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The $L(3,2,1)$-Labeling Problem for Trees

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Abstract An $L(3,2,1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all non-negative integers (labels) such that $|f(u)-f(v)|\geq 3$ if $d(u,v)=1$, $|f(u)-f(v)|\geq 2$ if $d(u,v)=2$ and $|f(u)-f(v)|\geq 1$ if $d(u,v)=3$. For a non-negative integer k, a k-$L(3,2,1)$-labeling is an $L(3,2,1)$-labeling such that no label is greater than k. The $L(3,2,1)$-labeling number of G, denoted by $\lambda_{3,2,1}(G)$, is the smallest number k such that G has a k-$L(3,2,1)$-labeling. In this article, we characterize the $L(3,2,1)$-labeling numbers of trees with diameter at most 6.

Keywords channel assignment; $L(3,2,1)$-labeling; trees; diameter

MR(2010) Subject Classification 05C15; 05C78

1. Introduction

Multilevel distance labeling is a generalization of distance two labeling, which is motivated by the channel assignment problem introduced by Hale [1]. The channel assignment problem is the assignment of frequencies to transmitters subject to satisfying certain distance restrictions to avoid interference between nearby transmitters. If there is a high usage of wireless communication networks, we have to find an appropriate channel assignment solution, so that the range of channels used is minimized.

Griggs and Yeh [2] firstly proposed the notation of distance two labeling of a graph, and they generalized it to p-levels of interference, specifically for given positive integers $k_1,k_2,\ldots,k_p$, an $L(k_1,k_2,\ldots,k_p)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all non-negative integers (labels), such that for all distinct vertices u,v, $|f(u)-f(v)|\geq k_t$ if $d(u,v)=t$, where $d(u,v)$ denotes the distance between u and v. For a non-negative integer k, a k-$L(k_1,k_2,\ldots,k_p)$-labeling is an $L(k_1,k_2,\ldots,k_p)$-labeling such that no label is greater than k. The $L(k_1,k_2,\ldots,k_p)$-labeling number of G, denoted by $\lambda_{k_1,k_2,\ldots,k_p}(G)$, is the smallest number k such that G has a k-$L(k_1,k_2,\ldots,k_p)$-labeling.

The $L(k_1,k_2,\ldots,k_p)$-labeling problem above is interesting in both theory and practical applications. For instance, when $p=1$, it becomes the ordinary vertex-coloring problem. When $p=2$, many interesting results [2–4] have been obtained for various families of finite graphs, especially for the case $(k_1,k_2)=(2,1)$. For more details, one may refer to the surveys [5, 6].

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More recently, researchers began to investigate the $L(3,2,1)$-labeling problem. For example, Shao [7] studied the $L(3,2,1)$-labeling of Kneser graphs, extremely irregular graphs, Halin graphs, and gave bounds for the $L(3,2,1)$-labeling numbers of these classes of graphs. Liu and Shao [8] studied the $L(3,2,1)$-labeling of planar graphs, and showed that $\lambda_{3,2,1}(G) \le 15(\Delta^2 - \Delta + 1)$ if G is a planar graph of maximum degree Δ. Clipperton et al. [9] determined the $L(3,2,1)$-labeling numbers for paths, cycles, caterpillars, n-ary trees, complete graphs and complete bipartite graphs, and showed that $\lambda_{3,2,1}(G) \le \Delta^3 + \Delta^2 + 3\Delta$ for any graph G with maximum degree Δ. In this article, we characterize the $L(3,2,1)$-labeling numbers of trees with diameter at most 6.

2. Preliminaries

In this article, we always suppose that T is a finite tree with diameter at least 3. A vertex u is said to be *k-vertex* if $d(u) = k$, where $d(u)$ is the degree of u. Let $N_1(u) = \{w \in N(u) : w$ is Δ-vertex$\}$, $N_0(u) = N(u) \setminus N_1(u)$ and $d_1(u) = |N_1(u)|$, $d_0(u) = |N_0(u)|$. Let $N[u] = N(u) \cup \{u\}$.

Sometimes it is convenient to consider one vertex of a tree as special; such a vertex is then called the root of this tree. And we denote by T_u the tree rooted at u. For a rooted tree T_u, define $L_i(u) := \{w \in V(T_u) : d(u,w) = i\}$ for $i = 0, 1, \ldots$. In particular, $L_0(u) = \{u\}$. Define $E_i(u) := \{xy : x \in L_{i-1}(u), y \in L_i(u)\}$ for $i = 1, 2, \ldots$. For $xy \in E(T_u)$, if $x \in L_{i-1}(u), y \in L_i(u)$, then we call x the parent of y, which is denoted by y_p.

