*Journal of Mathematical Research with Applications*
Jan., 2021, Vol. 41, No. 1, pp. 69–86
DOI:10.3770/j.issn:2095-2651.2021.01.008
Http://jmre.dlut.edu.cn

# Fine Regularity of Solutions to the Dirichlet Problem Associated with the Regional Fractional Laplacian

**Yanyan LI**
*Department of Mathematics and Physics, Shijiazhuang Tiedao University, Hebei* 050043, *P. R. China*

**Abstract** In this paper, we study the Hölder regularity of weak solutions to the Dirichlet problem associated with the regional fractional Laplacian $(-\Delta)^\alpha_\Omega$ on a bounded open set $\Omega \subset \mathbb{R}(N \geq 2)$ with $C^{1,1}$ boundary $\partial\Omega$. We prove that when $f \in L^p(\Omega)$, and $g \in C(\partial\Omega)$, the following problem $(-\Delta)^\alpha_\Omega u = f$ in $\Omega$, $u = g$ on $\partial\Omega$, admits a unique weak solution $u \in W^{\alpha,2}(\Omega) \cap C(\overline{\Omega})$, where $p > \frac{N}{2-2\alpha}$ and $\frac{1}{2} < \alpha < 1$. To solve this problem, we consider it into two special cases, i.e., $g \equiv 0$ on $\partial\Omega$ and $f \equiv 0$ in $\Omega$. Finally, taking into account the preceding two cases, the general conclusion is drawn.

**Keywords** regional fractional Laplacian; Dirichlet problem; Hölder regularity

**MR(2020) Subject Classification** 35B65; 35D30; 35R11

## 1. Introduction

This paper aims to analyze the Hölder regularity of weak solutions to the following Dirichlet problem associated with the regional fractional Laplacian

$$\begin{cases} (-\Delta)^\alpha_\Omega u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded open set with $C^{1,1}$ boundary $\partial\Omega$, $\frac{1}{2} < \alpha < 1$, and $N \geq 2$. Here $f \in L^p(\Omega)$ $(p > \frac{N}{2-2\alpha})$, $g \in C(\partial\Omega)$ and $(-\Delta)^\alpha_\Omega$ denotes the regional fractional Laplace operator, which is defined as the following singular integral

$$(-\Delta)^\alpha_\Omega u(x) := C_{N,\alpha} P.V. \int_\Omega \frac{u(x) - u(y)}{|x-y|^{N+2\alpha}} \mathrm{d}y, \quad x \in \Omega \tag{1.2}$$

with the normalized constant

$$C_{N,\alpha} := \frac{\alpha 2^{2\alpha} \Gamma(\frac{N+2\alpha}{2})}{\pi^{\frac{N}{2}} \Gamma(1-\alpha)},$$

where $\Gamma$ is the usual Gamma function. Based on definitions, we get the following relation

$$(-\Delta)^\alpha_\Omega u(x) = (-\Delta)^\alpha u(x) - V_\Omega(x) u(x), \tag{1.3}$$

where

$$V_\Omega(x) = C_{N,\alpha} P.V. \int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|x-y|^{N+2\alpha}} \mathrm{d}y, \quad \forall x \in \Omega. \tag{1.4}$$

Received November 19, 2019; Accepted March 19, 2020
Supported by the Natural Science Foundation of Hebei Province (Grant No. A2018210018) and the Science and Technology Research Program of Higher Educational in Hebei Province (Grant No. ZD2019047).
E-mail address: liyanyan425@yahoo.com

To have the computation of $(-\Delta)_\Omega^\alpha$ make sense, it is necessary to introduce the class of functions $u$. We give a completely rigorous definition of the regional fractional Laplacian in Appendix A.

In the Fokker-Plank equation for a stochastic differential equation, the classic Laplacian is thought of as a macroscopic manifestation of the Brownian motion. There are classical models in the literature dealing with classic Laplacian. However, several complex phenomena cannot be described appropriately by integer-order partial differential equations while fractional differential models are powerful for dealing with those challenging phenomena. So far, numerous fractional differential models have been proposed. From the long list of phenomena which is more appropriately modeled by fractional differential equations, we mention anomalous transport, long-range interactions, or from local to nonlocal dynamics, diffusion or dispersion [1,2], turbulent flows [3,4], hereditary phenomena with a long memory, nonlocal electrostatics, the latter being relevant to drug design [5, 6], systems of stochastic dynamics [5, 7], finance [8], and Levy motions which appear in important models in both applied mathematics and applied probability, as well as in models in biology and ecology [7].

