - \pi = (x_1 - y_1, -y_2) \in J(R_1 \oplus R_2)$ and $a\pi = (x_1 y_1, 0) = \pi a = (y_1 x_1, 0)$. Thus $x_1 - y_1 \in J(R_1)$, $x_1 y_1 = y_1 x_1$, and by Corollary 2.10, y_1 is a periodic element in R_1. So R_1 is a J-ring. Similarly, we can show that R_2 is a J-ring.

Assume that both R_1 and R_2 are J-rings. Let $a = (x_1, x_2) \in R_1 \oplus R_2$ be such that $a^s - a^t \in J(R_1 \oplus R_2)$ where $s \neq t \in \mathbb{N}$. Then we have $x_1^s - x_1^t \in J(R_1)$ and $x_2^s - x_2^t \in J(R_2)$. So there exist a periodic element $\pi_1 \in R_1$ and a periodic element $\pi_2 \in R_2$ such that $x_1 - \pi_1 \in J(R_1)$, $x_2 - \pi_2 \in J(R_2)$ and $x_1\pi_1 = \pi_1 x_1$, $x_2\pi_2 = \pi_2 x_2$. Let $\pi = (\pi_1, \pi_2)$. Then π is a periodic element in $R_1 \oplus R_2$, and $a - \pi = (x_1 - \pi_1, x_2 - \pi_2) \in J(R_1 \oplus R_2)$, $a\pi = \pi a$. So a lifts strongly modulo $J(R_1 \oplus R_2)$. Therefore, $R_1 \oplus R_2$ is a J-ring. □

Proposition 3.2 *Let R be a ring. If $T_n(R)$ is a J-ring, then so is $T_m(R)$ for each $1 \leq m \leq n$.*

Proof Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{mm} \end{pmatrix} \in T_m(R)$ be such that $A^s - A^t \in J(T_m(R))$ where $s \neq t \in \mathbb{N}$. Then we have $(A')^s - (A')^t \in J(T_n(R))$, where

$$A' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2m} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{mm} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} \in T_n(R).$$

So there exists a periodic matrix

$$B' = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix} \in T_n(R)$$

such that $A' - B' \in J(T_n(R))$ and $A'B' = B'A'$. Let

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ 0 & b_{22} & \cdots & b_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{mm} \end{pmatrix} \in T_m(R).$$

Then B is a periodic upper triangular matrix in $T_m(R)$, $A - B \in J(T_m(R))$ and $AB = BA$. Hence A lifts strongly modulo $J(T_m(R))$. Therefore, $T_m(R)$ is a J-ring, as required. □

Proposition 3.3 *Let A and B be rings and ${}_AV_B$ a bimodule. Let $R = \begin{pmatrix} A & V \\ 0 & B \end{pmatrix}$. If R is a J-ring, then so are A and B.*

Proof By using the same way as the proof of Proposition 3.2, we complete the proof. □

Proposition 3.4 *Let R be a ring. If $a \in R$ lifts strongly modulo $J(R)$, then the following statements are equivalent:*

(1) a is strongly J-semiclean;

(2) a^k is strongly J-semiclean for any $k \in \mathbb{N}$;

(3) a^k is strongly J-semiclean for some $k \in \mathbb{N}$.

Proof (1)$\Rightarrow$ (2). Suppose that a is strongly J-semiclean. Then there exist a periodic element $\pi \in R$ and an element $w \in J(R)$ such that $a = \pi + w$ and $\pi w = w\pi$. Then for any positive integer $k \in \mathbb{N}$, we have

$$a^k = (\pi + w)^k = \pi^k + C_k^1\pi^{k-1}w + C_k^2\pi^{k-2}w^2 + \cdots + w^k$$

where C_k^i is the binomial coefficient. Since π is a periodic element, there exist two distinct positive integers m and n such that $\pi^m = \pi^n$. Then $(\pi^k)^m = (\pi^k)^n$. So π^k is a periodic element. Clearly, $C_k^1\pi^{k-1}w + C_k^2\pi^{k-2}w^2 + \cdots + w^k \in J(R)$, and $\pi^k(C_k^1\pi^{k-1}w + C_k^2\pi^{k-2}w^2 + \cdots + w^k) = (C_k^1\pi^{k-1}w + C_k^2\pi^{k-2}w^2 + \cdots + w^k)\pi^k$. Therefore, a^k is strongly J-semiclean.

(2)$\Rightarrow$ (3). It is trivial.

(3)$\Rightarrow$ (1). Suppose that a^k is strongly J-semiclean for some $k \in \mathbb{N}$. Then $\overline{a^k} = (\overline{a})^k$ is a periodic element in $R/J(R)$, and so $\overline{a}$ is a periodic element in $R/J(R)$. Hence there exist some $s,\ t \in \mathbb{N}$ such that $a^s - a^t \in J(R)$. Since a lifts strongly modulo $J(R)$, there exist a periodic element $b \in R$ and an element $j \in J(R)$ such that $a = b + j$ and $ab = ba$. So $bj = jb$. Therefore, a is strongly J-semiclean. □

Proposition 3.5 *Let R be a ring. Then the following statements are equivalent:*

(1) R is a strongly J-semiclean ring;

(2) $R/J(R)$ is a periodic ring and R is a J-ring.

Proof (1)$\Rightarrow$ (2). It is trivial.

(2)$\Rightarrow$ (1). Let $x \in R$. Then $\overline{x} \in R/J(R)$ is a periodic element. So there exist some $k,\ l \in \mathbb{N}$ such that $x^k - x^l \in J(R)$. By hypothesis, there exists some periodic element $\pi \in R$ such that $x - \pi \in J(R)$ and $x\pi = \pi x$. Set $w = x - \pi$. Then $x = \pi + w$, $w \in J(R)$ and $\pi w = w\pi$. Hence x is strongly J-semiclean. Therefore, R is a strongly J-semiclean ring. □

Proposition 3.6 *Let R be a strongly J-semiclean ring with 2 being an invertible element. Then every element in R is a sum of two units.*

Proof For any $x \in R$, there exists a periodic element $\pi \in R$ and an element $w \in J(R)$ such that $\frac{x+1}{2} = \pi + w$ and $\pi w = w\pi$. According to [12, Lemma 5.1], there exist $e^2 = e \in R$ and $u \in U(R)$ such that $\pi = e + u$. Hence, $x = 2e - 1 + 2u + 2w$ where $2e - 1$ and $2u + 2w$ are units, as required. □

Let R be a ring and let $a \in R$. Let $\mathrm{ann}_l(a) = \{u \in R \mid ua = 0\}$ and $\mathrm{ann}_r(a) = \{u \in R \mid au = 0\}$.