The diameter of T, denoted by $\mathrm{diam}(T)$, is the length of the longest path of T. Note that if $\mathrm{diam}(T)$ is even, then there must exist a vertex, say u, such that every path of length $\mathrm{diam}(T)$ goes through u. Thus if we treat T as a rooted tree T_u, then

$$\begin{aligned} V(T) &= \{u\} \cup L_1(u) \cup L_2(u) \cup \cdots \cup L_{\frac{\mathrm{diam}(T)}{2}}(u), \\ E(T) &= E_1(u) \cup E_2(u) \cup \cdots \cup E_{\frac{\mathrm{diam}(T)}{2}}(u). \end{aligned}$$

Such a vertex u is called the *crossing vertex* of T.

If $\mathrm{diam}(T)$ is odd, then there must exist an edge, say uv, such that every path of length $\mathrm{diam}(T)$ goes through uv. Such an edge uv is called the crossing edge of T. Let T_u and T_v be the two rooted trees obtained from T by deleting the edge uv, respectively. Then

$$\begin{aligned} V(T) =& \{u, v\} \cup L_1(u) \cup L_2(u) \cup \cdots \cup L_{\frac{\mathrm{diam}(T)-1}{2}}(u) \cup \\ & L_1(v) \cup L_2(v) \cup \cdots \cup L_{\frac{\mathrm{diam}(T)-1}{2}}(v), \\ E(T) =& \{uv\} \cup E_1(u) \cup E_2(u) \cup \cdots \cup E_{\frac{\mathrm{diam}(T)-1}{2}}(u) \cup \\ & E_1(v) \cup E_2(v) \cup \ldots \cup E_{\frac{\mathrm{diam}(T)-1}{2}}(v). \end{aligned}$$

3. Some sufficient conditions for the lower and upper bounds

For integers i and j with $i \le j$, we denote $[i,j]$ as the set $\{i, i+1, \ldots, j-1, j\}$. Let $O_{[i,j]}$ and $E_{[i,j]}$ be the set of all odd numbers and all even numbers in $[i,j]$, respectively.

Lemma 3.1 ([10]) *Let T be a tree with diameter at least 3. Then* $2\Delta + 1 \le \lambda_{3,2,1}(T) \le 2\Delta + 3$.

Moreover, if $\lambda_{3,2,1}(T)=2\Delta+1$ and f is a $(2\Delta+1)$-$L(3,2,1)$-labeling of T, then $f(v)\in\{0,2\Delta+1\}$ for each Δ-vertex v.

Theorem 3.2 *Let T be a tree with diameter at least 3. If T contains one of the following configurations, then $\lambda_{3,2,1}(T)\geq 2\Delta+2$.*

(C1) There exist two Δ-vertices v_1,v_2 such that $d(v_1,v_2)=2$.

(C2) ([10]) There exist three Δ-vertices v_1,v_2,v_3 such that $d(v_i,v_j)\leq 3$ for all $1\leq i,j\leq 3$.

Proof Let f be a $(2\Delta+1)$-$L(3,2,1)$-labeling of T.

(C1) By Lemma 3.1, we know that $\{f(v_1),f(v_2)\}=\{0,2\Delta+1\}$. Without loss of generality, let $f(v_1)=0$ and $f(v_2)=2\Delta+1$. Then $f(N(v_1))=O_{[3,2\Delta+1]}$. Particularly, $2\Delta+1\in f(N(v_1))$. But this contradicts $f(v_2)=2\Delta+1$.

(C2) By Lemma 3.1, $f(v_i)\in\{0,2\Delta+1\}$ for $1\leq i\leq 3$. It is a contradiction since $d(v_i,v_j)\leq 3$ for all $1\leq i,j\leq 3$. □

Remark 3.3 The conditions in Theorem 3.2 are only sufficient but not necessary for $\lambda_{3,2,1}(T)=2\Delta+1$. For example, T is a tree with $\Delta=3$, shown in Figure 1. Suppose f is a 7-$L(3,2,1)$-labeling of T. This implies $f(u),f(u_1),f(x),f(y)\in\{0,7\}$. Without loss of generality, let $f(u)=0$. Then $f(u_1)=f(x)=f(y)=7$ and $\{f(u_2),f(u_3)\}=\{3,5\}$. Suppose $f(u_2)=3$. Then there is no proper label for x_p. Hence $\lambda_{3,2,1}(T)\geq 8$. But T does not contain (C1)–(C2) of Theorem 3.2.