The Dirichlet problem for the fractional Laplacian has been studied from probability, potential theory, and PDEs. The result of the one in our paper is based on [9], which develops a fractional analog of the Krylov boundary Harnack method, and establishes the Hölder regularity up to the boundary. Related regularity results up to the boundary have been proved in [3, 8, 10]. Some other results dealing with various aspects concerning the Dirichlet problem, see for example [11–13].

Our concern in this paper is the study of the local elliptic of weak solutions to the Dirichlet problem (1.1). For this purpose, we first introduce the following definition of weak solutions to the following Dirichlet problem (1.5). Throughout this paper, we assume that $\frac{1}{2} < \alpha < 1$, and $\Omega$ is a bounded open set in $\mathbb{R}^N$ $(N \geq 2)$ with $C^{1,1}$ boundary $\partial\Omega$ and $\rho = \operatorname{dist}(x, \partial\Omega)$.

**Definition 1.1** ([5, 14]) *Let $f \in W^{-\alpha,2}(\Omega)$. If the equality*

$$\frac{C_{N,\alpha}}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y = \langle f, v\rangle_{W^{-\alpha,2}(\Omega), W_0^{\alpha,2}(\Omega)}$$

*holds, for every $v \in W_0^{\alpha,2}(\Omega)$, then we say the function $u \in W_0^{\alpha,2}(\Omega)$ is a weak solution of the following Dirichlet problem*

$$\begin{cases} (-\Delta)_\Omega^\alpha u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

We notice that, when $1 < p < 2$, the Dirichlet problem (1.5) is not well defined. By virtue of duality, we first introduce an alternative definition.

**Definition 1.2** *Let $1 < p < 2$ and $f \in L^1(\Omega)$. If the equality*

$$\int_\Omega u\psi \mathrm{d}x = \int_\Omega f\phi \mathrm{d}x$$

*holds for every $\phi \in \mathcal{T}(\Omega)$ and $\psi \in \mathcal{D}(\Omega)$, then we say that $u \in L^1(\Omega)$ is a weak solution of the Dirichlet problem (1.5).*

Recall that $\mathcal{T}(\Omega)=\{\phi:(-\Delta)^{\alpha}_{\Omega}\phi=\psi \text{ in } \Omega,\ \phi=0 \text{ on } \partial\Omega,\ \psi\in\mathcal{D}(\Omega)\}$, and $\mathcal{D}(\Omega)=C_0^{\infty}(\Omega)$, which is the space of all continuously infinitely differentiable functions with compact support in $\Omega$.

Based on the preceding definition, it is not hard to see that, if $f\in L^p(\Omega)$ with $p\geq 2$, the weak solution to the Dirichlet problem (1.5) is well defined; if $1<p<2$, the weak solution will be understood in the case of transposition. Moreover, if $f\in L^p(\Omega)$ ($p\geq 2$), by the continuous embedding $L^p(\Omega)\hookrightarrow L^2(\Omega)\hookrightarrow W^{-\alpha,2}(\Omega)$, the property $f\in W^{-\alpha,2}(\Omega)$ is automatically guaranteed.

To define the weak solutions of the Dirichlet problem (1.1), by using the idea in [5], we first give the following notations.

Suppose that there exists a function $\widetilde{g}\in W^{\alpha,2}(\Omega)$ such that $\widetilde{g}=g$ on $\partial\Omega$, and let

$$K:=\{v\in W^{\alpha,2}(\Omega): v-\widetilde{g}\in W_0^{\alpha,2}(\Omega)\}.$$

It follows from [5, Theorem 9.17], that $K$ is independent of the choice of $\widetilde{g}$ and depends only on $g$. Moreover, $K$ is a nonempty closed convex set in $W_0^{\alpha,2}(\Omega)$.

**Definition 1.3** ([5,15]) *Let $f\in W^{-\alpha,2}(\Omega)$. If the equality*

$$\frac{C_{N,\alpha}}{2}\int_{\Omega}\int_{\Omega}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y=\langle f,v\rangle_{W^{-\alpha,2}(\Omega),W_0^{\alpha,2}(\Omega)}$$

*holds for every $v\in W_0^{\alpha,2}(\Omega)$, then we say the function $u\in K$ is a weak solution of the Dirichlet problem (1.1).*

*The following Hölder regularity is our first main result.*

**Theorem 1.4** *Let $g\in C(\partial\Omega)$ and $f\in L^p(\Omega)$ with $p>\frac{N}{2-2\alpha}$. Then the Dirichlet problem (1.1) has a unique weak solution $u\in W^{\alpha,2}(\Omega)\cap C(\overline{\Omega})$. Moreover, there is a constant $C>0$, such that*

$$\|u\|_{L^\infty(\Omega)}\leq C(\|f\|_{L^p(\Omega)}+\|g\|_{L^\infty(\partial\Omega)})$$

*and*

$$\|u\|_{W^{\alpha,2}(\Omega)}\leq C(\|f\|_{L^p(\Omega)}+\|g\|_{L^\infty(\partial\Omega)}).$$

To prove this result, we first show the Hölder regularity for the homogeneous Dirichlet problem (1.5).