Proposition 3.7 *Let R be a ring and let $a = \pi + w$ be a strongly J-semiclean decomposition of a in R. Then there exists some positive integer $k \in \mathbb{N}$ such that $\mathrm{ann}_l(a) \subseteq \mathrm{ann}_l(\pi^k)$ and $\mathrm{ann}_r(a) \subseteq \mathrm{ann}_r(\pi^k)$.*

Proof Let $r \in \mathrm{ann}_l(a)$. Then $ra = 0$. Write $a = \pi + w$, where π is a periodic element in R, $w \in J(R)$ and $\pi w = w\pi$. Then $r\pi = -rw$. Since π is a periodic element in R, by [13, Lemma 1], there exists some $k \in \mathbb{N}$ such that π^k is an idempotent. Hence $r\pi^k = r\pi^k\pi^k = -r\pi^k\pi^{k-1}w$.

It follows that $r\pi^k(1+\pi^{k-1}w)=0$ and so $r\pi^k=0$, that is, $r\in \mathrm{ann}_l(\pi^k)$. Therefore, $\mathrm{ann}_l(a)\subseteq \mathrm{ann}_l(\pi^k)$. A similar argument shows that $\mathrm{ann}_r(a)\subseteq \mathrm{ann}_r(\pi^k)$. $\square$

Proposition 3.8 *Let R be a ring, $f\in R$ an idempotent and $a\in fRf$ lifts strongly modulo $J(fRf)$. Then a is strongly J-semiclean in R if and only if a is strongly J-semiclean in fRf.*

Proof Suppose that a is strongly J-semiclean in fRf. Then $a=\pi+w$, where π is a periodic element in fRf and $w\in J(fRf)$ and $\pi w=w\pi$. Obviously, $w\in fJ(R)f\subseteq J(R)$. Hence $a\in fRf$ is strongly J-semiclean in R.

Conversely, assume that $a\in fRf$ is strongly J-semiclean in R. Then $a=\pi+w$, where π is a periodic element in R, $w\in J(R)$ and $\pi w=w\pi$. By [13, Lemma 1], there exists some $k\in\mathbb{N}$ such that $\pi^k=e$ is an idempotent. Thus $a^k=e+v$ where $v=C_k^1\pi^{k-1}w+C_k^2\pi^{k-2}w^2+\cdots+w^k\in J(R)$ and $ev=ve$. So $a^k\in fRf$ is strongly J-clean in R. By [9, Theorem 3.4], we obtain that a^k is strongly J-clean in fRf and so a^k is strongly J-semiclean in fRf. Then by Proposition 3.4, a is strongly J-semiclean in fRf. $\square$

As is well known, every corner of a strongly clean ring is strongly clean [3, Theorem 2.4], and every corner of a strongly J-clean ring is strongly J-clean [9, Corollary 3.5]. Analogously, we can derive the following corollary.

Corollary 3.9 *Let $e\in R$ be an idempotent and let eRe be a J-ring. If R is a strongly J-semiclean ring, then so is eRe.*

Proof Let $a\in eRe$. As R is strongly J-semiclean, we see that $a\in eRe$ is strongly J-semiclean in R. According to Proposition 3.8, $a\in eRe$ is strongly J-semiclean in eRe. Therefore, eRe is a strongly J-semiclean ring. $\square$

4. Triangular matrix ring

The main purpose of this section is to investigate the strongly J-semicleanness of $T_n(R)$. These results also hold for the ring of all lower triangular matrices by a similar route.

Let R be a ring, and let

$$S_n(R)=\left\{\begin{pmatrix} a & a_{12} & \cdots & a_{1n}\\ 0 & a & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & a\end{pmatrix} \mid a, a_{ij}\in R\right\},$$

$$U_n(R)=\left\{\begin{pmatrix} a_1 & a_2 & \cdots & a_n\\ 0 & a_1 & \cdots & a_{n-1}\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & a_1\end{pmatrix} \mid a_i\in R, 1\le i\le n\right\}.$$

Then $S_n(R)$ and $U_n(R)$ are subrings of $T_n(R)$ under usual matrix operations.

Proposition 4.1 *Let R be a commutative ring. Then the following statements are equivalent:*

(1) R is a strongly J-semiclean ring;

(2) $S_n(R)$ is a strongly J-semiclean ring;

(3) $U_n(R)$ is a strongly J-semiclean ring.

Proof (1)⇒(2). Suppose that R is a strongly J-semiclean ring. Then for any

$$A=\begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a \end{pmatrix}\in S_n(R),$$

there exist a periodic element $\pi\in R$ and an element $w\in J(R)$ such that $a=\pi+w$. Let

$$P=\begin{pmatrix} \pi & 0 & \cdots & 0 \\ 0 & \pi & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \pi \end{pmatrix},\quad W=\begin{pmatrix} w & a_{12} & \cdots & a_{1n} \\ 0 & w & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & w \end{pmatrix}.$$

Then P is a periodic element in $S_n(R)$ and $W\in J(S_n(R))$. By the condition that R is a commutative ring, we obtain that $PW=WP$. So $A=P+W$ is a strongly J-semiclean decomposition of A. Hence A is strongly J-semiclean in $S_n(R)$. Therefore, $S_n(R)$ is a strongly J-semiclean ring.

(2)⇒ (1). Assume that $S_n(R)$ is a strongly J-semiclean ring. Let

$$Q=\left\{\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\mid a_{ij}\in R\right\}.$$

Then Q is an ideal of $S_n(R)$ and $R\cong S_n(R)/Q$. So by Example 2.9, R is strongly J-semiclean.

(1)⇔ (3). The proof is similar to that of (1)⇔ (2). □

Based on Proposition 4.1, we derive the following corollary.