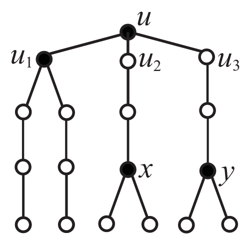

Figure 1 A tree T with $\Delta=3$, has $\lambda_{3,2,1}(T)\geq 2\Delta+2=8$. But T does not contain (C1)-(C2) of Theorem 3.2, where the black dots represent Δ-vertices.

Let f be an $L(3,2,1)$-labeling of T and $S\subseteq V(T)$. Define $f(S)=\{f(v):v\in S\}$.

Lemma 3.4 *Let f be a $(2\Delta+2)$-$L(3,2,1)$-labeling of T. Then $f(v)\in O_{[1,2\Delta+1]}\cup\{0,2\Delta+2\}$ for each Δ-vertex v. Moreover, if v is a Δ-vertex and $f(v)\in O_{[1,2\Delta+1]}$, then $f(N(v))=E_{[0,2\Delta+2]}\setminus\{f(v)\pm 1\}$.*

Proof Suppose to the contrary that there exists a Δ-vertex v such that $f(v)=i\in E_{[2,2\Delta]}$. Then $i,i\pm 1,i\pm 2\notin f(N(v))$. So $|f(N(v))|\leq \frac{i-2}{2}+\frac{(2\Delta+3)-(i+3)}{2}=\Delta-1$, a contradiction.

Next, if $f(v)=0$, then $f(N(v))\subseteq[3,2\Delta+2]$. By symmetry, $f(N(v))\subseteq[0,2\Delta-1]$ if $f(v)=2\Delta+2$. In either case, we can always choose Δ different labels such that any two labels are at least two apart.

If $f(v)=1$, then $f(N(v))=E_{[4,2\Delta+2]}$; if $f(v)=2\Delta+1$, then $f(N(v))=E_{[0,2\Delta-2]}$.

If $f(v)=i\in O_{[3,2\Delta-1]}$, then $f(N(v))=E_{[0,i-3]}\cup E_{[i+3,2\Delta+2]}=E_{[0,2\Delta+2]}\setminus\{i\pm 1\}$. □

Lemma 3.5 *Let $uv \in E(T)$ and $d(u) = d(v) = \Delta$. Let f be a $(2\Delta+2)$-$L(3,2,1)$-labeling of T. Then $f(u) \in \{0, 2\Delta+2\}$ or $f(v) \in \{0, 2\Delta+2\}$. Moreover, if there exist four Δ-vertices v_0, v_1, v_2, v_3 such that $v_0v_1, v_0v_2, v_0v_3 \in E(T)$, then $f(v_0) \in \{0, 2\Delta+2\}$.*

Proof Firstly, $f(u), f(v) \in O_{[1,2\Delta+1]} \cup \{0, 2\Delta+2\}$ by Lemma 3.4. Suppose to the contrary that $f(u) \in O_{[1,2\Delta+1]}$ and $f(v) \in O_{[1,2\Delta+1]}$. Then $f(N(u) \cup N(v)) \subseteq [0, 2\Delta+2] \setminus \{f(u) \pm 1, f(v) \pm 1\}$. Thus $|f(N(u) \cup N(v))| \le (2\Delta+3) - 4 = 2\Delta - 1$, a contradiction. Now, if $f(v_0) \in O_{[1,2\Delta+1]}$, then $f(v_i) \in \{0, 2\Delta+2\}$ for each $i \in \{1, 2, 3\}$. This is impossible. Therefore, $f(v_0) \in \{0, 2\Delta+2\}$. □

Lemma 3.6 *Let $u \in V(T)$ and $d_1(u) \ge 3$. Let f be a $(2\Delta+2)$-$L(3,2,1)$-labeling of T. Then $f(u) \in E_{[0,2\Delta+2]}$ and $f(N(u)) \subseteq O_{[1,2\Delta+1]}$.*

Proof Since $d_1(u) \ge 3$, there must exist some $w \in N_1(u)$ such that $f(w) \in O_{[1,2\Delta+1]}$. Then $f(N(w)) = E_{[0,2\Delta+2]} \setminus \{f(w) \pm 1\}$ by Lemma 3.4. Particularly, $f(u) \in E_{[0,2\Delta+2]}$. And $f(N(u)) \subseteq O_{[1,2\Delta+1]}$. □

In view of the above results, we now give some sufficient conditions for the upper bound.

