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Double Series Identity and Some Laurent Type Hypergeometric Generating Relations

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Abstract The main aim of this article is to obtain certain Laurent type hypergeometric generating relations. Using a general double series identity, Laurent type generating functions (in terms of Kampé de Fériet double hypergeometric function) are derived. Some known results obtained by the method of Lie groups and Lie algebras, are also modified here as special cases.

Keywords generalized hypergeometric functions; Laurent type generating relations; double series identity

MR(2010) Subject Classification 33B15; 33C10; 33C20

1. Introduction and preliminaries

There has been a great revival of interest in the study of hypergeometric functions in the last two decades. This newfound interest comes from the connections between hypergeometric functions and many areas of mathematics such as representation theory, algebraic geometry and Hodge theory, combinatorics, D-modules, number theory, mirror symmetry, etc. The integral representations played an important role in the study of the hypergeometric functions. The celebrated Euler's integral for the Gauss functions ${}_2F_1$ was probably the first among them.

The Pochhammer symbol $(\lambda)_n$ is defined as:

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1), & \text{if } n=1,2,3,\ldots,\\ 1, & \text{if } n=0,\\ n!, & \text{if } \lambda=1,\ n=0,1,2,3,\ldots.\end{cases}$$

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$${}_pF_q\left[\begin{array}{c}(\alpha_p);\\ \\ (\beta_q);\end{array} z\right]={}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \\ \beta_1,\beta_2,\ldots,\beta_q;\end{array} z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\,\frac{z^n}{n!} \tag{1.1}$$

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is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$ take on complex values, provided that

$$\beta_j \neq 0, -1, -2, \ldots; \ j = 1, 2, \ldots, q.$$

In contracted notation, the sequence of p numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ is denoted by (α_p) with similar interpretation for others throughout this paper.

Supposing that none of numerator parameters is zero or a negative integer and for $\beta_j \neq 0, -1, -2, \ldots; j = 1, 2, \ldots, q$, we note that the ${}_pF_q$ series defined by Eq. (1.1):

(i) converges for $|z| < \infty$, if $p \leq q$,

(ii) converges for $|z| < 1$, if $p = q + 1$ and

(iii) diverges for all z, $z \neq 0$, if $p > q + 1$.

Just as the Gaussian ${}_2F_1$ function was generalized to ${}_pF_q$ by increasing the number of the numerator and denominator parameters, the four Appell functions were unified and generalized by Kampé de Fériet [1] who defined a general hypergeometric function of two variables [2, p. 423, Eq.(26)].

The notation introduced by Kampé de Fériet for his double hypergeometric function of superior order was subsequently abbreviated by Burchnall and Chaundy [3, p.112]. We recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation [4, p.423, Eq.(26)]:

$$F^{p:\ q;\ k}_{\ell:\ m;\ n}\left[\begin{array}{l} (a_p)\ :\ (b_q)\ ;\ (c_k)\ ; \\ \\ (\alpha_\ell)\ :\ (\beta_m)\ ;\ (\gamma_n)\ ; \end{array} x,\ y\right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s} \prod_{j=1}^{q}(b_j)_r \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{\ell}(\alpha_j)_{r+s} \prod_{j=1}^{m}(\beta_j)_r \prod_{j=1}^{n}(\gamma_j)_s} \frac{x^r}{r!}\frac{y^s}{s!}, \tag{1.2}$$

where, for convergence,

$$\text{(i)}\ \ p + q < \ell + m + 1,\ p + k < \ell + n + 1,\ |x| < \infty,\ |y| < \infty,\ \text{or} \tag{1.3}$$

$$\text{(ii)}\ \ p + q = \ell + m + 1,\ p + k = \ell + n + 1\ \text{and} \tag{1.4}$$

$$\begin{cases} |x|^{1/(p-\ell)} + |y|^{1/(p-\ell)} < 1, & \text{if } p > \ell, \\ \max\{|x|, |y|\} < 1, & \text{if } p \leq \ell. \end{cases} \tag{1.5}$$

Although the double hypergeometric function defined by (1.2) reduces to the Kampé de Fériet function in the special case: $q = k$ and $m = n$, yet it is usually referred to in the literature as the Kampé de Fériet.

A multivariable hypergeometric function provides an interesting and useful unification of the generalized hypergeometric function ${}_pF_q$ of one variable (with p numerator and q denominator parameters).

