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Characterizations of Commutators of Singular Integral Operators on Variable Exponent Spaces

Chenglong FANG
Department of Mathematics, Yili Normal University, Xinjiang 835000*, P. R. China*

Abstract The main purpose of this paper is to characterize the Lipschitz space by the boundedness of commutators on Lebesgue spaces and Triebel-Lizorkin spaces with variable exponent. Based on this main purpose, we first characterize the Triebel-Lizorkin spaces with variable exponent by two families of operators. Immediately after, applying the characterizations of Triebel-Lizorkin space with variable exponent, we obtain that $b \in \dot{\Lambda}_\beta$ if and only if the commutator of Calderón-Zygmund singular integral operator is bounded, respectively, from $L^{p(\cdot)}(\mathbb{R}^n)$ to $\dot{F}^{\beta,\infty}_{p(\cdot)}$, from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$ with $1/p(\cdot)-1/q(\cdot)=\beta/n$. Moreover, we prove that the commutator of Riesz potential operator also has corresponding results.

Keywords commutator; Lipschitz space; Triebel-Lizorkin space; variable exponent; singular integral operator

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1. Introduction and some definitions

In this paper, for $\beta>0$ the Lipschits space $\dot{\Lambda}_\beta$ is the space of functions f such that

$$\|f\|_{\dot{\Lambda}_\beta}=\sup_{x,h\in\mathbb{R}^n,h\neq 0}\frac{|f(x+h)-f(x)|}{|h|^\beta}<\infty.$$

In 1978, Janson [1] proved that $b\in\dot{\Lambda}_\beta$ if and only if the commutator of Calderón-Zygmund integral operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, $1<p<q<\infty$ and $\beta=n(1/p-1/q)$. In 1982, Chanillo [2] obtained that $b\in \mathrm{BMO}$ if and only if the commutator of Riesz potential operator I_α from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, $1<p<q<\infty$ and $\alpha=n(1/p-1/q)$. In 1995, Paluszynski [3] extended and generalized the results from [2] and [1], using a complete proof of the result of Chanillo in [2], showed that $b\in\dot{\Lambda}_\beta$ if and only if commutators of Calderón-Zygmund singular integral operator and Riesz potential operator are bounded from Lebesgue spaces to Lebesgue spaces or Triebel-Lizorkin spaces. In this article, we prove that the above results still hold in variable exponent. Namely, we show that $b\in\dot{\Lambda}_\beta$ if and only if commutators of Calderón-Zygmund singular integral operator and Riesz potential operator are bounded from Lebesgue spaces to Lebesgue spaces or Triebel-Lizorkin spaces in variable exponent.

Firstly, we give the definition of Lebesgue spaces with variable exponent as follows.

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E-mail address: fangclmath@126.com

Definition 1.1 *Let $p:\mathbb{R}^n\to[1,\infty)$ be a measurable function. $L^{p(\cdot)}(\mathbb{R}^n)$ denotes the set of all measurable functions f on $\mathbb{R}^n$ such that for some $\lambda>0$,*

$$L^{p(\cdot)}(\mathbb{R}^n)=\Big\{f:\int_{\mathbb{R}^n}|\frac{f(x)}{\lambda}|^{p(x)}\mathrm{d}x<\infty\Big\}$$

and

$$\|f\|_{L^{p(\cdot)}}=\inf\Big\{\lambda>0:\ \int_{\mathbb{R}^n}(\frac{|f(x)|}{\lambda})^{p(x)}\mathrm{d}x\le 1\Big\}.$$

Then $L^{p(\cdot)}(\mathbb{R}^n)$ is Banach space with the norm $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions p on $\mathbb{R}^n$ with range in $[1,\infty)$ such that $1<p^-=\operatorname{ess\,inf}_{x\in\mathbb{R}^n}p(x)$, $\operatorname{ess\,sup}_{x\in\mathbb{R}^n}p(x)=p^+<\infty$. Moreover, we define $\mathcal{P}^0(\mathbb{R}^n)$ to be the set of all measurable functions p on $\mathbb{R}^n$ with range in $(0,\infty)$ such that $0<p^-=\operatorname{ess\,inf}_{x\in\mathbb{R}^n}p(x)$, $\operatorname{ess\,sup}_{x\in\mathbb{R}^n}p(x)=p^+<\infty$. Given $p(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$, one can define the space $L^{p(\cdot)}(\mathbb{R}^n)$ as above. This is equivalent to defining it to be the set of all functions f such that $|f|^{p_0}\in L^{q(\cdot)}(\mathbb{R}^n)$, where $0<p_0<p^-$, and $q(\cdot)=\frac{p(\cdot)}{p_0}\in\mathcal{P}(\mathbb{R}^n)$. Then one can define a quasi-norm on this space by

$$\|f\|_{L^{p(\cdot)}}=\||f|^{p_0}\|_{L^{q(\cdot)}}^{1/p_0}.$$

Now, we give some definitions of operators in this article. Meanwhile, some results of the boundedness of operators are given.

Definition 1.2 *Let $f\in L^1_{\rm loc}(\mathbb{R}^n)$. The standard Hardy-Littlewood maximal operator is defined by*

$$\mathcal{M}f(x)=\sup_{r>0}r^{-n}\int_{B_r(x)}|f(y)|\mathrm{d}y,$$

where $B_r(x)=B(x,r)=\{y\in\mathbb{R}^n:|x-y|<r\}$. The key tool we need is the boundedness of the Hardy-Littlewood maximal operator on variable exponent function spaces. There exist some sufficient conditions on $p(\cdot)$ such that the maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$; see for example [4,5]. $\mathcal{B}(\mathbb{R}^n)$ is the set of all $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

The standard fractional maximal operator is defined by

$$\mathcal{M}_\alpha f(x)=\sup_{r>0}r^{-n+\alpha}\int_{B_r(x)}|f(y)|\mathrm{d}y,$$

where $0<\alpha<n$. For $1/p(\cdot)-1/q(\cdot)=\alpha/n$, the fractional maximal operator $\mathcal{M}_\alpha$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$.

Definition 1.3 *(i) A continuous function $g:\mathbb{R}^n\to\mathbb{R}$ is called locally log-Hölder continuous, abbreviated $g\in\mathcal{C}^{\log}_{\rm loc}(\mathbb{R}^n)$, if there exists $c_{\log}>0$ such that for all $x,y\in\mathbb{R}^n$,*

$$|g(x)-g(y)|\le\frac{c_{\log}}{\log(e+1/|x-y|)}.$$

(ii) The function g is called globally log-Hölder continuous, abbreviated $g\in\mathcal{C}^{\log}(\mathbb{R}^n)$, if g is locally log-Hölder continuous and there exist $g_\infty\in\mathbb{R}$ and $C_{\log}>0$ such that for all $x\in\mathbb{R}^n$,

$$|g(x)-g_\infty|\le\frac{C_{\log}}{\log(e+|x|)},$$

where $g_\infty = \lim_{|x|\to\infty} g(x)$. *If* $q \in \mathcal{C}^{\log}(\mathbb{R}^n)$, *then for every* $q_0 < q^-$ *we have* $q(\cdot)/q_0 \in \mathcal{B}(\mathbb{R}^n)$. *The notation* $\mathcal{P}^{\log}(\mathbb{R}^n)$ *is used for those variable exponents* $p \in \mathcal{P}^0(\mathbb{R}^n)$ *with* $\frac{1}{p} \in \mathcal{C}^{\log}$. *If* $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, *then we have for every* $p_0 < p^-$ *that* $\mathcal{M}$ *is bounded on* $L^{p(\cdot)/p_0}(\mathbb{R}^n)$ *or, equivalently, that* $\mathcal{M}_t$ *is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$, *where* $t = \min(1, p_0)$.