Theorem 3.7 *Let $uv \in E(T)$. If $\min\{d_1(u), d_1(v)\} \ge 3$, then $\lambda_{3,2,1}(T) = 2\Delta + 3$.*

Proof Suppose f is a $(2\Delta+2)$-$L(3,2,1)$-labeling of T. By Lemma 3.6, we have $f(u), f(v) \in E_{[0,2\Delta+2]}$, $f(N(u)) \subseteq O_{[1,2\Delta+1]}$ and $f(N(v)) \subseteq O_{[1,2\Delta+1]}$ owing to $\min\{d_1(u), d_1(v)\} \ge 3$. But it is impossible. Hence $\lambda_{3,2,1}(T) = 2\Delta + 3$. □

Theorem 3.8 *Let $u \in V(T)$. If $d(u) \ge \Delta - 1, d_1(u) \ge 3$ and $d_1(u_i) = 2$ for all $u_i \in N(u)$, then $\lambda_{3,2,1}(T) = 2\Delta + 3$.*

Proof Suppose f is a $(2\Delta+2)$-$L(3,2,1)$-labeling of T. We consider the following two cases.

Case 1. $d(u) = \Delta$.

By Lemma 3.5, we have $f(u) \in \{0, 2\Delta+2\}$ since $d_1(u) \ge 3$. Without loss of generality, we may assume that $f(u) = 0$. Then $f(N(u)) = O_{[3,2\Delta+1]}$. This implies that there exists some $w \in N(u)$ such that $f(w) = 2\Delta + 1$. But now there is no proper label for $N_1(w) \setminus \{u\}$.

Case 2. $d(u) = \Delta - 1$.

Note that $f(N(u)) \subseteq O_{[1,2\Delta+1]}$ by Lemma 3.6. So $f(N_1(w)) = \{0, 2\Delta+2\}$ for each $w \in N_1(u)$. This implies $f(u) \in E_{[2,2\Delta]}$ and $f(N(u)) = O_{[1,2\Delta+1]} \setminus \{f(u) \pm 1\}$. Thus, there must exist some $w \in N(u)$ such that $f(w) = 1$ or $2\Delta + 1$. But then, no proper labels can be assigned to $N_1(w)$, also a contradiction. □

Remark 3.9 The conditions in Theorems 3.7 and 3.8 are only sufficient but not necessary for $\lambda_{3,2,1}(T) = 2\Delta + 3$. For example, T is a tree with $\Delta = 4$, shown in Figure 2. Suppose f is a 10-$L(3,2,1)$-labeling of T. This implies $f(u_1), f(u_2), f(v_1), f(v_2) \in \{0, 3, 5, 7, 10\}$, since $d_1(u_i) = d_1(v_i) = 2$ for each $i \in \{1, 2\}$. We may assume that $f(u_1) \in \{3, 5, 7\}$, since any two vertices in $\{u_1, u_2, v_1, v_2\}$ are of distance at most three. Then $f(N_1(u_1)) = \{0, 10\}$ and $f(u_2) \in \{3, 5, 7\}$. Hence $\{f(v_1), f(v_2)\} = \{0, 10\}$ and $f(v) \in \{3, 5, 7\}$. Now there is no proper label for u. Hence $\lambda_{3,2,1}(T) = 2\Delta + 3 = 11$. But T does not satisfy the conditions in Theorems 3.7 and 3.8.

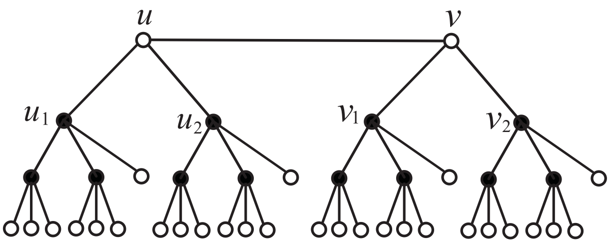

Figure 2 A tree T with $\Delta = 4$, has $\lambda_{3,2,1}(T) = 2\Delta + 3 = 11$. But T does not satisfy the conditions of Theorem 3.7-3.8, where the black dots represent Δ-vertices.