The following generalization of the hypergeometric function in several variables has been given by Srivastava and Daoust [5], which is referred to in the literature as the generalized

Lauricella function of several variables:

$$F^{A:B';B'';\ldots;B^{(n)}}_{C:D';D'';\ldots;D^{(n)}}\left(\begin{array}{l} [(a):\theta',\theta'',...,\theta^{(n)}]:\quad [(b'):\phi'];\quad [(b''):\phi''];\quad \cdots;\quad [(b^{(n)}):\phi^{(n)}];\\ \\ [(c):\psi',\psi'',...,\psi^{(n)}]:\quad [(d'):\delta'];\quad [(d''):\delta''];\quad \cdots;\quad [(d^{(n)}):\delta^{(n)}]; \end{array} z_1,z_2,\cdots,z_n\right)$$

$$= \sum_{m_1,m_2,\ldots,m_n=0}^{\infty} \Omega(m_1,m_2,...,m_n)\ \frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}\cdots\frac{z_n^{m_n}}{m_n!}, \tag{1.6}$$

where

$$\Omega(m_1,m_2,\ldots,m_n)$$
$$:= \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+m_2\theta''_j+\cdots+m_n\theta^{(n)}_j}\prod_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\prod_{j=1}^{B''}(b''_j)_{m_2\phi''_j}\ \cdots\ \prod_{j=1}^{B^{(n)}}(b^{(n)}_j)_{m_n\phi^{(n)}_j}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+m_2\psi''_j+\cdots+m_n\psi^{(n)}_j}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\prod_{j=1}^{D''}(d''_j)_{m_2\delta''_j}\ \cdots\ \prod_{j=1}^{D^{(n)}}(d^{(n)}_j)_{m_n\delta^{(n)}_j}}$$

and the coefficients

$$\theta^{(k)}_j, j=1,2,\ldots,A;\ \phi^{(k)}_j, j=1,2,\ldots,B^{(k)};\ \psi^{(k)}_j, j=1,2,\ldots,C;\ \delta^{(k)}_j, j=1,2,\ldots,D^{(k)};$$

for all $k\in\{1,2,\ldots,n\}$ are real and positive, (a) abbreviates the array of A parameters $a_1,a_2,\ldots,a_A$, $(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters $b^{(k)}_j, j=1,2,\ldots,B^{(k)}$; for all $k\in\{1,2,\ldots,n\}$ with similar interpretations for (c) and $(d^{(k)})$, $k=1,2,\ldots,n$; et cetera.

Generating functions play an important role in the investigation of various useful properties of the sequences which they generate. The Bessel functions $J_n(x)$ have the following generating function [6]:

$$\exp(\frac{x}{2}(t-\frac{1}{t})) = \sum_{n=-\infty}^{\infty} J_n(x)t^n,\quad t\neq 0;\ |x|<\infty. \tag{1.7}$$

The Bessel functions $J_n(x)$ are also defined by the series

$$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{x}{2})^{n+2k}}{k!\,(n+k)!},\ |x|<\infty \tag{1.8}$$

and

$$J_{-n}(x)=(-1)^nJ_n(x), \tag{1.9}$$

where n is a nonnegative integer.

Tricomi functions are Bessel like functions. The Tricomi functions $C_n(x)$ possess the following generating function [7]:

$$\exp(t-\frac{x}{t}) = \sum_{n=-\infty}^{\infty} C_n(x)t^n,\ t\neq 0;\ |x|<\infty \tag{1.10}$$

and have the following series definition:

$$C_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k\ x^k}{k!\ (n+k)!},\ n=0,1,2,\ldots;\ |x|<\infty \tag{1.11}$$

and

$$C_{-n}(x)=(-1)^nx^nC_n(x). \tag{1.12}$$

The Bessel functions $J_n(x)$ and Tricomi functions $C_n(x)$ have the following hypergeometric representations:

$$J_n(x) = \frac{(\frac{x}{2})^n}{\Gamma(n+1)} {}_0F_1[-; n+1; \frac{-x^2}{4}] \tag{1.13}$$

and

$$C_n(x) = \frac{1}{\Gamma(n+1)} {}_0F_1[-; n+1; -x], \tag{1.14}$$

respectively.