Definition 1.4 *Given a locally integrable function* K *defined on* $\mathbb{R}^n\backslash\{0\}$, *suppose that the Fourier transform of* K *is bounded, and* K *satisfies*

$$|K(x)| \le \frac{C}{|x|^n}, \quad |\nabla K(x)| \le \frac{C}{|x|^{n+1}}, \quad x \ne 0.$$

Then the singular integral operator T, *defined by* $Tf(x) = K * f(x)$, *is bounded on variable* $L^{p(\cdot)}(\mathbb{R}^n)$ *if* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ *(see [6]).*

Let $b \in L^1_{\text{loc}}$. The commutator $\big[b, T\big]$ is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x). \tag{1.1}$$

Definition 1.5 *For* $0 < \alpha < n$, *the Riesz potential operator* I_α *is defined by*

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \mathrm{d}y. \tag{1.2}$$

Further, if $1/q(\cdot) = 1/p(\cdot) - \alpha/n$, I_α *from the space* $L^{p(\cdot)}(\mathbb{R}^n)$ *into the space* $L^{q(\cdot)}(\mathbb{R}^n)$ *(see, [7]).*

Let $b \in L^1_{\text{loc}}$ and I_α be a Riesz potential operator. The commutator $[b, I_\alpha]$ is defined by

$$[b, I_\alpha]f(x) = b(x)I_\alpha f(x) - I_\alpha(bf)(x).$$

If $b \in \dot{\Lambda}_\beta$ and $1/q(\cdot) - 1/p(\cdot) = (\alpha+\beta)/n$, then $[b, I_\alpha] : L^{p(\cdot)}(\mathbb{R}^n) \to L^{q(\cdot)}(\mathbb{R}^n)$ (see, [8]).

The organization of this paper is as follows. In Section 1, some background material and definitions are given. In order to prove

$$b \in \dot{\Lambda}_\beta \Leftrightarrow [b, T] : L^{p(\cdot)}(\mathbb{R}^n) \to F^{\beta,\infty}_{p(\cdot)} \Leftrightarrow [b, I_\alpha] : L^{p(\cdot)}(\mathbb{R}^n) \to F^{\beta,\infty}_{q(\cdot)},$$

we first characterize the Triebel-Lizorkin spaces with variable exponents by two families of operators in Section 2. In Section 3, we characterize the Lipschitz space by the boundedness of commutators of singular integral operator and Riesz potential operator in variable exponent.

As usual, we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space. Use c as a generic positive constant, and denote simply by $A \lesssim B$ if there exists a constant $c_1 > 0$ such that $A \le c_1 B$. Further, $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. For a set A, χ_A denotes its characteristic function. The set $\mathcal{S}$ denotes the usual Schwartz class of infinitely differentiable rapidly decreasing complex-valued functions, by $\mathcal{S}'$ we denote its dual space. The Fourier transform of a tempered distribution f is denoted by $\hat{f}$ while its inverse transform is denoted by $\check{f}$.

2. Characterizations of Triebel-Lizorkin spaces with variable exponent

To discuss variable exponent Triebel-Lizorkin spaces, we first need to consider the following sequences function space. $L_{p(\cdot)}(l_q)$ is the space of all sequences $\{g_j\}$ of measurable functions on

$\mathbb{R}^n$ with finite quasi-norms

$$\|\{g_j\}\|_{L_{p(\cdot)}(l_q)} = \|\|\{g_j\}\|_{l_q}\|_{L^{p(\cdot)}} = \Big\|\Big(\sum_{j=1}^{\infty}(g_j)^q\Big)^{1/q}\Big\|_{L^{p(\cdot)}}.$$

We now recall the Fourier analytical approach to function spaces of Triebel-Lizorkin. Let $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi_0 \geq 0$ and satisfy the following conditions:

$$\varphi_0(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2. \end{cases}$$

Set $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$ with $x \in \mathbb{R}^n$. For $j \in \mathbb{N}$, let $\varphi_j(x) = \varphi(2^{-j}x)$. Then we call $\{\varphi_j\}_{j\in\mathbb{N}_0}$ a resolution of unity, it follows that

$$\sum_{j=0}^{\infty}\varphi_j(x) = 1.$$

Definition 2.1 *Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity, $0 < \beta, q \leq \infty$, $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$. The set*

$$\Big\{f \in \mathcal{S}^{'}(\mathbb{R}^n): \ \Big\|\Big(\sum_{j=0}^{\infty}|2^{j\beta}(\varphi_j * \widehat{f})^{\vee}|^q\Big)^{1/q}\Big\|_{L^{p(\cdot)}}\Big\}$$

is named to be the Triebel-Lizorkin space with variable exponent and denoted by $F^{\beta,q}_{p(\cdot)}$. The quasi-norm of f in this space is denoted by

$$\|f\|_{F^{\beta,q}_{p(\cdot)}} = \|2^{j\beta}(\varphi_j * \widehat{f})^{\vee}\|_{L_{p(\cdot)}(l_q)} = \Big\|\Big(\sum_{j=0}^{\infty}|2^{j\beta}(\varphi_j * \widehat{f})^{\vee}|^q\Big)^{1/q}\Big\|_{L^{p(\cdot)}}.$$

Remark 2.2 By [9, Proposition 6.4], we know that the Sobolev type embedding inequality of Triebel-Lizorkin spaces with variable exponent as follows.

If

$$\beta_0 \geq \beta_1 \ \text{ and } \ \frac{1}{p_0(\cdot)} - \frac{1}{p_1(\cdot)} = \frac{\beta_0 - \beta_1}{n},$$

then

$$F^{\beta_0,q_0}_{p_0(\cdot)}(\mathbb{R}^n) \hookrightarrow F^{\beta_1,q_1}_{p_1(\cdot)}(\mathbb{R}^n).$$

Thus, for $\beta_1 = 0$, $q_0 = \infty$ and $q_1 = 2$, then $F^{\beta_0,\infty}_{p_0(\cdot)}(\mathbb{R}^n) \hookrightarrow F^{0,2}_{p_1(\cdot)}(\mathbb{R}^n) = L^{p_1(\cdot)}$. This fact will be used in the next section.

Below, the characterizations of Triebel-Lizorkin spaces with variable exponent are given. Namely, we will characterize Triebel-Lizorkin spaces with variable exponent by two families of operators. To this aim we need the property of Peetre maximal operator on Triebel-Lizorkin spaces with variable exponent and the boundedness of Hardy-Littlewood maximal operator on sequences function space. Thus, we give some notations and facts as follows.

Let $\psi_0, \psi_1 \in \mathcal{S}(\mathbb{R}^n)$, $\varepsilon > 0$, integer $R \geq -1$ be such that

$$|\psi_0(x)| > 0 \text{ on } \{x \in \mathbb{R}^n : |x| < \varepsilon\}, \ \ |\psi_1(x)| > 0 \text{ on } \{x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon\} \tag{2.1}$$

and

$$D^{\beta}\psi(0) = 0, \text{ for } 0 \leq |\beta| < R. \tag{2.2}$$

Here (2.1) are Tauberian conditions, while (2.2) expresses moment conditions on ψ.