**Proposition 1.5** ([9,16]) *Let $f\in L^p(\Omega)$ with $p>\frac{N}{2-2\alpha}$. Then the Dirichlet problem (1.5) has a unique weak solution $u_f$ such that the estimate*

$$-C_1\|f_-\|_{L^p(\Omega)}\rho(x)^{2\alpha-1}\leq u_f(x)\leq C_1\|f_+\|_{L^p(\Omega)}\rho(x)^{2\alpha-1},\quad x\in\Omega \tag{1.6}$$

*holds for some $C_1>0$. Moreover, for $\theta\in(0,2\alpha-1)$ and an open set $\mathcal{O}\subset\Omega$ with $d_{\mathcal{O}}>0$, there exists a constant $C_2>0$ dependent on $d_{\mathcal{O}}$ and $\theta$, such that*

$$\|u_f\|_{C^\theta(\mathcal{O})}\leq C_2\|f\|_{L^p(\Omega)}. \tag{1.7}$$

*Especially, if* $f \geq 0$ *and* $f \not\equiv 0$*, we get* $u_f$ *is positive. Where* $f_\pm := \max\{\pm f, 0\}$*,* $\rho(x) := \operatorname{dist}(x, \partial\Omega)$*, and* $d_{\mathcal{O}} := \operatorname{dist}(\mathcal{O}, \partial\Omega)$*.*

Combining the boundary decay estimate (1.6) and the scaling property, we obtain the regularity up to the boundary of weak solutions to the Dirichlet problem (1.5) as follows.

**Theorem 1.6** *Let* $f \in L^p(\Omega)$ *with* $p > \frac{N}{2-2\alpha}$*, and let* $\theta \in (0, 2\alpha - 1)$*. Then the Dirichlet problem (1.5) has a unique weak solution* $u_f \in C^\theta(\overline{\Omega}) \cap W_0^{\alpha,2}(\Omega)$*. Moreover, there exists a constant* $C > 0$*, which is independent of* $f$ *such that*

$$\|u_f\|_{C^\theta(\Omega)} \leq C\|f\|_{L^p(\Omega)} \tag{1.8}$$

*and*

$$\|u_f\|_{W_0^{\alpha,2}(\Omega)} \leq C\|f\|_{L^p(\Omega)}. \tag{1.9}$$

Here $W_0^{\alpha,2}(\Omega)$ is the fractional-order Sobolev space which is the closure of $\mathcal{D}(\Omega)$ in the norm of $W^{\alpha,2}(\Omega)$, which is also a fractional-order Sobolev space and denoted by

$$W^{\alpha,2}(\Omega) := \Big\{u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\alpha}} \mathrm{d}x\mathrm{d}y < \infty\Big\}$$

and endow it with the norm

$$\|u\|_{W^{\alpha,2}(\Omega)} := \Big(\int_\Omega |u|^2 \mathrm{d}x + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\alpha}} \mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{2}}.$$

We define the dual space of $W_0^{\alpha,2}(\Omega)$ as $W^{-\alpha,2}(\Omega) := (W_0^{\alpha,2}(\Omega))^\star$, and equip it with the dual norm $\|f\|_{W^{-\alpha,2}(\Omega)} := \sup\{|\langle f, v\rangle_\Omega| : v \in W_0^{\alpha,2}(\Omega), \|v\|_{W_0^{\alpha,2}(\Omega)} = 1\}$. We will give a more exhaustive description of those spaces in Appendix A at the end of this paper.

We construct a sequence of $C^\infty(\overline{\Omega})$ function $\{g_n\}_{n=1}^\infty$ such that $g_n$ converges to $g$ uniformly in $\overline{\Omega}$ as $n \to \infty$. Then we change the correspondence inhomogeneous Dirichlet problem into the homogenous Dirichlet problem in a special case, by the limit property and the preceding theorem to get our results.