4. Results for trees with diameter at most 6

As we mentioned earlier that all the conditions in Theorems 3.2, 3.7 and 3.8 are just sufficient but not necessary. In this section, we will show that the sufficient conditions are also necessary for trees with diameter at most 6.

Theorem 4.1 *Let T be a tree with diameter 3. Then $\lambda_{3,2,1}(T) = 2\Delta + 1$.*

Proof Let uv be the crossing edge. It is enough to consider the case when u and v are Δ-vertices, since any subgraph of T has the $L(3,2,1)$-labeling number no more than T.

Now we define a $(2\Delta+1)$-$L(3,2,1)$-labeling f as follows:

(i) $f(u) = 0, f(v) = 2\Delta + 1$;

(ii) $f(N(u) \setminus \{v\}) = O_{[3,2\Delta-1]}, f(N(v) \setminus \{u\}) = E_{[2,2\Delta-2]}$.

Note that all the vertices of T have different labels. Next, $|f(u) - f(v)| = 2\Delta + 1 > 3$, $\min_{x\in L_1(u)} |f(u) - f(x)| = 3$ and $\min_{y\in L_1(v)} |f(v) - f(y)| = 2\Delta + 1 - (2\Delta - 2) = 3$. Finally, $\min_{x\in L_1(u)} |f(v) - f(x)| = 2\Delta + 1 - (2\Delta - 1) = 2$ and $\min_{y\in L_1(v)} |f(u) - f(y)| = 2$. So f is a $(2\Delta + 1)$-$L(3,2,1)$-labeling of T, which implies $\lambda_{3,2,1}(T) \le 2\Delta + 1$. Now from Lemma 3.1, we get $\lambda_{3,2,1}(T) = 2\Delta + 1$. □

Theorem 4.2 *Let T be a tree with diameter 4 and u be the crossing vertex. Then $2\Delta + 1 \le \lambda_{3,2,1}(T) \le 2\Delta + 2$. Furthermore, $\lambda_{3,2,1}(T) = 2\Delta + 2$ if and only if $d_1(u) \ge 2$.*

Proof Firstly, $\lambda_{3,2,1}(T) \ge 2\Delta + 1$ by Lemma 3.1. To prove the upper bound, we only need to consider the case when all vertices in $N[u]$ are Δ-vertices. Define

(i) $f(u) = 0$;

(ii) $f(N(u)) = O_{[3,2\Delta+1]}$;

(iii) $f(N(x) \setminus \{u\}) = E_{[2,2\Delta+2]} \setminus \{i-1, i+1\}$ if $f(x) = i \in O_{[3,2\Delta+1]}$ for each $x \in L_1(u)$.

It is clear that any pair of vertices of distance at most 3 have different labels. Secondly, $\min_{x\in L_1(u)} |f(u)-f(x)| = 3$ and $\min_{xy\in E_2(u)} |f(x)-f(y)| = 3$. Finally, $\min_{x,y\in V(T),d(x,y)=2} |f(x)-f(y)| = 2$. Therefore, f is a $(2\Delta + 2)$-$L(3,2,1)$-labeling of T, which implies $\lambda_{3,2,1}(T) \le 2\Delta + 2$.

Next, we prove the rest. The sufficiency follows from (C1) of Theorem 3.2. We now prove the necessity. Suppose to the contrary that $d_1(u) \le 1$. It is enough to consider the case when u is Δ-vertex and $d_1(u) = 1$. We define a $(2\Delta+1)$-$L(3,2,1)$-labeling of T as follows:

(i) $f(u) = 0$;

(ii) $f(N_1(u)) = \{2\Delta+1\}$, $f(N_0(u)) = O_{[3,2\Delta-1]}$;

(iii) $f(N(x) \setminus \{u\}) \subseteq E_{[2,2\Delta]} \setminus \{i-1, i+1\}$ if $f(x) = i \in O_{[3,2\Delta+1]}$ for each $x \in L_1(u)$.