Let us recall the classical Peetre maximal operator, introduced in [10]. In the following we define the system of maximal functions. Given a sequence of function $\{\Psi_j\}_j \subset \mathcal{S}(\mathbb{R}^n)$, a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ and a positive number $a > 0$, the Peetre's maximal functions are defined as

$$(\Psi_j^*)_a(x) = \sup_{y\in\mathbb{R}^n} \frac{|\Psi_j * f(y)|}{1+|2^j(x-y)|^a}, \quad x \in \mathbb{R}^n,\ j \in \mathbb{N}_0.$$

Now, we give the property of Peetre maximal operator on Triebel-Lizorkin spaces with variable exponent, and the boundedness of Hardy-Littlewood maximal operator on vector-valued function space as follows.

Lemma 2.3 ([11]) *Let $\beta < R+1$, $0 < q \le \infty$ and $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ with $p_0 < p^-$ such that $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$. If $n/a < p_0$, then for $f \in \mathcal{S}'(\mathbb{R}^n)$*

$$\|f\|_{F^{\beta,q}_{p(\cdot)}} \sim \|2^{j\beta}(\Psi_k^* f)_a\|_{L_{p(\cdot)}(l_q)} \sim \|2^{j\beta}(\Psi_k * f)\|_{L_{p(\cdot)}(l_q)}.$$

Lemma 2.4 ([6]) *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $1 < q \le \infty$, then there exists a positive constant C such that for all sequences $\{f_j\}_{j=0}^\infty$ of locally integrable functions,*

$$\|\{\mathcal{M}f_j\}_{j=0}^\infty\|_{L_{p(\cdot)}(l_q)} \le C\|\{f_j\}_{j=0}^\infty\|_{L_{p(\cdot)}(l_q)}.$$

To give our characterizations we define by Δ_h^k the difference operator. That is

$$\Delta_h^1 f(x) = \Delta_h f(x) = f(x+h) - f(x),$$

$$\Delta_h^{k+1} f(x) = \Delta_h^k f(x+h) - \Delta_h^k f(x), \quad k \ge 1.$$

$Q(x,t)$ denotes a cube centered at x, with side length t, sides parallel to the axes, we write $Q_x(t) = Q(x,t)$.

Consider the family of operators $S^\beta_{q,r,m}$, defined by

$$S^\beta_{q,r,m} f(x) = \Big(\int_0^\infty \Big(\frac{1}{|Q_0(t)|}\int_{Q_0(t)} |\Delta_h^m f(x)|^r \mathrm{d}h\Big)^{q/r} \frac{\mathrm{d}t}{t^{1+\beta q}}\Big)^{1/q}.$$

For a fixed cube $Q = Q_x(t)$, we define the oscillation

$$\mathrm{osc}_r^m(f,Q) = \mathrm{osc}_r^m(f,x,t) = \inf_{P\in P^m}\Big(\frac{1}{|Q|}\int_Q |f(y)-P(y)|^r \mathrm{d}y\Big)^{1/r},$$

where the infimum is taken over all polynomials of degree not exceeding m. Further, we define the family of operators

$$\mathcal{D}^\beta_{q,r,m} f(x) = \Big(\int_0^\infty (\mathrm{osc}_r^{m-1}(f,x,t))^q \frac{\mathrm{d}t}{t^{1+\beta q}}\Big)^{1/q}.$$

For $q = \infty$ or $r = \infty$ we have the usual modifications and replace integrations by sup-norms. Some properties of the above two families of operators can be found in [12] and [13].

In the following we use the above two families of operators to characterize Triebel-Lizorkin spaces with variable exponent.

Theorem 2.5 *For $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < q \le \infty$, $m > \beta/a_0$, $\nu \in \mathbb{R}$, $r \ge 1$, if*

$$\beta > \sigma_{p,q,r} = \max\{0, \nu(\frac{1}{p^-} - \frac{1}{r}), \nu(\frac{1}{q} - \frac{1}{r})\},$$

then $\|f\|_{\dot{F}^{\beta,q}_{p(\cdot)}} \sim \|S^{\beta}_{q,r,m} f\|_{L^{p(\cdot)}} \sim \|\mathcal{D}^{\beta}_{q,r,m} f\|_{L^{p(\cdot)}}$. Where $a_0 > 0$, ν is trace of a matrix [14].

Proof The proof follows the ideas in [14]. The whole proof is divided into three steps, which together give the proof of the theorem.

Step 1. To prove $\|S^{\beta}_{q,r,m} f\|_{L^{p(\cdot)}} \le c\|\mathcal{D}^{\beta}_{q,r,m} f\|_{L^{p(\cdot)}}$, we choose best approximants $P_t f$ in f in $L^1(Q_x(t))$. Since $\Delta^m_h P \equiv 0$ for all polynomials P of degree less than m, we may split

$$\Delta^m_h f(x) = (-1)^m (f(x) - P_t f(x)) + \sum_{j=1}^{m} (-1)^{m-j} C^j_k (f(x + jh) - P_t f(x + jh)).$$

Thus, we have

$$\begin{aligned} S^{\beta}_{q,r,m} f(x) \le & \Big(\int_0^\infty \Big(\frac{1}{|Q_0(t)|}\int_{Q_0(t)} |f(x) - P_t f(x)|^r \mathrm{d}h\Big)^{q/r} \frac{\mathrm{d}t}{t^{1+\beta q}}\Big)^{1/q} + \\ & \Big(\int_0^\infty \Big(\frac{1}{|Q_0(t)|}\int_{Q_0(t)} \Big|\sum_{j=1}^{m} C^j_k (f(x + jh) - P_t f(x + jh))\Big|^r \mathrm{d}h\Big)^{q/r} \frac{\mathrm{d}t}{t^{1+\beta q}}\Big)^{1/q} \\ = & I_0 + I_j. \end{aligned}$$

To estimate I_0, we use the following facts [13],

$$\lim_{l\to\infty} P_{t2^{-l}} f(x) = f(x), \quad |P_t f(x)| \le \frac{1}{|Q_x(t)|}\int_{Q_x(t)} |f(y)|\mathrm{d}y.$$

If $q \le 1$, we have

$$\begin{aligned} & \Big(\frac{1}{|Q_0(t)|}\int_{Q_0(t)} |f(x) - P_t f(x)|^r \mathrm{d}h\Big)^{q/r} \\ \le & \Big(\frac{1}{|Q_0(t)|}\int_{Q_0(t)} \Big(\sum_{l=0}^{\infty} |P_{2^{-l-1}t} f(x) - P_{2^{-l}t} f(x)|\Big)^r \mathrm{d}h\Big)^{q/r} \\ \le & \Big(\frac{1}{|Q_0(t)|}\int_{Q_0(t)} \Big(\sum_{l=0}^{\infty} \frac{1}{|Q_x(2^{-l}t)|}\int_{Q_x(2^{-l}t)} |f(y) - P_{2^{-l}t} f(y)|\mathrm{d}y\Big)^r \mathrm{d}h\Big)^{q/r} \\ \le & \Big(\frac{1}{|Q_x(t)|}\int_{Q_x(t)} |f(y) - P_t f(y)|\mathrm{d}y\Big)^q. \end{aligned}$$

Thus, we obtain

$$I_0 \le c\mathcal{D}^{\beta}_{q,1,m} f(x) \le c\mathcal{D}^{\beta}_{q,r,m} f(x).$$

If $q \ge 1$, we apply Minkowski's inequality to get the same result.