**Theorem 1.7** *Let* $f \equiv 0$ *and* $g \in C(\partial\Omega)$*. Then there exists a unique function* $u \in W^{\alpha,2}(\Omega) \cap C(\overline{\Omega})$ *satisfying the Dirichlet problem (1.1). Moreover, there is a constant* $C > 0$*, such that* $\|u\|_{L^\infty(\Omega)} \leq C\|g\|_{L^\infty(\partial\Omega)}$ *and* $\|u\|_{W^{\alpha,2}(\Omega)} \leq C\|g\|_{L^\infty(\partial\Omega)}$*.*

Based on Theorems 1.6 and 1.7, it is natural to obtain the Hölder regularity for the general Dirichlet problem.

**Lemma 1.8** ([17]) *Assume that* $g_i : \partial\Omega \to \mathbb{R}$ *and* $f_i : \Omega \to \mathbb{R}$ *are continuous functions with* $i = 1, 2$ *satisfying*

$$g_1 \geq g_2 \text{ on } \partial\Omega, \text{ and } f_1 \geq f_2 \text{ in } \Omega.$$

*Let* $u_1$ *and* $u_2$ *be two weak solutions of the Dirichlet problem (1.1), with* $f = f_1$*,* $f_2$ *and* $g = g_1$*,* $g_2$*, respectively. Then*

$$u_1 \geq u_2 \text{ in } \Omega.$$

*Furthermore, if* $f \equiv 0$ *and* $g \equiv 0$*, then this problem has only zero as the weak solution.*

This paper is organized as follows. In Section 2, we give the proof of Proposition 1.5 and Theorem 1.6, which is the Hölder regularity for the homogeneous Dirichlet problem (1.3). In Section 3, by using the limit property and results in Section 2, we show Theorem 1.7, which is the Hölder regularity for the Dirichlet problem (1.1) in case of $f \equiv 0$. Finally, in Section 4, taking into account the results in Sections 2 and 3, our main result (Theorem 1.4) is proved.

## 2. Hölder regularity of solutions to the homogenous Dirichlet problem

In this section, we will prove that the Hölder regularity of weak solutions to Dirichlet problem (1.5). To show it, we first prove that $u \in C^\theta$ in an open set $\mathcal{O} \subset \Omega$, for $\theta \in (0, 2\alpha - 1)$. Then we extend the regularity up to the boundary.

**Lemma 2.1** *Let $f \in L^p(\Omega)$ with $p > \frac{N}{2-2\alpha}$, $N \geq 2$, and $\alpha \in (\frac{1}{2}, 1)$. Then $\mathbb{G}_{\Omega,\alpha}[f]$ is the unique weak solution to Dirichlet problem (1.5). Moreover*

$$|\mathbb{G}_{\Omega,\alpha}[f](x)| \leq C\rho^{2\alpha-1}(x)\|f\|_{L^p(\Omega)}, \quad x \in \Omega, \tag{2.1}$$

*for some $C > 0$ and $\rho(x) = \mathrm{dist}(x, \partial\Omega)$.*

**Proof** Assume $f \in L^p(\Omega)$ with $p > \frac{N}{2-2\alpha}$, $N \geq 2$, and $\alpha \in (\frac{1}{2}, 1)$, then $p > 2$.

Existence. Let $G_{\Omega,\alpha}$ be the green kernel of $(-\Delta)^\alpha_\Omega$. By the integration by parts formula in [18] and Definition 1.1, it is easy to show that $\mathbb{G}_{\Omega,\alpha}[f]$ is a weak solution of Dirichlet problem (1.5).

Uniqueness. Let $u$ and $w$ be two weak solutions of Dirichlet problem (1.5). By Definition 1.1

$$\frac{C_{N,\alpha}}{2}\int_\Omega\int_\Omega \frac{[(u-w)(x)-(u-w)(y)][v(x)-v(y)]}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y = 0, \quad \forall\, v \in W^{\alpha,2}_0(\Omega).$$

By taking $v = u - w$ as a test function, we get $u \equiv w$ in $\Omega$.

At last, we prove Inequality (2.1). By [9, theorem], there exists $C > 0$ such that

$$G_{\Omega,\alpha}[f] \leq C \min\left\{\frac{1}{|x-y|^{N-2\alpha}}, \frac{\rho(x)^{2\alpha-1}\rho(y)^{2\alpha-1}}{|x-y|^{N-2+2\alpha}}\right\}$$

holds for any $(x, y) \in \Omega \times \Omega$ with $x \neq y$. By the above inequality and Hölder inequality. For every $x \in \Omega$, the inequalities