Firstly, any pair of vertices of distance at most 3 have different labels. Next, $\min_{x\in L_1(u)} |f(u)-f(x)| = 3$, $\min_{xy\in E_2(u)} |f(x)-f(y)| = 3$. Finally, $\min_{x,y\in V(T),d(x,y)=2} |f(x)-f(y)| = 2$. Then, clearly, f is a $(2\Delta+1)$-$L(3,2,1)$-labeling of T. Thus, $\lambda_{3,2,1}(T) = 2\Delta+1$ by Lemma 3.1, a contradiction. $\Box$

Theorem 4.3 *Let T be a tree with diameter 5 and uv be the crossing edge of T. Then*

(1) $\lambda_{3,2,1}(T) = 2\Delta+1$ *if and only if* $\max\{d_1(u), d_1(v)\} \le 1$.

(2) $\lambda_{3,2,1}(T) = 2\Delta+3$ *if and only if* $\min\{d_1(u), d_1(v)\} \ge 3$.

Proof (1) If $\max\{d_1(u), d_1(v)\} \ge 2$, then there must exist two vertices v_1, v_2 with $d(v_1, v_2) = 2$. So $\lambda_{3,2,1}(T) \ge 2\Delta+2$ by Theorem 3.2. On the other hand, suppose $d_1(u) = d_1(v) = 1$. We will give a $(2\Delta+1)$-$L(3,2,1)$-labeling of T. We consider three cases depending on values of $d(u)$ and $d(v)$.

Case 1. $d(u) = d(v) = \Delta$.

Consider the following labeling f:

(i) $f(u) = 0, f(v) = 2\Delta+1$;

(ii) $f(N_0(u)) = O_{[3,2\Delta-1]}, f(N_0(v)) = E_{[2,2\Delta-2]}$;

(iii) $f(N(x)) \subseteq E_{[2,2\Delta]} \setminus \{i-1, i+1\}$ if $f(x) = i \in O_{[3,2\Delta-1]}$ for each $x \in L_1(u)$;

(iv) $f(N(x)) \subseteq O_{[1,2\Delta-1]} \setminus \{i-1, i+1\}$ if $f(x) = i \in E_{[2,2\Delta-2]}$ for each $x \in L_1(v)$.

Case 2. $d(u) = \Delta, d(v) < \Delta$.

Consider the following labeling f:

(i) $f(u) = 0, f(v) = 3$;

(ii) $f(N_1(u)) = \{2\Delta+1\}$, $f(N_0(u) \setminus \{v\}) = O_{[5,2\Delta-1]}, f(N(v) \setminus \{u\}) \subseteq E_{[6,2\Delta]}$;

(iii) $f(N(x)) \subseteq E_{[2,2\Delta]} \setminus \{i-1, i+1\}$ if $f(x) = i \in O_{[5,2\Delta+1]}$ for each $x \in L_1(u)$;

(iv) $f(N(x)) \subseteq O_{[1,2\Delta+1]} \setminus \{3, i-1, i+1\}$ if $f(x) = i \in E_{[6,2\Delta]}$ for each $x \in L_1(v)$.

Case 3. $d(u) < \Delta, d(v) < \Delta$.

Consider the following labeling f:

(i) $f(u) = 3, f(v) = 6$;

(ii) $f(N_1(u)) = \{0\}$, $f(N_0(u) \setminus \{v\}) \subseteq E_{[8,2\Delta]}$, $f(N_1(v)) = \{2\Delta+1\}$, $f(N_0(v) \setminus \{u\}) \subseteq O_{[9,2\Delta-1]} \cup \{1\}$;

(iii) $f(N(x)) \subseteq O_{[1,2\Delta+1]} \setminus \{3, i-1, i+1\}$ if $f(x) = i \in E_{[8,2\Delta]} \cup \{0\}$ for each $x \in L_1(u)$;

(iv) $f(N(x)) \subseteq E_{[0,2\Delta]} \setminus \{6, i-1, i+1\}$ if $f(x) = i \in O_{[9,2\Delta+1]} \cup \{1\}$ for each $x \in L_1(v)$.

For the above three cases, it is easy to verify that f is a $(2\Delta+1)$-$L(3,2,1)$-labeling of T. Therefore, $\lambda_{3,2,1}(T) = 2\Delta+1$ by Lemma 3.1.

(2) The sufficiency follows from Theorem 3.7. Now we prove the necessity. Suppose, to the contrary, $\min\{d_1(u), d_1(v)\} \le 2$. It is sufficient to consider the following two cases:

(I) $d(u) = d(v) = \Delta$ and $d_1(u) = 2$.

(II) $d(u) = \Delta, d(v) = \Delta - 1$ and $d_1(u) = 2$.

Case 1. (I) holds.