We recall the following Laurent type generating relations which are obtained by Lie algebraic techniques [8, p-325; Eqs. (9) and (10)]:

$$(1-xt)^{-\alpha}(1-\frac{y}{t})^{-\beta} = \sum_{p=-\infty}^{\infty} \frac{\Gamma(\alpha+p)(xt)^p}{\Gamma(\alpha)\Gamma(1+p)} \; {}_2F_1\left[\begin{array}{lc} \alpha+p,\ \beta \;\; ; & \\ & xy \\ 1+p \quad\;\; ; & \end{array}\right],\ |y| < |t| < |x|^{-1} \tag{1.15}$$

and

$$(1-xt)^{-\beta} \; {}_1F_1\left[\begin{array}{lc} \alpha \quad ; & \\ & \frac{y}{t}(1-xt) \\ 1-\beta \;\; ; & \end{array}\right]$$

$$= \sum_{p=-\infty}^{\infty} \frac{\Gamma(\beta+p)(xt)^p}{\Gamma(\beta)\Gamma(1+p)} \; {}_1F_1\left[\begin{array}{lc} \alpha \quad ; & \\ & -xy \\ 1+p \;\; ; & \end{array}\right],\ 0 < |t| < |x|^{-1}. \tag{1.16}$$

Also, the simplified form of another Laurent type generating relation [8, p-330, Eqs. (27) and (28)] in terms of other parameters and variables is as follows:

$${}_1F_0\left[\begin{array}{lc} -\nu \;\; ; & \\ & \frac{-c}{\beta t} \\ - \;\; ; & \end{array}\right] {}_1F_0\left[\begin{array}{lc} \alpha+\nu+1 \;\; ; & \\ & \frac{-\beta\gamma t}{1+\gamma c} \\ - \;\; ; & \end{array}\right]$$

$$= \sum_{p=-\infty}^{\infty} \frac{(1+\alpha+\nu)_p}{p!} \; {}_2F_1\left[\begin{array}{lc} -\nu,\ \alpha+\nu+p+1 \;\; ; & \\ & \frac{\gamma c}{1+\gamma c} \\ 1+p \quad\quad ; & \end{array}\right] (\frac{-\beta\gamma}{1+\gamma c})^p t^p, \tag{1.17}$$

provided $|\frac{\beta\gamma t}{1+\gamma c}| < 1$, $|\frac{c}{\beta t}| < 1$, $-\pi < \arg(\beta)$, $\arg(\frac{1+\gamma c}{\beta}) < \pi$.

In this article, the following lemma has been used:

Lemma 1.1 ([9, p.4, Eq (12)]) *Let $\{f(s)\}_{s=0}^{\infty}$ be a bounded sequence of arbitrary real or complex numbers. Then*

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,r=0}^{\infty} f(n+r)\frac{x^n}{n!}\frac{y^r}{r!}, \tag{1.18}$$

provided that each of the series involved is absolutely convergent.

Proof of Lemma 1.1 On replacing n by $n-r$ in r.h.s of Eq. (1.18) and then using the result

$\sum_{n,r=0}^{\infty}\Psi(n,r)=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\Psi(n-r,r)$ (see [8, pp.100]), we get

$$\sum_{n,r=0}^{\infty} f(n+r)\frac{x^n}{n!}\frac{y^r}{r!}=\sum_{n=0}^{\infty}\sum_{r=0}^{n} f(n)\frac{x^{n-r}}{(n-r)!}\frac{y^r}{r!}$$

$$=\sum_{n=0}^{\infty} f(n)\frac{x^n}{n!}\sum_{r=0}^{n}\frac{(-n)_r(-\frac{y}{x})^r}{r!}=\sum_{n=0}^{\infty} f(n)\frac{x^n}{n!}\ {}_1F_0\left[\begin{array}{lll} -n & ; & \\ & & -\frac{y}{x} \\ - & ; & \end{array}\right]. \tag{1.19}$$

Now, applying binomial theorem

$${}_1F_0\left[\begin{array}{lll} a & ; & \\ & & z \\ - & ; & \end{array}\right]=(1-z)^{-a};\ a\in\mathbb{C},\ |z|<1,$$

we get

$$\sum_{n,r=0}^{\infty} f(n+r)\frac{x^n}{n!}\frac{y^r}{r!}=\sum_{n=0}^{\infty} f(n)\frac{x^n}{n!}(1+\frac{y}{x})^n=\sum_{n=0}^{\infty} f(n)\frac{(x+y)^n}{n!}. \tag{1.20}$$