Corollary 4.2 *Let R be a commutative ring. Then the following statements are equivalent:*

(1) R is a strongly J-semiclean ring;

(2) The trivial extension $R\bowtie R$ of R by R is a strongly J-semiclean ring.

Let R be a ring and let

$$W(R)=\left\{\begin{pmatrix} a & 0 & 0 \\ a_{21} & a & a_{23} \\ 0 & 0 & a \end{pmatrix}\mid a, a_{ij}\in R\right\}.$$

Then $W(R)$ is a ring under usual matrix operations. By using the same way as the proof of Proposition 4.1, we obtain the following proposition.

Proposition 4.3 *Let R be a commutative ring. Then the following statements are equivalent:*

(1) R is a strongly J-semiclean ring;

(2) $W(R)$ is a strongly J-semiclean ring.

Proposition 4.4 *Let R be a ring. Then the following statements are equivalent:*

(1) $T_n(R)$ is a strongly J-semiclean ring;

(2) $T_m(R)$ is a strongly J-semiclean ring for all $1 \leq m \leq n$.

Proof (1)⇒ (2). Suppose that $T_n(R)$ is a strongly J-semiclean ring. Then by Proposition 3.5, $T_n(R)$ is a J-ring. In view of Proposition 3.2, $T_m(R)$ is a J-ring for all $1 \leq m \leq n$. Let $e = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{m}, 0, \ldots, 0) \in T_n(R)$. Then $T_m(R) \cong eT_n(R)e$. It follows from Corollary 3.9 that $T_m(R)$ is strongly J-semiclean for all $1 \leq m \leq n$.

(2)⇒ (1). It is trivial. □

Corollary 4.5 *Let A and B be rings and ${}_AV_B$ a bimodule. Let $R = \left(\begin{smallmatrix} A & V \\ 0 & B \end{smallmatrix}\right)$. If R is a strongly J-semiclean ring, then so are A and B.*

Proof By using the same way as the the proof of Proposition 4.4, we complete the proof. □

Let $a \in R$, $l_a : R \longrightarrow R$ and $r_a : R \longrightarrow R$ denote, respectively, the abelian group endomorphisms given by $l_a(r) = ar$ and $r_a(r) = ra$ for all $r \in R$. Thus $l_a - r_b$ is an abelian group endomorphism such that $(l_a - r_b)(r) = ar - rb$ for any $r \in R$. Following Diesl [4], a local ring R is bleached provided that for any $a \in U(R)$, $b \in J(R)$, $l_a - r_b$, and $l_b - r_a$ are both surjective.

Lemma 4.6 *Let R be a local ring, and suppose that $A \in T_n(R)$. Then for any set $\{e_{ii}\}$ of idempotents in R such that $e_{ii} = e_{jj}$ whenever $l_{A_{ii}} - r_{A_{jj}}$ is not a surjective abelian group endomorphism of R, there exists an idempotent $E \in T_n(R)$ such that $AE = EA$ and $E_{ii} = e_{ii}$ for any $i \in \{1, 2, \ldots, n\}$.*

Proof See [1, Lemma 7]. □

Proposition 4.7 *Let R be a local ring, and let $n \geq 2$. Then the following statements are equivalent:*

(1) $T_n(R)$ is strongly J-semiclean;

(2) $T_n(R)$ is a J-ring, R is bleached and $R/J(R)$ is a periodic ring.

Proof (1)⇒ (2). Suppose that $T_n(R)$ is strongly J-semiclean. In view of Proposition 3.5, $T_n(R)$ is a J-ring. According to Proposition 4.4, R is a strongly J-semiclean ring. Then it follows from Proposition 3.5, $R/J(R)$ is a periodic ring. Now we show that R is bleached. In view of Proposition 4.4, $T_2(R)$ is strongly J-semiclean. Let $a \in U(R)$ and $b \in J(R)$. We will show that $l_a - r_b : R \longrightarrow R$ is surjective. For any $v \in R$, it suffices to find some $x \in R$ such that $ax - xb = v$. Let $r = \left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right)$. Since $T_2(R)$ is strongly J-semiclean, there exist a periodic element $\pi = \left(\begin{smallmatrix} e & g \\ 0 & f \end{smallmatrix}\right) \in T_2(R)$, and an element $w \in J(T_2(R))$ such that $r = \pi + w$ and $\pi w = w\pi$. So $r\pi = \pi r$. Since R is a local ring, for periodic elements e, $f \in R$, there exists some $l \in \mathbb{N}$ such that $e^l = 1$ or $e^l = 0$ and $f^l = 1$ or $f^l = 0$. As $J(R)$ is a maximal ideal of R, $R/J(R)$ is a periodic ring implies that $R/J(R)$ is a periodic field. So for $a \in U(R)$, there exists some $m \in \mathbb{N}$

such that $a^m \in 1+J(R)$. Thus we can find some $k \in \mathbb{N}$ such that $r^k \in \left(\begin{smallmatrix} 1+J(R) & R \\ 0 & J(R) \end{smallmatrix}\right)$ and π^k is one of the following:

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

Since $r^k - \pi^k = (C_k^1 \pi^{k-1} + C_k^2 \pi^{k-2} w + \cdots + w^{k-1}) w \in J(T_2(R))$, we have $\pi^k = \left(\begin{smallmatrix} 1 & x \\ 0 & 0 \end{smallmatrix}\right)$, otherwise $r^k - \pi^k \notin J(T_2(R))$. It follows from $r\pi = \pi r$ that $r\pi^k = \pi^k r$. Thus we deduce that $ax - xb = v$. Thus $l_a - r_b : R \longrightarrow R$ is surjective. Analogously, we show that $l_b - r_a : R \longrightarrow R$ is surjective. Therefore, R is bleached.