Consider the following labeling f:

(i) $f(u) = 3, f(v) = 0$;

(ii) $f(N_1(u) \setminus \{v\}) = \{2\Delta + 2\}$, $f(N_0(u)) = E_{[6,2\Delta]}$, $f(N(v) \setminus \{u\}) = O_{[5,2\Delta+1]}$;

(iii) $f(N(x)) \subseteq O_{[1,2\Delta+1]} \setminus \{3, i-1, i+1\}$ if $f(x) = i \in E_{[6,2\Delta+2]}$ for each $x \in L_1(u)$;

(iv) $f(N(x)) \subseteq E_{[2,2\Delta+2]} \setminus \{i-1, i+1\}$ if $f(x) = i \in O_{[5,2\Delta+1]}$ for each $x \in L_1(v)$.

Case 2. (II) holds.

Consider the following labeling f:

(i) $f(u) = 3, f(v) = 6$;

(ii) $f(N_1(u)) = \{0, 2\Delta + 2\}$, $f(N_0(u) \setminus \{v\}) = E_{[8,2\Delta]}$, $f(N(v) \setminus \{u\}) = O_{[9,2\Delta+1]} \cup \{1\}$;

(iii) $f(N(x)) \subseteq O_{[1,2\Delta+1]} \setminus \{3, i-1, i+1\}$ if $f(x) = i \in E_{[8,2\Delta+2]} \cup \{0\}$ for each $x \in L_1(u)$;

(iv) $f(N(x)) \subseteq E_{[0,2\Delta+2]} \setminus \{6, i-1, i+1\}$ if $f(x) = i \in O_{[9,2\Delta+1]} \cup \{1\}$ for each $x \in L_1(v)$.

For the above two cases, it is straightforward to check that f is a $(2\Delta+2)$-$L(3,2,1)$-labeling of T. Thus, $\lambda_{3,2,1}(T) \le 2\Delta + 2$, a contradiction. □

In the theorem below we give a complete characterization of trees with diameter 6.

Theorem 4.4 *Let T be a tree with diameter 6 and u be the crossing vertex. Then*

(1) $\lambda_{3,2,1}(T) = 2\Delta + 1$ if and only if T does not contain (C1)–(C2) of Theorem 3.2.

(2) $\lambda_{3,2,1}(T) = 2\Delta + 3$ if and only if T contains one of the following configurations:

(C1) $\min\{d_1(u), d_1(u_i)\} \ge 3$ for some $u_i \in N(u)$.

(C2) $d(u) \ge \Delta - 1, d_1(u) \ge 3$ and $d_1(u_i) = 2$ for all $u_i \in N(u)$.

Proof (1) The necessity follows from Theorem 3.2. We now prove the sufficiency. If T does not contain (C1)–(C2) of Theorem 3.2, then we only need to consider the following two cases.

Case 1. $d(u) = \Delta$ and $d_1(u) = 1$.

Consider the following labeling f:

(i) $f(u) = 0$;

(ii) $f(N_1(u)) = \{2\Delta + 1\}$, $f(N_0(u)) = O_{[3,2\Delta-1]}$;

(iii) $f(N(x)) \subseteq E_{[2,2\Delta]} \setminus \{i-1, i+1\}$ if $f(x) = i \in O_{[3,2\Delta+1]}$ for each $x \in L_1(u)$;

(iv) $f(N(x)) \subseteq O_{[1,2\Delta+1]} \setminus \{f(x_p), i-1, i+1\}$ if $f(x) = i \in E_{[2,2\Delta]}$ for each $x \in L_2(u)$.

Case 2. $d(u) < \Delta$, $d_1(u) = 1$ and $d_1(x) = 1$ for all $x \in L_1(u)$.

Consider the following labeling f:

(i) $f(u) = 2$;

(ii) $f(N_1(u)) = \{2\Delta + 1\}$, $f(N_0(u)) \subseteq O_{[5,2\Delta-1]}$;

(iii) $f(N_1(x)) = \{0\}$, $f(N_0(x) \setminus \{u\}) \subseteq E_{[4,2\Delta]} \setminus \{i-1, i+1\}$ if $f(x) = i \in O_{[5,2\Delta+1]}$ for each $x \in L_1(u)$;

(iv) $f(N(x)) \subseteq O_{[1,2\Delta+1]} \setminus \{f(x_p), i-1, i+1\}$ if $f(x) = i \in E_{[4,2\Delta]} \cup \{0\}$ for each $x \in L_2(u)$.