Since n is a dummy index, so we can write above equation as follows:

$$\sum_{n,r=0}^{\infty} f(n+r)\frac{x^n}{n!}\frac{y^r}{r!}=\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!}. \tag{1.21}$$

This completes the proof. □

Throughout this article, the meaning of $p^\star$ is as follows:

$$p^\star=\max\{0,-p\}=\begin{cases}-p, & \text{when } p=-1,-2,-3,\dots\\ 0, & \text{when } p=0,1,2,\dots.\end{cases} \tag{1.22}$$

The recent development in the field of hypergeometric functions has been studied by many authors. The basic properties of the extended τ-Gauss hypergeometric function, including integral and derivative formulas involving the Mellin transform and the operators of fractional calculus, are derived in [10]. Also, the introduction of extended Pochhammer symbol by using a known extension of the gamma function involving the modified Bessel (or Macdonald) function are recent investigations, see for example, [11, 12].

The purpose of this note is to obtain the Laurent type hypergeometric generating relations.

This paper is organized as follows. In Section 2, a general double series identity is derived. Section 3 is dedicated to obtain the Laurent type hypergeometric generating relations. In Section 4, some special cases of the obtained results are presented.

We are now in a position to prove the following general double series identity and hypergeometric generating relations, using series rearrangement techniques.

2. General double series identity

Lemma 2.1 *Let $\{\Phi(m,n)\}$ be a bounded multiple sequence of arbitrary real or complex num-*

bers. Then

$$\sum_{m,n=0}^{\infty}\Phi(m,n)\frac{a^m b^n}{m!n!}t^{m-n}=\sum_{p=-\infty}^{\infty}\sum_{n=0}^{\infty}\Phi(n+p+p^\star,n+p^\star)\frac{a^{n+p+p^\star}b^{n+p^\star}}{(n+p+p^\star)!(n+p^\star)!}t^p, \tag{2.1}$$

provided that each of double series involved is absolutely convergent, for suitable values of a, b and $t(t\neq 0)$ and $p^\star$ is defined by Eq. (1.22).

Proof Suppose the l.h.s. of Eq. (2.1) is denoted by Λ. Then, we have

$$\Lambda=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Phi(m,n)\frac{a^m b^n}{m!n!}t^{m-n}. \tag{2.2}$$

On replacing m by $n+p$ in Eq. (2.2), we get

$$\begin{aligned}
\Lambda&=\sum_{p=-\infty}^{\infty}\sum_{n=0}^{\infty}\Phi(n+p,n)\frac{a^{n+p}\ b^n}{(n+p)!n!}t^p\\
&=\sum_{p=-\infty}^{-1}\sum_{n=0}^{\infty}\Phi(n+p,n)\frac{a^{n+p}\ b^n}{(n+p)!n!}t^p+\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\Phi(n+p,n)\frac{a^{n+p}\ b^n}{(n+p)!n!}t^p\\
&=\cdots+\sum_{n=0}^{\infty}\Phi(n-4,n)\frac{a^{n-4}\ b^n}{(n-4)!n!}t^{-4}+\sum_{n=0}^{\infty}\Phi(n-3,n)\frac{a^{n-3}\ b^n}{(n-3)!n!}t^{-3}+\\
&\quad\sum_{n=0}^{\infty}\Phi(n-2,n)\frac{a^{n-2}\ b^n}{(n-2)!n!}t^{-2}+\sum_{n=0}^{\infty}\Phi(n-1,n)\frac{a^{n-1}\ b^n}{(n-1)!n!}t^{-1}+\\
&\quad\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\Phi(n+p,n)\frac{a^{n+p}\ b^n}{(n+p)!n!}t^p\\
&=\cdots+\sum_{n=4}^{\infty}\Phi(n-4,n)\frac{a^{n-4}\ b^n}{(n-4)!n!}t^{-4}+\sum_{n=3}^{\infty}\Phi(n-3,n)\frac{a^{n-3}\ b^n}{(n-3)!n!}t^{-3}+\\
&\quad\sum_{n=2}^{\infty}\Phi(n-2,n)\frac{a^{n-2}\ b^n}{(n-2)!n!}t^{-2}+\sum_{n=1}^{\infty}\Phi(n-1,n)\frac{a^{n-1}\ b^n}{(n-1)!n!}t^{-1}+\\
&\quad\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\Phi(n+p,n)\frac{a^{n+p}\ b^n}{(n+p)!n!}t^p\\
&=\sum_{p=-\infty}^{-1}\sum_{n=-p}^{\infty}\Phi(n+p,n)\frac{a^{n+p}\ b^n}{(n+p)!n!}t^p+\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\Phi(n+p,n)\frac{a^{n+p}\ b^n}{(n+p)!n!}t^p\\
&=\sum_{p=-\infty}^{\infty}\sum_{n=p^\star}^{\infty}\Phi(n+p,n)\frac{a^{n+p}\ b^n}{(n+p)!n!}t^p,\ \text{ where } p^\star=\max\{0,-p\}.
\end{aligned} \tag{2.3}$$