To estimate I_j, clearly, if $j \ge 1$,

$$I_j \le c\mathcal{D}^{\beta}_{q,r,m} f.$$

Step 2. To prove $\|\mathcal{D}^{\beta}_{q,r,m} f\|_{L^{p(\cdot)}} \le c\|f\|_{\dot{F}^{\beta,q}_{p(\cdot)}}$, by the proof of [14, Theorem 1], we can obtain

$$\|\mathcal{D}^{\beta}_{q,r,m} f\|_{L^{p(\cdot)}} \le c\|\mathcal{D}^{\beta}_{q,\tau,m} f\|_{L^{p(\cdot)}},$$

where $\tau > 0$ with $\tau < \min(1, q, p^-)$.

Now we decompose

$$f = f_{0,t} + f_{1,t}, \quad f_{0,t} = \sum_{2^k t \geq 1} \psi_k * f.$$

Then

$$\begin{aligned}\mathcal{D}^{\beta}_{q,\tau,m} f(x) \leq & \Big(\int_0^\infty (\mathrm{osc}^{m-1}_{\tau}(f_{0,t}, x, t))^q \frac{\mathrm{d}t}{t^{1+\beta q}} \Big)^{1/q} + \Big(\int_0^\infty (\mathrm{osc}^{m-1}_{\tau}(f_{1,t}, x, t))^q \frac{\mathrm{d}t}{t^{1+\beta q}} \Big)^{1/q} \\ =: & I + II.\end{aligned}$$

To estimate I, let $a > 0$. Since $\mathrm{osc}^{m-1}_{\tau}(f_{0,t}, x, t) \leq (\mathcal{M}(f^{\tau}_{0,t}))^{1/\tau}$ and $\tau < q$, we may apply Lemma 2.4 to get

$$\|I\|_{L^{p(\cdot)}} \leq c \Big\| \Big(\int_0^\infty \Big(\sum_{2^k t \geq 1} \Psi_k * f \Big)^q \frac{\mathrm{d}t}{t^{1+\beta q}} \Big)^{1/q} \Big\|_{L^{p(\cdot)}} \leq c \Big\| \Big(\sum_k 2^{kaq} |\Psi_k * f|^q \Big)^{1/q} \Big\|_{L^{p(\cdot)}}.$$

To estimate II, obviously, we can get

$$\mathrm{osc}^{m-1}_{\tau}(f_{1,t}, x, t) \leq c \Big(\int_{Q_x(t)} \Big| \sum_{2^k t \leq 1} (2^k t)^{m a_0} (\Psi^*_k f)_a(z) \Big|^{\tau} \mathrm{d}z \Big)^{1/\tau},$$

where $Q_x(t)$ is a cube with x as its center and t as its side-length.

According to Lemma 2.3, we have

$$\begin{aligned}\|II\|_{L^{p(\cdot)}} \leq & c \Big\| \Big(\int_0^\infty \Big| \sum_{2^k t \leq 1} (2^k t)^{m a_0} (\Psi^*_k f)_a \Big|^q \frac{\mathrm{d}t}{t} \Big)^{1/q} \Big\|_{L^{p(\cdot)}} \\ \leq & c \Big\| \Big(\sum_k 2^{kaq} (\Psi^*_k f)^q_a \Big)^{1/q} \Big\|_{L^{p(\cdot)}} \\ \leq & c \|f\|_{\dot{F}^{\beta,q}_{p(\cdot)}}.\end{aligned}$$

Step 3. By the proof of [14, Theorem 1] and the lines of Triebel [15, pp.82, 103], it is obvious that $\|f\|_{\dot{F}^{\beta,q}_{p(\cdot)}} \leq c \|S^{\beta}_{q,r,m} f\|_{L^{p(\cdot)}}$. $\square$

Remark 2.6 Through the above theorem, for $q = \infty$, $0 < \beta < 1$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, we have

$$\|f\|_{\dot{F}^{\beta,\infty}_{p(\cdot)}} \approx \Big\| \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f - f_Q| \Big\|_{p(\cdot)}.$$

For a cube Q, we let

$$f_Q = \frac{1}{|Q|} \int_Q f(x) \mathrm{d}x.$$

3. Characterizations of the Lipschitz space

In this section, we characterize the Lipschitz space by using the boundedness of Commutators of Calderón-Zygmund singular integrals and Riesz potential operator. Firstly, we recall some lemmas, then give the main conclusions in this article (see Theorems 3.5 and 3.6). The proof of the Lemma 3.1 may be found in [13].

Lemma 3.1 *For $0<\beta<1$, $1<q\le\infty$, we have*

$$\|f\|_{\dot\Lambda_\beta}\approx\sup_Q\frac{1}{|Q|^{1+\beta/n}}\int_Q|f-f_Q|\approx\sup_Q\frac{1}{|Q|^{\beta/n}}\Big(\frac{1}{|Q|}\int_Q|f-f_Q|^q\Big)^{1/q},$$

for $q=\infty$ the formula should be interpreted appropriately.

According to the proof in [13, pp.71-72], under the certain conditions we have

$$\Big\|\sup_Q\frac{1}{|Q|^{1+\gamma/n}}\int_Q|h^Q|\Big\|_q\le C\Big\|\sup_Q\frac{1}{|Q|^{1+\gamma/n+\alpha/n}}\int_Q|h^Q|\Big\|_p,$$

where h^Q represent the function of defined on the cube Q.

By the argument same in the proof of the above fact with $L^{p(\cdot)}$ replaced by L^p, obviously, we can obtain the following lemma.

Lemma 3.2 *Let $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $1/p(\cdot)-1/q(\cdot)=\alpha/n$. Suppose for cach cube Q we have a function h^Q, defined on this cube. Then, for $\gamma\ge0$,*

$$\Big\|\sup_Q\frac{1}{|Q|^{1+\gamma/n}}\int_Q|h^Q|\Big\|_{q(\cdot)}\le C\Big\|\sup_Q\frac{1}{|Q|^{1+\gamma/n+\alpha/n}}\int_Q|h^Q|\Big\|_{p(\cdot)},$$

where the constant C depends only on p,q,α and n.

Very often we have to deal with the norm of characteristic functions on balls (or cubes) when studying the behavior of various exponents. In classical Lebesgue spaces the norm of such functions is easily calculated, but this is not the case when we consider variable exponents. The following lemma takes into account the norm of characteristic functions in the variable exponents.

Lemma 3.3 ([16]) *Let $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$. Then*

$$\|\chi_Q\|_{p(\cdot)}\sim\begin{cases}|Q|^{\frac{1}{p(\cdot)}}, & \text{if } |Q|\le 2^n \text{ and } x\in Q,\\ |Q|^{\frac{1}{p_\infty}}, & \text{if } |Q|\ge 1\end{cases}$$

for every cube (or ball) $Q\subset\mathbb{R}^n$.

For the norm of characteristic functions in the variable exponents we have simple norm estimates as follows.

Lemma 3.4 *Let $x_0,z_0\in\mathbb{R}^n$, $t>1$, and let $Q=Q(x_0,t)$, $Q^0=Q(x_0+z_0t,t)$.*

(i) If $p(\cdot),q'(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$, $\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)}=\frac{\beta}{n}$ with $\frac{1}{p^+}>\frac{\beta}{n}$, then

$$\|\chi_{Q^0}\|_{L^{p(\cdot)}}\|\chi_Q\|_{L^{q'(\cdot)}}\sim t^{n+\beta}.$$

(ii) If $p(\cdot),r'(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$, $\frac{1}{p(\cdot)}-\frac{1}{r(\cdot)}=\frac{\alpha+\beta}{n}$ with $\frac{1}{p^+}>\frac{\alpha+\beta}{n}$, then

$$\|\chi_{Q^0}\|_{L^{p(\cdot)}}\|\chi_Q\|_{L^{r'(\cdot)}}\sim t^{n+\alpha+\beta}.$$

Proof We just need to prove (i), and the proof of (ii) is similar. The case $|Q|\le2^n$ is obvious by Lemma 3.3.