(2)$\Rightarrow$ (1). Let $A = (a_{ij}) \in T_n(R)$. As R is a local ring and $R/J(R)$ is a periodic ring, there exists some $k \in \mathbb{N}$ such that $\{a_{ii}^k \mid i = 1, 2, \ldots, n\} \subseteq J(R) \cup (1 + J(R))$. In order to show that A^k is strongly J-semiclean, it suffices to construct a periodic element $E \in T_n(R)$ such that $EA^k = A^k E$ and such that $A^k - E \in J(T_n(R))$. Begin by constructing the main diagonal of E. Set $e_{ii} = 0$ if $a_{ii}^k \in J(R)$, and set $e_{ii} = 1$ otherwise. Thus $a_{ii}^k - e_{ii} \in J(R)$ for every i. If $e_{ii} \neq e_{jj}$, then it must be the case (without loss of generality) that $a_{ii}^k \in U(R)$ and $a_{jj}^k \in J(R)$. As R is bleached, $l_{a_{ii}^k} - r_{a_{jj}^k} : R \longrightarrow R$ is surjective. According to Lemma 4.6, there exists an idempotent $E \in T_n(R)$ such that $A^k E = EA^k$ and $E_{ii} = e_{ii}$ for every $i \in \{1, 2, \ldots, n\}$. In addition, $A^k - E \in J(T_n(R))$. Hence A^k is strongly J-clean and so A^k is strongly J-semiclean. Since $T_n(R)$ is a J-ring, by Proposition 3.4, A is a strongly J-semiclean element. Therefore, $T_n(R)$ is a strongly J-semiclean ring. $\square$

Corollary 4.8 *Let R be a commutative local ring and let $n \geq 2$. Then the following statements are equivalent:*

(1) $T_n(R)$ is strongly J-semiclean;

(2) $R/J(R)$ is a periodic ring and $T_n(R)$ is a J-ring.

Proof (1)$\Rightarrow$ (2) is obvious from Proposition 4.7.

(2)$\Rightarrow$ (1). As R is a commutative local ring, it is bleached. Therefore, the result follows from Proposition 4.7. $\square$

Corollary 4.9 *Let R be a local ring. Then $T_2(R)$ is strongly J-semiclean if and only if:*

(1) $R/J(R)$ is a periodic ring and $T_2(R)$ is a J-ring,

(2) For any $a \in U(R)$, $b \in J(R)$ and $v \in R$, there exists $P, Q \in U(T_2(R))$ such that $P^{-1}\left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right)P = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ and $Q^{-1}\left(\begin{smallmatrix} b & v \\ 0 & a \end{smallmatrix}\right)Q = \left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right)$.

Proof Suppose that $T_2(R)$ is strongly J-semiclean. By virtue of Proposition 4.7, $T_2(R)$ is a J-ring, R is bleached and $R/J(R)$ is a periodic ring. Then by using the same way as the proof of Corollary 4.6 in [9], it is easy to see that there exist $P, Q \in U(T_2(R))$ such that $P^{-1}\left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right)P = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ and $Q^{-1}\left(\begin{smallmatrix} b & v \\ 0 & a \end{smallmatrix}\right)Q = \left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right)$.

Conversely, assume that (1) and (2) hold. Let $A = \left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right) \in T_2(R)$.

Case I. Both a and b are in $J(R)$. Then $\left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right) \in J(T_2(R))$ and so $A = \left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right)$ is strongly J-semiclean.

Case II. Both a and b are in $U(R)$. Since R is a local ring and $R/J(R)$ is a periodic ring, we can find some $k \in \mathbb{N}$ such that $a^k \in 1 + J(R)$ and $b^k \in 1 + J(R)$. Then it is easy to see that $A^k = \left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right)^k$ is strongly J-semiclean. As $T_2(R)$ is a J-ring, by Proposition 3.4, $A = \left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right)$ is strongly J-semiclean.

Case III. $a \in U(R)$ and $b \in J(R)$. Then there exists some $P \in U(T_2(R))$ such that $P^{-1}\left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right)P = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$. As $T_2(R)$ is a J-ring, by Proposition 3.2, R is a J-ring. Then by Proposition 3.5, R is a strongly J-semiclean ring. Thus we have a periodic element $\pi \in R$ and an element $w \in J(R)$ such that $a = \pi + w$ and $\pi w = w\pi$. Then $P^{-1}\left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right)P = \left(\begin{smallmatrix} \pi & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} w & 0 \\ 0 & b \end{smallmatrix}\right)$, and so $\left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right) = P\left(\begin{smallmatrix} \pi & 0 \\ 0 & 0 \end{smallmatrix}\right)P^{-1} + P\left(\begin{smallmatrix} w & 0 \\ 0 & b \end{smallmatrix}\right)P^{-1}$. Clearly, $P\left(\begin{smallmatrix} \pi & 0 \\ 0 & 0 \end{smallmatrix}\right)P^{-1}$ is a periodic element in $T_2(R)$, $P\left(\begin{smallmatrix} w & 0 \\ 0 & b \end{smallmatrix}\right)P^{-1} \in J(T_2(R))$ and

$$P\begin{pmatrix} \pi & 0 \\ 0 & 0 \end{pmatrix}P^{-1}P\begin{pmatrix} w & 0 \\ 0 & b \end{pmatrix}P^{-1} = P\begin{pmatrix} w & 0 \\ 0 & b \end{pmatrix}P^{-1}P\begin{pmatrix} \pi & 0 \\ 0 & 0 \end{pmatrix}P^{-1}.$$

Thus $\left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right)$ is a strongly J-semiclean element in $T_2(R)$.

Case IV. $a \in J(R)$ and $b \in U(R)$. Then there exists some $Q \in U(T_2(R))$ such that $Q^{-1}\left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right)Q=\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$. By using the same way as the proof of Case III, we can show that $\left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right)$ is strongly J-semiclean.

In any case, we conclude that $A = \left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right)$ is a strongly J-clean element. Therefore, $T_2(R)$ is a strongly J-semiclean ring. $\square$

Proposition 4.10 *Let A and B be local rings and ${}_AV_B$ a bimodule. Let $R = \left(\begin{smallmatrix} A & V \\ 0 & B \end{smallmatrix}\right)$. Then $\pounds = \left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right) \in R$ is strongly J-semiclean if and only if :*

(1) $\pounds$ lifts strongly modulo $J(R)$;

(2) There exist some s, $t \in \mathbb{N}$ such that $a^s \in J(A)$ and $b^t \in J(B)$, or

(3) There exist some u, $v \in \mathbb{N}$ and some invertible periodic elements $p \in A$ and $q \in B$ such that $a^u \in p + J(A)$ and $b^v \in q + J(B)$, or

(4) There exist $x \in V$, $k \in \mathbb{N}$ and an invertible periodic element $c \in A$ such that $c(a^{k-1}v + a^{k-2}vb + \cdots + vb^{k-1}) = a^kx - xb^k$ and that $ca - a^{k+1} \in J(A)$, $b \in J(B), a^kc = ca^k$, or

(5) There exist $y \in V$, $l \in \mathbb{N}$ and an invertible periodic element $d \in A$ such that $(a^{l-1}v + a^{l-2}vb + \cdots + vb^{l-1})d = yb^l - a^ly$ and that $bd - b^{l+1} \in J(B)$, $a \in J(A), b^ld = db^l$.