On replacing n by $n+p^\star$ in Eq. (2.3), assertion (2.1) follows. □

In the next section, using the double series identity, we derive the Laurent type generating relations in terms of Kampé de Fériet double hypergeometric function.

3. Laurent type generating relations

The result is given in the form of the following theorem:

Theorem 3.1 *The following generating function (in terms of Kampé de Fériet double hypergeometric function) for Srivastava-Daoust hypergeometric function holds true:*

$$F^{D:G;K}_{E:H;L}\begin{bmatrix}(d_D) & : & (g_G) & ; & (k_K) & ; & \\ & & & & & & at,\ \frac{b}{t}+c\\ (e_E) & : & (h_H) & ; & (\ell_L) & ; & \end{bmatrix}$$

$$= \sum_{p=-\infty}^{\infty}\frac{\prod_{i=1}^{D}(d_i)_{p+2p^\star}\prod_{i=1}^{K}(k_i)_{p^\star}\prod_{i=1}^{G}(g_i)_{p+p^\star}}{\prod_{i=1}^{E}(e_i)_{p+2p^\star}\prod_{i=1}^{L}(\ell_i)_{p^\star}\prod_{i=1}^{H}(h_i)_{p+p^\star}}\frac{a^{p+p^\star}b^{p^\star}}{(p+p^\star)!(p^\star)!}\times$$

$$F^{D+K:G+1;0}_{E+L:H+2;0}\left(\begin{array}{l} [(d_D)+p+2p^\star:2,1],\ [(k_K)+p^\star:1,1] \ : \ [(g_G)+p+p^\star:1],\ [1:1] \ ; \ \text{——} \ ; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ab,\ c \\ [(e_E)+p+2p^\star:2,1],\ [(l_L)+p^\star:1,1] \ : \ [(h_H)+p+p^\star:1],\ [1+p+p^\star:1],\ [1+p^\star:1] \ ; \ \text{——} \ ; \end{array}\right)t^p, \tag{3.1}$$

where, for convergence, $t\neq 0$,

$$D+G<E+H+1,\ D+K<E+L+1,\ |at|<\infty,\ |\frac{b}{t}+c|<\infty,\ \textit{or}$$

$$D+G=E+H+1,\ D+K=E+L+1\ \text{and}$$

$$\begin{cases}|at|^{1/(D-E)}+|\frac{b}{t}+c|^{1/(D-E)}<1, & \textit{if}\ D>E\\ \max\left\{|at|,|\frac{b}{t}+c|\right\}<1, & \textit{if}\ D\leq E\end{cases}$$

and $p^\star$ *is defined by Eq. (1.22).*

Proof Suppose the power series form of l.h.s. of Eq. (3.1) is denoted by Ω. Then, we have

$$\Omega=\sum_{m=0}^{\infty}\sum_{N=0}^{\infty}\frac{\prod_{i=1}^{D}(d_i)_{m+N}\prod_{i=1}^{G}(g_i)_m\prod_{i=1}^{K}(k_i)_N}{\prod_{i=1}^{E}(e_i)_{m+N}\prod_{i=1}^{H}(h_i)_m\prod_{i=1}^{L}(\ell_i)_N}\frac{(at)^m}{m!}\frac{(\frac{b}{t}+c)^N}{N!}. \tag{3.2}$$