In the case $|Q|>2^n$, according to $\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)}=\frac{\beta}{n}$, we have

$$\frac{1}{p_\infty}-\frac{1}{q_\infty}=\frac{\beta}{n}.$$

Thus, by Lemma 3.3, we obtain

$$\|\chi_{Q^0}\|_{L^{p(\cdot)}}\|\chi_Q\|_{L^{q'(\cdot)}} \sim t^{n+\frac{n}{p_\infty}-\frac{n}{q_\infty}} \sim t^{n+\beta}.$$

The proof is completed. □

The following two theorems are the main results of this paper, mainly to characterize the Lipschitz space by using the boundedness of Commutators of Calderón-Zygmund singular integrals and Riesz potential operator.

Theorem 3.5 *Let $0 < \beta < 1$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $p(\cdot), q'(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$. $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\beta}{n}$ with $\frac{1}{p^+} > \frac{\beta}{n}$. Then, the following conditions are equivalent:*

(a) $b \in \dot{\Lambda}_\beta$;

(b) $[b, T]$ is a bounded operator from $L^{p(\cdot)}(\mathbb{R}^n)$ to $F^{\beta,\infty}_{p(\cdot)}$;

(c) $[b, T]$ is a bounded operator from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$.

Proof Let $0 < \beta < 1$, $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\beta}{n}$ with $\frac{1}{p^+} > \frac{\beta}{n}$. Now, we go to prove that the (a) is equivalent to (b) and (c). Firstly, fix a cube $Q = Q(x_Q, t)$ and $x \in Q$. For $f \in L^{p(\cdot)}$ and let $f^0 = f\chi_{2Q}$, $f^\infty = f - f^0$.

(a)$\Rightarrow$ (b). According to (1.1), we have

$$[b, T]f = [b - b_Q, T]f,$$

so

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q |[b, T]f - ([b, T]f)_Q| \\
&= \frac{1}{|Q|^{1+\beta/n}}\int_Q |[b - b_Q, T]f - ([b - b_Q, T]f)_Q| \\
&\lesssim \frac{2}{|Q|^{1+\beta/n}}\int_Q |[b - b_Q, T]f - T((b - b_Q)f^\infty)(x_Q)| \\
&\lesssim \frac{1}{|Q|^{1+\beta/n}}\int_Q |(b - b_Q)Tf| + \frac{1}{|Q|^{1+\beta/n}}\int_Q |T((b - b_Q)f^0)| \\
&\quad \frac{1}{|Q|^{\beta/n}}\sup_{y\in Q} |T((b - b_Q)f^\infty)(y) - T((b - b_Q)f^\infty)(x_Q)| + \\
&= D_1 + D_2 + D_3.
\end{aligned}$$

First, we estimate D_1, using Lemma 3.1, it follows that

$$\begin{aligned}
\frac{1}{|Q|^{1+\beta/n}}\int_Q |(b - b_Q)Tf| &\lesssim \frac{1}{|Q|^{\beta/n}}\sup_{y\in Q}|b(y) - b_Q|(\frac{1}{|Q|}\int_Q |Tf|) \\
&\lesssim \|b\|_{\dot{\Lambda}_\beta} M(Tf)(x).
\end{aligned}$$

Thus, we obtain $D_1 \lesssim \|b\|_{\dot{\Lambda}_\beta} M(Tf)(x)$.

To estimate D_2, let $0 < t < p^-$, according to the boundedness of T, we have

$$D_2 \lesssim \frac{1}{|Q|^{1+\beta/n}}\Big(\int_Q |T((b - b_Q)f^0)|^t\Big)^{1/t}|Q|^{1-1/t}$$

$$
\begin{aligned}
&\lesssim |Q|^{-\beta/n-1/t}\Big(\int_Q |(b-b_Q)f^0|^t\Big)^{1/t}\\
&\lesssim |Q|^{-\beta/n}\sup_{y\in Q}|b(y)-b_Q|\Big(\frac{1}{|Q|}\int_Q |f|^t\Big)^{1/t}\\
&\lesssim \|b\|_{\dot\Lambda_\beta}(M(|f|^t))^{1/t}(x).
\end{aligned}
$$

We now estimate D_3. First, we need the following well-known fact. Let $Q^*\subset Q$. Then

$$|b_{Q^*}-b_Q|\lesssim C\|b\|_{\dot\Lambda_\beta}|Q|^{\beta/n}.$$

Further, we have

$$
\begin{aligned}
&|T(y(b-b_Q)f^\infty)(y)-T((b-b_Q)f^\infty)(x_Q)|\\
&=\Big|\int_{R^n}(yK(y-z)-K(x_Q-z))(b(z)-b_Q)f^\infty(z)\mathrm{d}z\Big|\\
&\lesssim \int_{(2Q)^c}\frac{|y-x_Q|}{(|x_Q-z|)^{n+1}}|b(z)-b_Q||f(z)|\mathrm{d}z\\
&\lesssim \sum_{m=2}^\infty\int_{2^mQ\backslash 2^{m-1}Q}2^{-m}|2^mQ|^{-1}(|b(z)-b_{2^kQ}|+|b_{2^kQ}-b_Q|)|f(z)|\mathrm{d}z\\
&\lesssim \sum_{m=2}^\infty 2^{-m}|2^mQ|^{\beta/n}\|b\|_{\dot\Lambda_\beta}M(f)(x)+\sum_{m=2}^\infty 2^{-m}|2^mQ|^{\beta/n}\|b\|_{\dot\Lambda_\beta}M(f)(x)\\
&\lesssim \|b\|_{\dot\Lambda_\beta}|Q|^{\beta/n}\sum_{m=2}^\infty 2^{-m+\beta m}M(f)(x)\\
&\lesssim \|b\|_{\dot\Lambda_\beta}|Q|^{\beta/n}M(f)(x).
\end{aligned}
$$

Thus, we can obtain $D_3\lesssim \|b\|_{\dot\Lambda_\beta}M(f)(x)$.

We finally obtain

$$
\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q |[b,T]f-([b,T]f)_Q|\\
&\quad\lesssim \|b\|_{\dot\Lambda_\beta}(yM(yTf)(x)+(M(|f|^t))^{1/t}(x)+Mf(x)).
\end{aligned}
$$

We now take the supremum over all Q such that $x\in Q$, and the norm of $L^{p(\cdot)}$ on both sides. Since $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$, by Remark 2.6 and the boundedness of M, we conclude that

$$
\begin{aligned}
\|[b,T]f\|_{\dot F^{\beta,\infty}_{p(\cdot)}}&\lesssim \|b\|_{\dot\Lambda_\beta}\|M(Tf)\|_{p(\cdot)}+\|b\|_{\dot\Lambda_\beta}\Big\|(M(|f|^t))^{1/t}\|_{p(\cdot)}+\|b\|_{\dot\Lambda_\beta}\|Mf\Big\|_{p(\cdot)}\\
&\lesssim \|b\|_{\dot\Lambda_\beta}\|f\|_{p(\cdot)}.
\end{aligned}
$$

(b)$\Rightarrow$(a). We know $K(y,z)$ is a homogeneous kernel of degree $-n$. Choose $z_0\in\mathbb{R}^n$, $Q(yz_0,\delta\sqrt{n})\subset\mathbb{R}^n$ and take $|z_0|>\sqrt{n}$, $\delta<1$ small so that $\bar Q\bigcap\{0\}=\varnothing$ is the ball for which we can express $\frac{1}{K(x,y)}$ as an absolutely convergent Fourier series of the form

$$\frac{1}{K(x,y)}=\sum_{m=0}^\infty a_m e^{i\langle\nu_m,(x-y)\rangle},\tag{3.1}$$

where above and in what follows, $\nu_m\in R^n$ are the specific vectors, and $\sum_{m=0}^\infty|a_m|<\infty$.