Proof $(\Rightarrow)$. Suppose that $\pounds$ is strongly J-semiclean. Then $\pounds$ lifts strongly modulo $J(R)$. If there exists some $m \in \mathbb{N}$ such that $\pounds^m \in J(R)$, then $a^m \in J(A)$, $b^m \in J(B)$ and so the condition (2) holds. If there exist some invertible periodic elements $p \in A$ and $q \in B$, and a positive integer $n \in \mathbb{N}$ such that $\left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right) - \left(\begin{smallmatrix} a & v \\ 0 & b \end{smallmatrix}\right)^n \in J(R)$, then $a^n \in p + J(A)$, $b^n \in q + J(B)$, and so the condition (3) holds. Now we assume that for any $m \in \mathbb{N}$, $\pounds^m \notin J(R)$, and for any invertible periodic elements $p \in A$, $q \in B$, any $n \in \mathbb{N}$, $\left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right) - \pounds^n \notin J(R)$. Since $\pounds$ is strongly J-semiclean, there exists some periodic element $E = \left(\begin{smallmatrix} e & w \\ 0 & f \end{smallmatrix}\right) \in R$ such that $\pounds - E \in J(R)$ and $\pounds E = E\pounds$. Since E is a periodic element in R, we obtain that e and f are periodic elements in A and B, respectively. If both e and f are invertible periodic elements, then $\left(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right) - \pounds \in J(R)$, a contradiction. If both e and f are nilpotent elements, the $E \in J(R)$ and so $\pounds \in J(R)$, a contradiction. So e is an invertible periodic element in A and f is a nilpotent element in B, or e is a nilpotent element in A and f is an invertible element in B. Firstly, we assume that e is an invertible periodic element in A and f is a nilpotent element in B. It follows from $\pounds - E \in J(R)$ that $b \in J(B)$. Let $k \in \mathbb{N}$ be such that $f^k = 0$. Then $E^k = \begin{pmatrix} e^k & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} c & x \\ 0 & 0 \end{pmatrix}$ where $c = e^k$ is an invertible periodic element in A and $x \in V$. It follows from $\pounds E = E\pounds$ that $\pounds^k E^k = E^k \pounds^k$. So $\begin{pmatrix} a^k & a^{k-1}v+a^{k-2}vb+\cdots+vb^{k-1} \\ 0 & b^k \end{pmatrix}\begin{pmatrix} c & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} c & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a^k & a^{k-1}v+a^{k-2}vb+\cdots+vb^{k-1} \\ 0 & b^k \end{pmatrix}$. Thus $c(a^{k-1}v + a^{k-2}vb + \cdots + vb^{k-1}) = a^k x - xb^k$ and $a^k c = ca^k$. Since $\pounds - E \in J(R)$ and $\pounds E = E\pounds$, there exists $U \in J(R)$ such that $\pounds = E + U$ and $EU = UE$. Then $\pounds^k = E^k + U'$ where $U' = C_k^1 E^{k-1}U + C_k^2 E^{k-2}U^2 + \cdots + U^k \in J(R)$. Hence $e^k - a^k = c - a^k \in J(A)$ and so $ca - a^{k+1} \in J(A)$. Secondly, assume that e is a nilpotent element in A and f is an invertible element in B. Then similarly, we show that there exist $y \in V$, $l \in \mathbb{N}$ and an invertible periodic element $d \in A$ such that $(a^{l-1}v + a^{l-2}vb + \cdots + vb^{l-1})d = yb^l - a^l y$ and that $bd - b^{l+1} \in J(B)$, $a \in J(A)$, $b^l d = db^l$.

($\Leftarrow$). Suppose that $\pounds = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in R$ lifts strongly modulo $J(R)$.

Case I. There exist some $s, t \in \mathbb{N}$ such that $a^s \in J(A)$ and $b^t \in J(B)$. Then $\pounds^{st} = \begin{pmatrix} a^{st} & x \\ 0 & b^{st} \end{pmatrix} \in J(R)$ where $x \in V$. Then $\pounds^{st}$ is strongly J-semiclean and so $\pounds$ is strongly J-semiclean by Proposition 3.4.

Case II. There exist some u, $v \in \mathbb{N}$ and some invertible periodic elements $p \in A$ and $q \in B$ such that $a^u \in p + J(A)$ and $b^v \in q + J(B)$. Let $k \in \mathbb{N}$ be such that $p^k = 1_A$ and $q^k = 1_B$. Then $\pounds^{uvk} = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}^{uvk} = \begin{pmatrix} a^{uvk} & x \\ 0 & b^{uvk} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + W'$ where $W' \in J(R)$ and $x \in V$. Hence $\pounds^{uvk}$ is strongly J-semiclean and so $\pounds$ is strongly J-semiclean by Proposition 3.4.

Case III. There exist $x \in V$, $k \in \mathbb{N}$ and an invertible periodic element $c \in A$ such that $c(a^{k-1}v + a^{k-2}vb + \cdots + vb^{k-1}) = a^k x - xb^k$ and that $ca - a^{k+1} \in J(A)$, $b \in J(B)$ and $a^k c = ca^k$. Since $J(A)$ is a maximal ideal of A, $a \in J(A)$ or $c - a^k \in J(A)$. If $a \in J(A)$, then $\pounds \in J(R)$ and so $\pounds$ is strongly J-semiclean. Now assume that $c - a^k \in J(A)$, $b \in J(B)$, $a^k c = ca^k$ and $c(a^{k-1}v + a^{k-2}vb + \cdots + vb^{k-1}) = a^k x - xb^k$. Choose $E = \begin{pmatrix} c & x \\ 0 & 0 \end{pmatrix}$. Since c is an invertible periodic element, there exists $l \in \mathbb{N}$ such that $c^l = 1_A$ and so $E^{2l} = E^l$. Hence E is a periodic matrix in R. Since $\pounds^k - E \in J(R)$ and $\pounds^k E = E\pounds^k$, we obtain that $\pounds^k$ is strongly J-semiclean, and so $\pounds$ is strongly J-semiclean.