Now, using Lemma 1.1 in Eq. (3.2), we get

$$\Omega=\sum_{m=0}^{\infty}\sum_{n,r=0}^{\infty}\frac{\prod_{i=1}^{D}(d_i)_{m+n+r}\prod_{i=1}^{G}(g_i)_m\prod_{i=1}^{K}(k_i)_{n+r}}{\prod_{i=1}^{E}(e_i)_{m+n+r}\prod_{i=1}^{H}(h_i)_m\prod_{i=1}^{L}(\ell_i)_{n+r}}\frac{(at)^m}{m!}\frac{(\frac{b}{t})^n c^r}{n!r!}$$

$$=\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{D}(d_i)_r\prod_{i=1}^{K}(k_i)_r c^r}{\prod_{i=1}^{E}(e_i)_r\prod_{i=1}^{L}(\ell_i)_r r!}\sum_{m,n=0}^{\infty}\frac{\prod_{i=1}^{D}(d_i+r)_{m+n}\prod_{i=1}^{G}(g_i)_m\prod_{i=1}^{K}(k_i+r)_n}{\prod_{i=1}^{E}(e_i+r)_{m+n}\prod_{i=1}^{H}(h_i)_m\prod_{i=1}^{L}(\ell_i+r)_n}\frac{a^m}{m!}\frac{b^n}{n!}t^{m-n}. \tag{3.3}$$

On applying general double series identity (2.1) in Eq. (3.3), we obtain

$$\Omega = \sum_{r=0}^{\infty} \frac{\prod_{i=1}^{D}(d_i)_r \prod_{i=1}^{K}(k_i)_r c^r}{\prod_{i=1}^{E}(e_i)_r \prod_{i=1}^{L}(\ell_i)_r r!} \sum_{p=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{D}(d_i+r)_{2n+p+2p^\star} \prod_{i=1}^{K}(k_i+r)_{n+p^\star}}{\prod_{i=1}^{E}(e_i+r)_{2n+p+2p^\star} \prod_{i=1}^{L}(\ell_i+r)_{n+p^\star}} \frac{\prod_{i=1}^{G}(g_i)_{n+p+p^\star}}{\prod_{i=1}^{H}(h_i)_{n+p+p^\star}} \times$$

$$\frac{a^{n+p+p^\star}}{(n+p+p^\star)!} \frac{b^{n+p^\star}}{(n+p^\star)!} t^p$$

$$= \sum_{p=-\infty}^{\infty} \sum_{n,r=0}^{\infty} \frac{\prod_{i=1}^{D}(d_i)_{2n+r+p+2p^\star} \prod_{i=1}^{K}(k_i)_{n+r+p^\star}}{\prod_{i=1}^{E}(e_i)_{2n+r+p+2p^\star} \prod_{i=1}^{L}(\ell_i)_{n+r+p^\star}} \frac{\prod_{i=1}^{G}(g_i)_{n+p+p^\star}}{\prod_{i=1}^{H}(h_i)_{n+p+p^\star}} \frac{a^{n+p+p^\star}}{(n+p+p^\star)!} \frac{b^{n+p^\star}}{(n+p^\star)!} t^p$$

$$= \sum_{p=-\infty}^{\infty} \frac{\prod_{i=1}^{D}(d_i)_{p+2p^\star} \prod_{i=1}^{K}(k_i)_{p^\star} \prod_{i=1}^{G}(g_i)_{p+p^\star}}{\prod_{i=1}^{E}(e_i)_{p+2p^\star} \prod_{i=1}^{L}(\ell_i)_{p^\star} \prod_{i=1}^{H}(h_i)_{p+p^\star}} \frac{a^{p+p^\star} b^{p^\star}}{(p+p^\star)!(p^\star)!} \times$$

$$\sum_{n,r=0}^{\infty} \frac{\prod_{i=1}^{D}(d_i+p+2p^\star)_{2n+r} \prod_{i=1}^{K}(k_i+p^\star)_{n+r} \prod_{i=1}^{G}(g_i+p+p^\star)_n}{\prod_{i=1}^{E}(e_i+p+2p^\star)_{2n+r} \prod_{i=1}^{L}(\ell_i+p^\star)_{n+r} \prod_{i=1}^{H}(h_i+p+p^\star)_n} \frac{(1)_n}{(1+p+p^\star)_n(1+p^\star)_n} \frac{(ab)^n}{n!} \frac{c^r}{r!} t^p.$$

Now, using the definition of Srivastava-Daoust double hypergeometric function (1.6), we obtain assertion (3.1). $\square$

Remark 3.2 Taking $c = 0$ in Eq. (3.1), the following consequence of Theorem 3.1 is deduced:

Corollary 3.3 *The following generating function (in terms of Kampé de Fériet double hypergeometric function) for generalized hypergeometric function holds true:*

$$F^{D:G;K}_{E:H;L}\left[\begin{array}{ccccccc} (d_D) & : & (g_G) & ; & (k_K) & ; & \\ & & & & & & at \ , \ \frac{b}{t} \\ (e_E) & : & (h_H) & ; & (\ell_L) & ; & \end{array}\right]$$

$$= \sum_{p=-\infty}^{\infty} \frac{\prod_{i=1}^{D}(d_i)_{p+2p^\star} \prod_{i=1}^{G}(g_i)_{p+p^\star} \prod_{i=1}^{K}(k_i)_{p^\star}}{\prod_{i=1}^{E}(e_i)_{p+2p^\star} \prod_{i=1}^{H}(h_i)_{p+p^\star} \prod_{i=1}^{L}(\ell_i)_{p^\star}} \frac{a^{p+p^\star} b^{p^\star}}{(p+p^\star)!(p^\star)!} \times$$

$${}_{2D+G+K+1}F_{2E+H+L+2}\left[\begin{array}{l} \frac{d_1+p+2p^\star}{2}, \ldots, \frac{d_D+p+2p^\star}{2}, \frac{d_1+p+2p^\star+1}{2}, \ldots, \frac{d_D+p+2p^\star+1}{2}, \\ \\ \frac{e_1+p+2p^\star}{2}, \ldots, \frac{e_E+p+2p^\star}{2}, \frac{e_1+p+2p^\star+1}{2}, \ldots, \frac{e_E+p+2p^\star+1}{2}, \end{array}\right.$$

$$\left.\begin{array}{ll} g_1+p+p^\star, \ldots, g_G+p+p^\star, k_1+p^\star, \ldots, k_K+p^\star, \ 1 & ; \\ & 2^{2(D-E)}ab \\ h_1+p+p^\star, \ldots, h_H+p+p^\star, \ell_1+p^\star, \ldots, \ell_L+p^\star, \ 1+p+p^\star, \ 1+p^\star & ; \end{array}\right] t^p, \tag{3.4}$$

where, for convergence, $t \neq 0$,

$$D+G < E+H+1,\ D+K < E+L+1,\ |at| < \infty,\ |\frac{b}{t}| < \infty,\ \text{or}$$
$$D+G = E+H+1,\ D+K = E+L+1 \text{ and}$$

$$\begin{cases} |at|^{1/(D-E)} + \left|\frac{b}{t}\right|^{1/(D-E)} < 1, & \text{if } D > E \\ \max\left\{|at|, \left|\frac{b}{t}\right|\right\} < 1, & \text{if } D \leq E \end{cases}$$

and $p^\star$ *is defined by Eq. (1.22).*

In the next section, some special cases of the results obtained in Section 3 are discussed.

4. Special cases

(1) Taking $D = E = H = L = 0$, $G = K = 1$; $g_1 = \alpha$, $k_1 = \beta$; $a = x$, $b = y$ in Eq. (3.4), we obtain

$$(1-xt)^{-\alpha}(1-\frac{y}{t})^{-\beta} = \sum_{p=-\infty}^{\infty} \frac{(\alpha)_{p+p^\star}(\beta)_{p^\star}}{(p+p^\star)!(p^\star)!} x^{p+p^\star} y^{p^\star}\ {}_3F_2\left[\begin{array}{c} \alpha+p+p^\star,\ \beta+p^\star,\ 1\ ; \\ \quad\quad xy \\ 1+p+p^\star,\ 1+p^\star\ ; \end{array}\right] t^p, \tag{4.1}$$

provided that $|y| < |t| < |x|^{-1}$; $t \neq 0$, which is the modified form of Eq. (1.15).

(2) Taking $D = E = G = H = K = L = 0$ in Eq. (3.4), we get

$$\exp(at + \frac{b}{t}) = \sum_{p=-\infty}^{\infty} \frac{a^{p+p^\star} b^{p^\star}}{(p+p^\star)!(p^\star)!}\ {}_1F_2\left[\begin{array}{c} 1\ ; \\ \quad\quad ab \\ 1+p+p^*\ ,\ 1+p^*\ ; \end{array}\right] t^p, \tag{4.2}$$

where $t \neq 0$ and $\forall$ finite values of a, b and t.