Choose $x_0 \in \mathbb{R}^n,\ t>1$, and let $Q=Q(x_0,t),\ Q^0=Q(x_0+z_0t,t)$. For $x\in Q,\ y\in Q^0$ with $(y-x)/t\in Q(z_0,\delta\sqrt{n})$. Take on $s(x)=\overline{\operatorname{sgn}(b(x)-b_{Q^0})}$. Applying (3.1), Remark 2.2 and Lemma 3.4 (i), we can obtain

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q |(b(x)-b_Q)|\mathrm{d}x \lesssim \frac{2}{|Q|^{1+\beta/n}}\int_Q |b(x)-b_{Q^0}|\mathrm{d}x\\
&\lesssim \frac{1}{|Q|^{1+\beta/n}}\frac{1}{|Q^0|}\int_Q s(x)\Big(\int_{Q^0}(yb(x)-b(y))\mathrm{d}y\Big)\mathrm{d}x\\
&=\frac{1}{t^{2n+\beta}}\int_Q s(x)\Big(\int_{Q^0}(b(x)-b(y))\frac{K(x-y)}{K(x-y)}\mathrm{d}y\Big)\mathrm{d}x\\
&=\frac{1}{t^{n+\beta}}\sum_{m=0}^{\infty}a_m\int_Q s(x)\Big(\int_{Q^0}(yb(x)-b(y))K(x-y)e^{i\langle\nu_m,y/t\rangle}\mathrm{d}y\Big)e^{-i\langle\nu_m,x/t\rangle}\mathrm{d}x\\
&\lesssim \frac{1}{t^{n+\beta}}\sum_{m=0}^{\infty}|a_m|\int_{R^n}([b,T](y\chi_{Q^0}e^{i\langle\nu_m,\cdot/t\rangle})(x))(\chi_Q(x)e^{-i\langle\nu_m,x/t\rangle}s(x))\mathrm{d}x\\
&\lesssim \frac{1}{t^{n+\beta}}\sum_{m=0}^{\infty}|a_m|\|[b,T](yy\chi_{Q^0}e^{i\langle\nu_m,\cdot/t\rangle})\|_{L^{q(\cdot)}}\|\chi_Q\|_{L^{q'(\cdot)}}\\
&\lesssim \frac{1}{t^{n+\beta}}\sum_{m=0}^{\infty}|a_m|\|[b,T](\chi_{Q^0}e^{i\langle\nu_m,\cdot/t\rangle})\|_{\dot F^{\beta,\infty}_{p(\cdot)}}\|\chi_Q\|_{L^{q'(\cdot)}}\\
&\lesssim \frac{1}{t^{n+\beta}}\sum_{m=0}^{\infty}|a_m|\|[b,T]\|_{L^{p(\cdot)}\to\dot F^{\beta,\infty}_{p(\cdot)}}\|\chi_{Q^0}\|_{L^{p(\cdot)}}\|\chi_Q\|_{L^{q'(\cdot)}}\\
&\lesssim \|[b,T]\|_{L^{p(\cdot)}\to\dot F^{\beta,\infty}_{p(\cdot)}}.
\end{aligned}$$

Thus, (b)⇒(a) is proved.

(a)⇒(c). For $b\in\dot\Lambda_\beta$, $0<\beta<1$, $\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)}=\beta/n$. Since $K(x-y)$ is Calderón-Zygmund kernel, we have

$$|b(x)-b(y)||K(x-y)|\lesssim \frac{A\|b\|_{\dot\Lambda_\beta}}{|x-y|^{n-\beta}}.$$

Using I_β is a bounded operator from $L^{p(\cdot)}$ to $L^{q(\cdot)}$, it follows that

$$\begin{aligned}
\|[b,T]f\|_{q(\cdot)}\le&\Big\|\int_{R^n}|b(x)-b(y)||K(x-y)||f(y)|\mathrm{d}y\Big\|_{q(\cdot)}\\
\lesssim&\|b\|_{\dot\Lambda_\beta}\Big\|\int_{R^n}\frac{|f(y)|}{|x-y|^{n-\beta}}\mathrm{d}y\Big\|_{q(\cdot)}\\
\lesssim&\|b\|_{\dot\Lambda_\beta}\|I_\beta|f|\|_{q(\cdot)}\lesssim\|b\|_{\dot\Lambda_\beta}\|f\|_{p(\cdot)}.
\end{aligned}$$

(c)⇒(a). Proceeding in the method as (b)⇒(a), if $\frac{1}{q(\cdot)}+\frac{1}{q'(\cdot)}=1$, we can obtain

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q |(b(x)-b_Q)|\mathrm{d}x \lesssim \frac{2}{|Q|^{1+\beta/n}}\int_Q |b(x)-b_{Q^0}|\mathrm{d}x\\
&\lesssim \frac{1}{|Q|^{1+\beta/n}}\frac{1}{|Q^0|}\int_Q s(x)\Big(\int_{Q^0}(b(x)-b(y))\mathrm{d}y\Big)\mathrm{d}x\\
&\lesssim\frac{1}{t^{2n+\beta}}\int_Q s(x)\Big(\int_{Q^0}(b(x)-b(y))\frac{K(x-y)}{K(x-y)}\mathrm{d}y\Big)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \frac{1}{t^{n+\beta}}\sum_{m=0}^{\infty} a_m \int_Q s(x)\Big(\int_{Q^0}(b(x)-b(y))K(x-y)e^{i\langle \nu_m, y/t\rangle}\mathrm{d}y\Big)e^{-i\langle \nu_m, x/t\rangle}\mathrm{d}x\\
&\lesssim \frac{1}{t^{n+\beta}}\sum_{m=0}^{\infty} |a_m| \int_{R^n}([b,T](\chi_{Q^0}e^{i\langle \nu_m, \cdot/t\rangle})(x))(\chi_Q(x)e^{-i\langle \nu_m, x/t\rangle}s(x))\mathrm{d}x\\
&\lesssim \frac{1}{t^{n+\beta}}\sum_{m=0}^{\infty} |a_m| \|[b,T](\chi_{Q^0}e^{i\langle \nu_m, \cdot/t\rangle})\|_{L^{q(\cdot)}}\|\chi_Q\|_{L^{q'(\cdot)}}\\
&\lesssim \frac{1}{t^{n+\beta}}\sum_{m=0}^{\infty} |a_m| \|[b,T]\|_{L^{p(\cdot)}\to L^{q(\cdot)}}\|\chi_{Q^0}\|_{L^{p(\cdot)}}\|\chi_Q\|_{L^{q'(\cdot)}}\\
&\lesssim \|[b,T]\|_{L^{p(\cdot)}\to L^{q(\cdot)}}.
\end{aligned}
$$

$[b,T]$ is a bounded operator from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}$. Thus, (c)⇒(a) is proved. which completes the proof of Theorem 3.5. □

Theorem 3.6 *Let $0<\beta<1$, $p(\cdot),q(\cdot),r(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and $p(\cdot),r'(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$. $\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)}=\frac{\alpha}{n}$, $\frac{1}{p(\cdot)}-\frac{1}{r(\cdot)}=\frac{\alpha+\beta}{n}$ with $\frac{1}{p^+}>\frac{\alpha+\beta}{n}$. Then, the following conditions are equivalent:*

(a) $b\in\dot{\Lambda}_\beta$;

(b) $[b,I_\alpha]$ is a bounded operator from $L^{p(\cdot)}(\mathbb{R}^n)$ to $F^{\beta,\infty}_{q(\cdot)}$;

(c) $[b,I_\alpha]$ is a bounded operator from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{r(\cdot)}(\mathbb{R}^n)$.