Case IV. There exist $y \in V$, $l \in \mathbb{N}$ and an invertible periodic element $d \in A$ such that $(a^{l-1}v + a^{l-2}vb + \cdots + vb^{l-1})d = y^l b - a^l y$ and that $bd - b^{l+1} \in J(B)$, $a \in J(A)$, $db^l = b^l d$. By using the same way as the proof of Case III, we can show that $\pounds$ is strongly J-semiclean.

Therefore in any case, we conclude that $\pounds \in R$ is strongly J-semiclean. $\square$

Proposition 4.11 *Let A and B be local rings and ${}_AV_B$ a bimodule. Let $R = \begin{pmatrix} A & V \\ 0 & B \end{pmatrix}$. Then R is strongly J-semiclean if and only if:*

(1) R is a J-ring;

(2) Both $A/J(A)$ and $B/J(B)$ are periodic rings;

(3) If $a \in 1_A + J(A)$, $b \in J(B)$, and $v \in V$, there exists $x \in V$ such that $v = ax - xb$.

Proof ($\Rightarrow$). Suppose that R is strongly J-semiclean. Then by Proposition 3.5, R is a J-

ring. In view of Corollary 4.5, A and B are strongly J-semiclean rings. Then by Proposition 3.5, both $A/J(A)$ and $B/J(B)$ are periodic rings. Suppose that $a \in 1_A + J(A)$, $b \in J(B)$, and $v \in V$. Then $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in R$ is a strongly J-semiclean element in R. So there exists a periodic element $E = \begin{pmatrix} e & w \\ 0 & f \end{pmatrix} \in R$ such that $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} - E \in J(R)$ and $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} E = E \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$. Since E is a periodic element in R, we obtain that both e and f are periodic elements. If f is an invertible periodic element in B, then $b - f$ is an invertible element in B and so $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} - E \notin J(R)$, a contradiction. If e is a nilpotent element in A, then $a - e$ is an invertible element in A and so $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} - E \notin J(R)$, a contradiction. Hence e is an invertible periodic element in A and f is a nilpotent element in B. Let $k \in \mathbb{N}$ be such that $e^k = 1_A$ and $f^k = 0$. Then $E^k = \begin{pmatrix} 1_A & x \\ 0 & 0 \end{pmatrix}$ where $x = e^{k-1}w + e^{k-2}wf + \cdots + wf^{k-1} \in V$. It follows from $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} E = E \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$ that $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} E^k = E^k \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$. Hence $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}\begin{pmatrix} 1_A & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1_A & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$. Thus $v = ax - xb$.

($\Leftarrow$). Suppose that the conditions (1), (2) and (3) are satisfied. Let $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in R$. Since R is a J-ring, by Proposition 3.3, A and B are J-rings. As both $A/J(A)$ and $B/J(B)$ are periodic rings, we obtain that $a \in J(A)$ or $a \in p + J(A)$, that $b \in J(B)$ or $b \in q + J(B)$, where p is an invertible periodic element in A and q is an invertible periodic element in B.

Case I. $a \in J(A)$ and $b \in J(B)$. Then $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in J(R)$, and so $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in R$ is strongly J-semiclean.

Case II. $a \in p + J(A)$, $b \in q + J(B)$, where p and q are invertible periodic elements. Let $s \in \mathbb{N}$ be such that $p^s = 1_A$ and $q^s = 1_B$. Then $a^s \in 1_A + J(A)$ and $q^s \in 1_B + J(B)$ and so $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}^s = \begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix} + \begin{pmatrix} a^s - 1_A & a^{s-1}v + a^{s-2}vb + \cdots + vb^{s-1} \\ 0 & b^s - 1_B \end{pmatrix}$, where $\begin{pmatrix} a^s - 1_A & a^{s-1}v + a^{s-2}vb + \cdots + vb^{s-1} \\ 0 & b^s - 1_B \end{pmatrix} \in J(R)$. Hence $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}^s$ is strongly J-semiclean and so $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$ is strongly J-semiclean.

Case III. $a \in p + J(A)$, $b \in J(B)$ where p is an invertible periodic element. Let $s \in \mathbb{N}$ be such that $p^s = 1_A$. Then $a^s \in 1_A + J(A)$, $b^s \in J(B)$. Consider $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}^s = \begin{pmatrix} a^s & v' \\ 0 & b^s \end{pmatrix} \in R$ where $v' = a^{s-1}v + a^{s-2}vb + \cdots + vb^{s-1} \in V$. By hypothesis there exists some $x \in V$ such that $v' = a^s x - xb^s$. Let $E = \begin{pmatrix} 1_A & x \\ 0 & 0 \end{pmatrix}$. Then E is a periodic element in R, $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}^s - E \in J(R)$ and $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}^s E = E \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}^s$. Hence $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}^s$ is strongly J-semiclean and so $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$ is strongly J-semiclean.

Case IV. $a \in J(A)$, $b \in q + J(B)$, where q is an invertible periodic element in B. Let $l \in \mathbb{N}$ be such that $q^l = 1_B$. Then $a^l \in J(A)$, $b^l \in 1_B + J(B)$, and so $1_A - a^l \in 1_A + J(A)$ and $1_B - b^l \in J(B)$. Consider $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}^l = \begin{pmatrix} a^l & u \\ 0 & b^l \end{pmatrix} \in R$ where $u = a^{l-1}v + a^{l-2}vb + \cdots + vb^{l-1} \in V$. By the hypothesis there exists $x \in V$ such that $u = (1_A - a^l)x - x(1_B - b^l)$, i.e., $u = a^l(-x) - (-x)b^l$. Let $E = \begin{pmatrix} 0 & -x \\ 0 & 1_B \end{pmatrix} \in R$. Then E is a periodic element in R, $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}^l - E \in J(R)$ and $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}^l E = E \begin{pmatrix} a & v \\ 0 & b \end{pmatrix}^l$. Hence $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}^l$ is strongly J-semiclean and so $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$ is strongly J-semiclean. Therefore, R is a strongly J-semiclean ring. $\square$

Let R be a local ring. As an immediate consequence, we deduce that $T_2(R)$ is strongly J-semiclean if and only if R is a J-ring, $R/J(R)$ is a periodic ring and R is bleached.