For the above two cases, it is easy to verify that f is a $(2\Delta+1)$-$L(3,2,1)$-labeling of T. Therefore, $\lambda_{3,2,1}(T)=2\Delta+1$ by Lemma 3.1.

(2) The sufficiency follows from Theorem 3.7-3.8. We now prove the necessity. Suppose, to the contrary, the conclusion is false. We may assume that one of the followings holds:

(I) $d_1(u)\le 2$.

(II) $d_1(u)\ge 3$, $d_1(u_i)\le 2$ for all $u_i\in N(u)$ and there exists some $w\in N(u)$ such that $d_1(w)\le 1$.

(III) $d_1(u)\ge 3, d(u)\le \Delta-2$ and $d_1(u_i)=2$ for all $u_i\in N(u)$.

Case 1. (I) holds.

Consider the following labeling f:

(i) $f(u)=3$;

(ii) $f(N_1(u))\subseteq\{0,2\Delta+2\}$, $f(N_0(u))\subseteq E_{[0,2\Delta+2]}\setminus(\{2,4\}\cup f(N_1(u)))$;

(iii) $f(N(x))\subseteq O_{[1,2\Delta+1]}\setminus\{3,i-1,i+1\}$ if $f(x)=i\in E_{[0,2\Delta+2]}\setminus\{2,4\}$ for each $x\in L_1(u)$;

(iv) $f(N(x))\subseteq E_{[0,2\Delta+2]}\setminus\{f(x_p),i-1,i+1\}$ if $f(x)=i\in O_{[1,2\Delta+1]}\setminus\{3\}$ for each $x\in L_2(u)$.

Case 2. (II) holds.

We consider the following two subcases.

Subcase 2.1. $d(u)=\Delta$.

Consider the following labeling f:

(i) $f(u)=0$;

(ii) $f(w)=2\Delta+1$, $f(N(u)\setminus\{w\})=O_{[3,2\Delta-1]}$;

(iii) $f(N(x))\subseteq E_{[2,2\Delta+2]}\setminus\{i-1,i+1\}$ if $f(x)=i\in O_{[3,2\Delta+1]}$ for each $x\in L_1(u)$ and make $f(N_1(x)\setminus\{u\})=\{2\Delta+2\}$ if $N_1(x)\setminus\{u\}\ne\emptyset$;

(iv) $f(N(x))\subseteq O_{[1,2\Delta+1]}\setminus\{f(x_p),i-1,i+1\}$ if $f(x)=i\in E_{[2,2\Delta+2]}$ for each $x\in L_2(u)$.

Subcase 2.2. $d(u)<\Delta$.

Consider the following labeling f:

(i) $f(u)=2$;

(ii) $f(w)=2\Delta+1$, $f(N(u)\setminus\{w\})\subseteq O_{[5,2\Delta-1]}$;

(iii) $f(N(x))\subseteq E_{[0,2\Delta+2]}\setminus\{2,i-1,i+1\}$ if $f(x)=i\in O_{[5,2\Delta+1]}$ for each $x\in L_1(u)$ and make $f(N_1(x))\subseteq\{0,2\Delta+2\}$ if $N_1(x)\ne\emptyset$;

(iv) $f(N(x))\subseteq O_{[1,2\Delta+1]}\setminus\{f(x_p),i-1,i+1\}$ if $f(x)=i\in E_{[0,2\Delta+2]}\setminus\{2\}$ for each $x\in L_2(u)$.

Case 3. (III) holds.

Consider the following labeling f:

(i) $f(u)=2$;

(ii) $f(N(u))\subseteq O_{[5,2\Delta-1]}$;

(iii) $f(N(x))\subseteq E_{[0,2\Delta+2]}\setminus\{2,i-1,i+1\}$ if $f(x)=i\in O_{[5,2\Delta-1]}$ for each $x\in L_1(u)$ and make $f(N_1(x))=\{0,2\Delta+2\}$;

(iv) $f(N(x))\subseteq O_{[1,2\Delta+1]}\setminus\{f(x_p),i-1,i+1\}$ if $f(x)=i\in E_{[0,2\Delta+2]}\setminus\{2\}$ for each $x\in L_2(u)$.

For the above three cases, we check that f is a $(2\Delta+2)$-$L(3,2,1)$-labeling of T. Thus, $\lambda_{3,2,1}(T)\le 2\Delta+2$, a contradiction. $\square$

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