Proof Let $0<\beta<1$, $\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)}=\frac{\alpha}{n}$, $\frac{1}{p(\cdot)}-\frac{1}{r(\cdot)}=\frac{\alpha+\beta}{n}$ with $\frac{1}{p^+}>\frac{\alpha+\beta}{n}$. Fix to x_Q as the center of a cube Q, for $g\in L^{p(\cdot)}$, let $g^0=g\chi_{2Q}$ and $g^\infty=g-g^0$.

(a)⇒ (b). By Remark 2.6 and Lemma 3.2, we can obtain

$$
\begin{aligned}
&\|[b,I_\alpha](g)\|_{F^{\beta,\infty}_{q(\cdot)}}\lesssim\Big\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q|[b,I_\alpha](g)-([b,I_\alpha](g))_Q|\Big\|_{q(\cdot)}\\
&\lesssim\Big\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q|[b-b_Q,I_\alpha](g)-([b-b_Q,I_\alpha](g))_Q|\Big\|_{q(\cdot)}\\
&\lesssim\Big\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q|[b-b_Q,I_\alpha](g)-I_\alpha((b-b_Q)g^\infty)(x_Q)|\Big\|_{q(\cdot)}\\
&\lesssim\Big\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q|(b-b_Q)I_\alpha(g)|\Big\|_{q(\cdot)}+\\
&\quad\Big\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\alpha/n+\beta/n}}\sup_{y\in Q}|I_\alpha((b-b_Q)g^0)(y)|\Big\|_{p(\cdot)}+\\
&\quad\Big\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{\alpha/n+\beta/n}}\sup_{y\in Q}|I_\alpha((b-b_Q)g^\infty)(y)-I_\alpha((b-b_Q)g^\infty)(x_Q)|\Big\|_{p(\cdot)}\\
&=F_1+F_2+F_3.
\end{aligned}
$$

Firstly, to estimate F_1, for each $x\in Q$, we get by Lemma 3.1,

$$
\begin{aligned}
\frac{1}{|Q|^{\beta/n}}\frac{1}{|Q|}\int_Q|(b-b_Q)I_\alpha(g)|&\lesssim\frac{1}{|Q|^{\beta/n}}\sup_{y\in Q}|(b(y)-b_Q)|\Big(y\frac{1}{|Q|}\int_Q|I_\alpha(g)|\Big)\\
&\lesssim\|b\|_{\dot{\Lambda}_\beta}M(I_\alpha(g))(x).
\end{aligned}
$$

According to the boundedness of I_α, we then obtain

$$F_1 \lesssim \|b\|_{\dot{\Lambda}_\beta} \|I_\alpha(g)\|_{q(\cdot)} \lesssim \|b\|_{\dot{\Lambda}_\beta} \|g\|_{p(\cdot)}.$$

To estimate F_2, choose r, $1 < r < p^-$, and $\bar{r}$ such that $(1/r - 1/\bar{r}) = (\alpha/n)$. Such $\bar{r}$ exists, since $r < p^- < n/\alpha$, it follows that

$$\begin{aligned}
&\frac{1}{|Q|^{1+(\alpha+\beta)/n}} \int_Q |I_\alpha((b-b_Q)g^0)| \lesssim \frac{1}{|Q|^{1+(\alpha+\beta)/n}} \|I_\alpha((b-b_Q)g^0)\|_{\bar{r}} |Q|^{1/\bar{r}'} \\
&\quad \lesssim |Q|^{-1-(\alpha+\beta)/n+1-1/\bar{r}} \|(b-b_Q)g^0\|_r \lesssim \|b\|_{\dot{\Lambda}_\beta} (yM(|g|^r))^{1/r}.
\end{aligned}$$

Thus, we obtain $F_2 \lesssim \|b\|_{\dot{\Lambda}_\beta} \|g\|_{p(\cdot)}$.

We now estimate F_3. Analogously to the estimate of D_3, we have

$$\begin{aligned}
&\frac{1}{|Q|^{\alpha/n+\beta/n}} |I_\alpha(y(b-b_Q)g^\infty)(y) - I_\alpha((b-b_Q)g^\infty)(x_Q)| \\
&\quad \lesssim \frac{1}{|Q|^{\alpha/n+\beta/n}} \int_{(2Q)^c} \frac{|y-x_Q||b(z)-b_Q||g(z)|}{|x_Q-z|^{n+1-\alpha}} \mathrm{d}z \\
&\quad \lesssim \frac{1}{|Q|^{\alpha/n+\beta/n}} \sum_2^\infty \int_{2^kQ\setminus 2^{k-1}Q} 2^{-k} |2^kQ|^{-1+\alpha/n} |g(z)||b(z)-b_Q| \mathrm{d}z \\
&\quad \lesssim \sum_2^\infty 2^{-k+k\alpha+k\beta} \frac{1}{|2^kQ|^{\beta/n}} \frac{1}{|2^kQ|} \int_{2^kQ} |b(z)-b_{2^kQ}||g(z)|\mathrm{d}z + \\
&\quad\quad \sum_2^\infty 2^{-k+k\alpha} \frac{1}{|Q|^{\beta/n}} |2^kQ|^{\beta/n} \|b\|_{\dot{\Lambda}_\beta} \frac{1}{|2^kQ|} \int_{2^kQ} |g(z)|\mathrm{d}z \\
&\quad \lesssim \|b\|_{\dot{\Lambda}_\beta} M(g)(x).
\end{aligned}$$

So, $F_3 \lesssim \|b\|_{\dot{\Lambda}_\beta} \|g\|_{p(\cdot)}$. Thus, (a)$\Rightarrow$(b) is proved.

(b)$\Rightarrow$(a). We know $\frac{1}{(|x-y|)^{n-\alpha}}$ is a homogeneous kernel of degree $-n+\alpha$. Choose $x_0 \in \mathbb{R}^n, t > 0$, and let $Q = Q(x_0, t)$, $Q^0 = Q(x_0 + z_1 t, t)$, for $x \in Q, y \in Q^0$. According to (3.1), we can get

$$\frac{1}{|x-y|^{n-\alpha}} = \sum_{m=0}^\infty a_m e^{i\langle \nu_m, x-y\rangle}.$$