Corollary 4.12 *Let A and B be local rings and ${}_AV_B$ a bimodule. Let $R = \begin{pmatrix} A & V \\ 0 & B \end{pmatrix}$. Then R is strongly J-semiclean if and only if:*

(1) R is a J-ring;

(2) Both $A/J(A)$ and $B/J(B)$ are periodic rings;

(3) R is strongly clean.

Proof According to [7, Example 2], R is strongly clean if and only if whenever $a-1_A\in J(A)$, $b\in J(B)$, and $v\in V$, there exists $x\in V$ such that $v=ax-xb$. Therefore we complete the proof by Proposition 4.11. $\square$

5. 2×2 matrix rings

The main purpose of this section is to investigate the strong J-semicleanness of a single 2×2 matrix over a commutative local rings.

Lemma 5.1 *Let R be a commutative local ring, and let $E\in M_2(R)$ be a periodic matrix. Then there exists some $k\in\mathbb{N}$ such that E^k is similar to a periodic diagonal matrix $\left(\begin{smallmatrix} f_1 & 0\\ 0 & f_2\end{smallmatrix}\right)$, where f_1 and f_2 are periodic elements in R.*

Proof Since E is a periodic matrix, by [13, Lemma 1], there exists some $k\in\mathbb{N}$ such that E^k is an idempoten [9, Lemma 5.1], we complete the proof. $\square$

Proposition 5.2 *Let R be a commutative local ring. Then $A\in M_2(R)$ is strongly J-semiclean if and only if A lifts strongly modulo $J(M_2(R))$ and there exist some $k,\ l,\ m\in\mathbb{N}$ and some invertible periodic element p and q in R such that $A^k\in J(M_2(R))$, or $pI_2-A^l\in J(M_2(R))$, or A^m is similar to a matrix $\left(\begin{smallmatrix} q+w_1 & 0\\ 0 & w_2\end{smallmatrix}\right)$, where I_2 is the identity matrix of $M_2(R)$ and $w_1,\ w_2\in J(R)$.*

Proof If either $A^k\in J(M_2(R))$ or $pI_2-A^l\in J(M_2(R))$ for some $k,\ l\in\mathbb{N}$ and some invertible periodic element $p\in R$, then A^k or A^l is strongly J-semiclean, and so A is strongly J-semiclean by Proposition 3.4. For $w_1,\ w_2\in J(R)$, $\left(\begin{smallmatrix} q+w_1 & 0\\ 0 & w_2\end{smallmatrix}\right)=\left(\begin{smallmatrix} q & 0\\ 0 & 0\end{smallmatrix}\right)+\left(\begin{smallmatrix} w_1 & 0\\ 0 & w_2\end{smallmatrix}\right)$ is strongly J-semiclean. Thus A^m is strongly J-semiclean and so A is strongly J-semiclean. Hence, one direction is obvious.

Conversely, assume that $A\in M_2(R)$ is strongly J-semiclean. Then A lifts strongly modulo $J(M_2(R))$ and there exist a periodic matrix $E\in M_2(R)$ and a $W\in J(M_2(R))$ such that $A=E+W$ with $EW=WE$. Suppose that for any $k,\ l\in\mathbb{N}$ and any invertible periodic element $p\in R$, A^k and pI_2-A^l are not in $J(M_2(R))$. According to Lemma 5.1, there exist some $t\in\mathbb{N}$ and $J\in GL_2(R)$ such that $JE^tJ^{-1}=\left(\begin{smallmatrix} f_1 & 0\\ 0 & f_2\end{smallmatrix}\right)$, where $f_1,\ f_2$ are periodic elements. If both $f_1,\ f_2$ are nilpotent periodic elements, then there exists $s\in\mathbb{N}$ such that $f_1^s=f_2^s=0$. Then $JE^{st}J^{-1}=0$. It follows from $A=E+W$ and $EW=WE$ that

$$JA^{st}J^{-1}=J(C_{st}^1E^{st-1}W+C_{st}^2E^{st-2}W^2+\cdots+W^{st})J^{-1}\in J(M_2(R)).$$

Hence $A^{st}\in J(M_2(R))$. This contradicts the hypothesis that for any $k\in\mathbb{N}$, A^k is not in $J(M_2(R))$. If both f_1 and f_2 are invertible periodic elements, then there exists some $u\in\mathbb{N}$ such that $f_1^u=f_2^u=1$ and so $JE^{ut}J^{-1}=I_2$. Thus $E^{ut}=I_2$ and so $I_2-A^{ut}=C_{ut}^1E^{ut-1}W+C_{ut}^2E^{ut-2}W^2+\cdots+W^{ut}\in J(M_2(R))$. This contradicts the fact that for any $l\in\mathbb{N}$ and any invertible periodic element $p\in R$, $pI_2-A^l\notin J(M_2(R))$. Thus f_1 is an invertible periodic element, f_2 is a nilpotent periodic element, or f_1 is a nilpotent periodic element, f_2 is an invertible periodic element. So there exist some $v,\ v'\in\mathbb{N}$ such that $JE^{tv}J^{-1}=\left(\begin{smallmatrix} f_1^v & 0\\ 0 & 0\end{smallmatrix}\right)$ or $JE^{tv}J^{-1}=\left(\begin{smallmatrix} 0 & 0\\ 0 & f_2^{v'}\end{smallmatrix}\right)$, where

f_1^v and $f_2^{v'}$ are invertible periodic elements. Therefore there exist some $m \in \mathbb{N}$, $H \in GL_2(R)$ and some invertible periodic element $q \in R$ such that $HA^mH^{-1} = \left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right) + HW'H^{-1}$, where $W' = C_m^1 E^{m-1}W + C_m^2 E^{m-2}W^2 + \cdots + W^m \in J(M_2(R))$. Set $V = (v_{ij}) = HW'H^{-1} \in J(M_2(R))$. It follows from $EW = WE$ that $\left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right)V = V\left(\begin{smallmatrix} q & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Hence $v_{12} = v_{21} = 0$ and v_{11}, $v_{22} \in J(R)$. Therefore A^m is similar to a matrix $\left(\begin{smallmatrix} q+w_1 & 0 \\ 0 & w_2 \end{smallmatrix}\right)$ where w_1, $w_2 \in J(R)$, q is an invertible periodic element. $\square$

Let R be a commutative local ring. Then $M_2(R)$ is not strongly J-clean by [9, Corollary 5.4]. But the following example shows that this is not true for strongly J-semiclean rings.