(i) Putting $a = \frac{x}{2}$, $b = -\frac{x}{2}$ in Eq. (4.2), we get

$$\exp(\frac{x}{2}(t - \frac{1}{t})) = \sum_{p=-\infty}^{\infty} \frac{(-1)^{p^\star}(\frac{x}{2})^{p+2p^\star}}{(p+p^\star)!(p^\star)!}\ {}_1F_2\left[\begin{array}{c} 1\ ; \\ \quad\quad -\frac{x^2}{4} \\ 1+p+p^\star, 1+p^*\ ; \end{array}\right] t^p, \tag{4.3}$$

which is the modified form of generating relation (1.7) associated with Bessel function $J_n(x)$ and $t \neq 0$, $|x| < \infty$.

(ii) Putting $a = 1$, $b = -x$ in Eq. (4.2), we get

$$\exp(t - \frac{x}{t}) = \sum_{p=-\infty}^{\infty} \frac{(-x)^{p^\star}}{(p+p^\star)!(p^\star)!}\ {}_1F_2\left[\begin{array}{c} 1\ ; \\ \quad\quad -x \\ 1+p+p^\star, 1+p^*\ ; \end{array}\right] t^p, \tag{4.4}$$

which is the modified form of generating relation (1.10) associated with Tricomi function $C_n(x)$ and $t \neq 0$, $|x| < \infty$.

(3) Taking $D = E = H = L = 0$, $G = K = 1$; $g_1 = 1 + \alpha + \nu$, $k_1 = -\nu$, $a = \frac{-\beta\gamma}{1+\gamma c}$, $b = \frac{-c}{\beta}$ in

Eq. (3.4), we obtain

$$ {}_1F_0\left[\begin{array}{lc} -\nu\ ; & \\ & -\frac{c}{\beta t} \\ -; & \end{array}\right] {}_1F_0\left[\begin{array}{lc} \alpha+\nu+1\ ; & \\ & \frac{-\beta\gamma t}{1+\gamma c} \\ -\ ; & \end{array}\right] = \sum_{p=-\infty}^{\infty} \frac{(1+\alpha+\nu)_{p+p^\star}(-\nu)_{p^\star}}{(p+p^\star)!(p^\star)!}\times $$

$$ {}_3F_2\left[\begin{array}{lc} 1+\alpha+\nu+p+p^\star,\ -\nu+p^\star,\ 1\ \ ; & \\ & \frac{\gamma c}{1+\gamma c} \\ 1+p+p^\star,\ 1+p^\star\ \ ; & \end{array}\right] (\frac{\gamma c}{1+\gamma c})^{p^\star} (\frac{-\beta\gamma}{1+\gamma c})^{p} t^{p}, \tag{4.5} $$

provided that

$$ |\frac{\beta\gamma t}{1+\gamma c}|<1,\ |\frac{c}{\beta t}|<1,\ -\pi<\arg(\beta),\ \arg(\frac{1+\gamma c}{\beta})<\pi, $$

which is the modified form of Eq. (1.17).

(4) Taking $D=E=H=0$, $G=K=L=1$; $g_1=\beta$, $k_1=\alpha$, $l_1=1-\beta$; $a=x$, $b=y$ and $c=-xy$ in Eq. (3.1), we obtain

$$ (1-xt)^{-\beta}\ {}_1F_1\left[\begin{array}{lc} \alpha\ ; & \\ & \frac{y(1-xt)}{t} \\ 1-\beta\ ; & \end{array}\right] = \sum_{p=-\infty}^{\infty} \frac{(\alpha)_{p^\star}(\beta)_{p+p^\star}}{(1-\beta)_{p^\star}} \frac{x^{p+p^\star}y^{p^\star}}{(p+p^\star)!(p^\star)!}\times $$

$$ F^{1:2;0}_{1:2;0}\left[\begin{array}{lc} \alpha+p^\star\ :\ \beta+p+p^\star,\ 1\ ;\ \text{—}\ ; & \\ & xy,\ -xy \\ 1-\beta+p^\star\ :\ 1+p+p^\star,\ 1+p^\star\ ;\ \text{—}\ ; & \end{array}\right] t^p, \tag{4.6} $$

provided that $|xy|<\infty$; $t\neq 0$, which is the modified form of Eq. (1.16).

5. Conclusion

We conclude our present investigation by observing that several other Laurent type hypergeometric generating relations and its applications can also be deduced in an analogous manner.

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