Like the method of (b)$\Rightarrow$(a) in Theorem 3.5, by Lemma 3.4 (ii), we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}} \int_Q |(b(x)-b_Q)|\mathrm{d}x \lesssim \frac{2}{|Q|^{1+\beta/n}} \int_Q |b(x)-b_{Q^0}|\mathrm{d}x \\
&\quad \lesssim \frac{1}{|Q|^{1+\beta/n}} \frac{1}{|Q^0|} \int_Q s(x)\Big(\int_{Q^0} (b(x)-b(y))\mathrm{d}y\Big)\mathrm{d}x \\
&\quad = \frac{1}{t^{2n+\beta}} \int_Q s(x)\Big(\int_{Q^0} (b(x)-b(y)) \frac{|x-y|^{n-\alpha}}{|x-y|^{n-\alpha}} \mathrm{d}y\Big)\mathrm{d}x \\
&\quad = \frac{1}{t^{n+\beta+\alpha}} \sum_{m=0}^\infty a_m \int_Q s(x)\Big(\int_{Q^0} (b(x)-b(y))|x-y|^{n-\alpha} e^{i\langle \nu_m, y/t\rangle}\mathrm{d}y\Big) e^{-i\langle \nu_m, x/t\rangle}\mathrm{d}x \\
&\quad \lesssim \frac{1}{t^{n+\beta+\alpha}} \sum_{m=0}^\infty |a_m| \int_{R^n} ([b, I_\alpha](\chi_{Q^0} e^{i\langle \nu_m, \cdot/t\rangle})(x))(\chi_Q(x) e^{-i\langle \nu_m, x/t\rangle} s(x))\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \frac{1}{t^{n+\beta+\alpha}} \sum_{m=0}^{\infty} |a_m| \|[b, I_\alpha](\chi_{Q^0} e^{i\langle \nu_m, \cdot/t\rangle})\|_{L^{r(\cdot)}} \|\chi_Q\|_{L^{r'(\cdot)}} \\
&\lesssim \frac{1}{t^{n+\beta+\alpha}} \sum_{m=0}^{\infty} |a_m| \|[b, I_\alpha](\chi_{Q^0} e^{i\langle \nu_m, \cdot/t\rangle})\|_{\dot F^{\beta,\infty}_{q(\cdot)}} \|\chi_Q\|_{L^{r'(\cdot)}} \\
&\lesssim \frac{1}{t^{n+\beta+\alpha}} \sum_{m=0}^{\infty} |a_m| \|[b, I_\alpha]\|_{L^{p(\cdot)} \to \dot F^{\beta,\infty}_{q(\cdot)}} \|\chi_{Q^0}\|_{L^{p(\cdot)}} \|\chi_Q\|_{L^{r'(\cdot)}} \\
&\lesssim \|[b, I_\alpha]\|_{L^{p(\cdot)} \to \dot F^{\beta,\infty}_{q(\cdot)}}.
\end{aligned}
$$

$[b, I_\alpha]$ is a bounded operator from $L^{p(\cdot)}(\mathbb{R}^n)$ to $\dot F^{\beta,\infty}_{q(\cdot)}$. So, (b)$\Rightarrow$(a) is proved.

(a)$\Rightarrow$ (c) was proved in [8, Theorem 5].

(c)$\Rightarrow$(a). By the same argument as (b)$\Rightarrow$(a) in the proof of this theorem, we have

$$
\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}} \int_Q |(b(x) - b_Q)| \mathrm{d}x \lesssim \frac{2}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_{Q^0}| \mathrm{d}x \\
&\lesssim \frac{1}{|Q|^{1+\beta/n}} \frac{1}{|Q^0|} \int_Q s(x) \Big(\int_{Q^0} (b(x) - b(y)) \mathrm{d}y \Big) \mathrm{d}x \\
&= \frac{1}{t^{2n+\beta}} \int_Q s(x) \Big(\int_{Q^0} (b(x) - b(y)) \frac{|x-y|^{n-\alpha}}{|x-y|^{n-\alpha}} \mathrm{d}y \Big) \mathrm{d}x \\
&= \frac{1}{t^{n+\beta+\alpha}} \sum_{m=0}^{\infty} a_m \int_Q s(x) \Big(\int_{Q^0} (b(x) - b(y)) |x-y|^{n-\alpha} e^{i\langle \nu_m, y/t\rangle} \mathrm{d}y \Big) e^{-i\langle \nu_m, x/t\rangle} \mathrm{d}x \\
&\lesssim \frac{1}{t^{n+\beta+\alpha}} \sum_{m=0}^{\infty} |a_m| \int_{R^n} ([b, I_\alpha](\chi_{Q^0} e^{i\langle \nu_m, \cdot/t\rangle})(x)) (\chi_Q(x) e^{-i\langle \nu_m, x/t\rangle} s(x)) \mathrm{d}x \\
&\lesssim \frac{1}{t^{n+\beta+\alpha}} \sum_{m=0}^{\infty} |a_m| \|[b, I_\alpha](\chi_{Q^0} e^{i\langle \nu_m, \cdot/t\rangle})\|_{L^{r(\cdot)}} \|\chi_Q\|_{L^{r'(\cdot)}} \\
&\lesssim \frac{1}{t^{n+\beta+\alpha}} \sum_{m=0}^{\infty} |a_m| \|[b, I_\alpha]\|_{L^{p(\cdot)} \to L^{r(\cdot)}} \|\chi_{Q^0}\|_{L^{p(\cdot)}} \|\chi_Q\|_{L^{r'(\cdot)}} \\
&\lesssim \|[b, I_\alpha]\|_{L^{p(\cdot)} \to L^{r(\cdot)}}.
\end{aligned}
$$

The proof of Theorem 3.6 is completed. □

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References

[1] S. JANSON. *Mean oscillation and commutators of singular integral operators*. Ark. Mat., 1978, **16**(2): 263–270.

[2] S. CHANILLO. *A note on commutators*. Indiana Univ. Math. J., 1982, **31**(1): 7–16.

[3] M. PALUSZYNSKI. *Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss*. Indiana. Univ. Math. J., 1995, **44**(1): 1–17.

[4] A. NEKVINDA. *Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$*. Math. Inequal. Appl., 2004, **7**(2): 255–265.

[5] L. PICK, M. RUŽiČKA. *An example of a space $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded*. Expo. Math., 2001, **19**(4): 369–371.

[6] D. CRUZ-URIBE, A. FIORENZA, J. M. MARTELL, et al. *The boundedness of classical operators on variable L^p spaces*. Ann. Acad. Sci. Fen. Math., 2006, **31**(1): 239–264.

[7] S. SAMKO. *On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators*. Integral Transform. Spec. Funct., 2005, **16**(5-6): 461–482.

[8] Huiling WU, Jiacheng LAN. *Lipschitz estimates for fractional multilinear singular integral on variable exponent Lebesgue spaces*. Abstr. Appl. Anal., 2013, Art. ID 632384, 6 pp.

[9] L. DIENING, P. HÄSTÖ, S. ROUDENKO. *Function spaces of variable smoothness and integrability*. J. Funct. Anal., 2009, **256**(6): 1731–1768.

[10] J. PEETRE. *On spaces of Triebel-Lizorkin type*. Ark. Mat., 1975, **13**: 123–130.

[11] Jingshi XU. *Variable Besov and Triebel-Lizorkin spacess*. Ann. Acad. Sci. Fenn. Math., 2008, **33**(2): 511–522.

[12] M. CHRIST. *The extension problem for certain function spaces involving fractional orders of differentiability*. Ark. Mat., 1984, **22**(1): 63–81.

[13] D. CRUZ-URIBE, R. C. SHARPLEY. *Maximal functions measuring smoothness*. Mem. Amer. Math. Soc., 1984, **47**(293): viii+115 pp.

[14] A. SEEGER. *A Note on Triebel-Lizorkin Spaces, Approximation and Function Spaces*. PWN, Warsaw, 1989.

[15] H. TRIEBEL. *Theory of Function Spaces*. Birkhäuser, Basel, 1983.

[16] L. DIENING, P. HARJULEHTO, P. HÄSTÖ, et al. *Lebesgue and Sobolev Spaces with Variable Exponents*. Springer, Heidelberg, 2011.