Example 5.3 Let $\mathbb{Z}_4$ denote the ring of integers modulo 4. Then $\mathbb{Z}_4$ is a commutative local ring. Since $M_2(\mathbb{Z}_4)$ is a finite ring, each matrix in $M_2(\mathbb{Z}_4)$ is a periodic matrix. Hence $M_2(\mathbb{Z}_4)$ is a strongly J-semiclean ring.

Based on Example 5.3, we obtain the following proposition.

Proposition 5.4 *If R is a finite ring, then $M_n(R)$ is a strongly J-semiclean ring.*

Lemma 5.5 *Let R be a commutative local ring, and let $A \in M_2(R)$ be strongly J-semiclean. Then there exist some k, l, $m \in \mathbb{N}$ and some invertible periodic element $p \in R$ such that $A^k \in J(M_2(R))$, or $pI_2 - A^l \in J(M_2(R))$, or A^m is similar to a matrix $\left(\begin{smallmatrix} 0 & \lambda \\ 1 & \mu \end{smallmatrix}\right)$, where $\lambda \in J(R)$, $\mu \in 1 + J(R)$.*

Proof If for any k, $l \in \mathbb{N}$ and any invertible periodic element $p \in R$, $A^k \notin J(M_2(R))$ and $pI_2 - A^l \notin J(M_2(R))$, then it follows from Proposition 5.2 that there exist some $s \in \mathbb{N}$, some $P \in GL_2(R)$ and some invertible periodic element $q \in R$ such that $P^{-1}A^sP = \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)$ where $u \in q + J(R)$, $v \in J(R)$. Let $t \in \mathbb{N}$ be such that $q^t = 1$. Then $u^t \in 1 + J(R)$, and so $P^{-1}A^{st}P = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$ where $\alpha = u^t \in 1 + J(R)$, $\beta = v^t \in J(R)$. So $\alpha - \beta$ is an invertible element in R. Note that $\left(\begin{smallmatrix} -\beta & -\alpha \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} (\alpha-\beta)^{-1} & \alpha(\alpha-\beta)^{-1} \\ -(\alpha-\beta)^{-1} & -\beta(\alpha-\beta)^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and

$$\begin{pmatrix} -\beta & -\alpha \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} (\alpha-\beta)^{-1} & \alpha(\alpha-\beta)^{-1} \\ -(\alpha-\beta)^{-1} & -\beta(\alpha-\beta)^{-1} \end{pmatrix} = \begin{pmatrix} 0 & -\alpha\beta \\ 1 & \alpha+\beta \end{pmatrix}.$$

Let $m = st$. Then A^m is similar to $\left(\begin{smallmatrix} 0 & \lambda \\ 1 & \mu \end{smallmatrix}\right)$, where $\lambda = -\alpha\beta \in J(R)$, $\mu = \alpha + \beta \in 1 + J(R)$. $\square$

Let R be a commutative ring with an identity, and let $A = (a_{ij}) \in M_2(R)$. Denote $Tr(A) = a_{11} + a_{22}$ and $det(A) = a_{11}a_{22} - a_{12}a_{21}$. The following proposition shows that the strong J-semicleanness of a 2×2 matrix over a commutative local ring can be characterize by a kind of quadratic equation.

Proposition 5.6 *Let R be a commutative local ring. Then the following statements are equivalent:*

(1) $A \in M_2(R)$ is strongly J-semiclean.

(2) A lifts strongly modulo $J(M_2(R))$ and there exist some k, l, $m \in \mathbb{N}$ and some invertible periodic element $p \in R$ such that $A^k \in J(M_2(R))$, or $pI_2 - A^l \in J(M_2(R))$, or the equation $x^2 - TrA^mx + detA^m = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.

Proof (1)$\Rightarrow$ (2). Let $A \in M_2(R)$ be strongly J-semiclean. Then A lifts strongly modulo $J(M_2(R))$. Assume that for any k, $l \in \mathbb{N}$ and any invertible periodic element $p \in R$, $A^k \notin J(M_2(R))$ and $pI_2 - A^l \notin J(M_2(R))$. In view of Lemma 5.5, there exists some $m \in \mathbb{N}$ such that A^m is similar to a matrix $B = \left(\begin{smallmatrix} 0 & \lambda \\ 1 & \mu \end{smallmatrix}\right)$, where $\lambda \in J(R)$, $\mu \in 1 + J(R)$. By using the same way as the proof of [9, Theorem 6.3], we obtain that the equation $x^2 - TrA^m x + detA^m = 0$ has a root in $J(R)$ and a root in $1 + J(R)$.

(2)$\Rightarrow$ (1). Suppose that $A \in M_2(R)$ lifts strongly modulo $J(R)$. If there exists some k, $l \in \mathbb{N}$ and some invertible periodic element $p \in R$ such that $A^k \in J(M_2(R))$ or $pI_2 - A^l \in J(M_2(R))$, then A^k or A^l are strongly J-semiclean and so A is strongly J-semiclean. Otherwise, we obtain that $A^m \notin J(M_2(R))$ and $I_2 - A^m \notin J(M_2(R))$ for all $m \in \mathbb{N}$. Then by [9, Theorem 6.3], we obtain that A^m is strongly J-clean and so A^m is strongly J-semiclean. In view of Proposition 3.4, A is strongly J-semiclean, as desired. $\square$

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