$$\begin{aligned}
|\mathbb{G}_{\Omega,\alpha}[f](x)| \leq& C\int_\Omega \frac{\rho(x)^{2\alpha-1}\rho(y)^{2\alpha-1}}{|x-y|^{N-2+2\alpha}}|f(y)|\mathrm{d}y \\
\leq& C\|f\|_p\Big[\int_\Omega(\frac{\rho(x)^{2\alpha-1}\rho(y)^{2\alpha-1}}{|x-y|^{N-2+2\alpha}})^q\mathrm{d}y\Big]^{\frac{1}{q}}(\frac{1}{p}+\frac{1}{q}=1) \\
\leq& C\|f\|_p\rho(x)^{2\alpha-1}\Big[\int_\Omega(\frac{1}{|x-y|^{N-2+2\alpha}})^q\mathrm{d}y\Big]^{\frac{1}{q}} \\
\leq& C\|f\|_p\rho(x)^{2\alpha-1}\Big[\int_{B_{d_0(x)}}(\frac{1}{|x-y|^{N-2+2\alpha}})^q\mathrm{d}y\Big]^{\frac{1}{q}} \\
\leq& C\|f\|_p\rho(x)^{2\alpha-1}
\end{aligned}$$

hold, where $d_0 = \sup_{x,y\in\Omega}|x-y|$, and $\Omega \subset B_{d_0}(x)$. In the previous estimates, we use the fact that $\rho(y)$ is bounded for $y \in \Omega$. The integral

$$\int_{B_{d_0(x)}} (\frac{1}{|x-y|^{N-2+2\alpha}})^q \mathrm{d}y = \int_0^{d_0} \frac{1}{r^{q(N-2+2\alpha)-(N-1)}} \mathrm{d}r$$

will be finite if $0 < q(N-2+2\alpha) - (N-1) < 1$. The result is obvious under the condition $p > \frac{N}{2-2\alpha}$ and $\frac{1}{p} + \frac{1}{q} = 1$. The proof is completed. □

**Lemma 2.2** ([8]) *We have $V_\Omega \in C^{0,1}_{loc}(\Omega)$ and*

$$\frac{1}{C}\rho(x)^{-2\alpha} \le V_\Omega(x) \le C\rho(x)^{-2\alpha}, \quad \forall x \in \Omega,$$

*for some $C > 0$, here $V_\Omega$ is the same as in Equality (1.4) and $\rho(x) = \mathrm{dist}(x, \partial\Omega)$.*

**Proposition 2.3** ([16]) *Let $w \in L^\infty(B_1)$, and let $u \in L^\infty(B_1)$ be a weak solution of $w = (-\Delta)^\sigma u$ with $\sigma > 0$. Then we have the following results:*

*(1) If $2\sigma \le 1$, then $u \in C^{0,\alpha}(B_{\frac{1}{2}})$ for every $\alpha < 2\sigma$. Moreover*

$$\|u\|_{C^{0,\alpha}(B_{\frac{1}{2}})} \le C(\|u\|_{L^\infty(B_1)} + \|w\|_{L^\infty(B_1)}),$$

*where $C$ is a constant number depending on $N, \alpha$ and $\sigma$, and $B_1 = \{x : |x-0| < 1\}$, $B_{\frac{1}{2}} = \{x : |x-0| < \frac{1}{2}\}$.*

*(2) If $2\sigma > 1$, then $u \in C^{1,\alpha}(B_{\frac{1}{2}})$ for every $\alpha < 2\sigma - 1$. Moreover*

$$\|u\|_{C^{1,\alpha}(B_{\frac{1}{2}})} \le C(\|u\|_{L^\infty(B_1)} + \|w\|_{L^\infty(B_1)}),$$

*where $C$ is a constant number depending on $N, \alpha$ and $\sigma$.*

**Proof** We only show the first part, the proof of the second part is similar. Assume that $2\sigma \le 1$ and $\alpha < 2\sigma$. Then $0 < 1 - (2\sigma - \alpha) < 1$. Let $w \in L^\infty(B_1)$, and let $u \in L^\infty(B_1)$ be a weak solution of $w = (-\Delta)^\sigma u$, then

$$u = (-\Delta)^{-\sigma} w = (-\Delta)^{1-\sigma} \circ (-\Delta)^{-1} w.$$

By [19, Theorem 4.16], we get that $(-\Delta)^{-1} w \in C^{1,1-(2\sigma-\alpha)}(B_{\frac{1}{2}})$, and

$$\|(-\Delta)^{-1} w\|_{C^{1,1-(2\sigma-\alpha)}(B_{\frac{1}{2}})} \le C\|w\|_{L^\infty(B_1)}.$$

Considering the definition of seminorm and the property in [16, Proposition 2.1.8], we obtain that $(-\Delta)^{1-\sigma} \circ (-\Delta)^{-1} w \in C^\alpha(B_{\frac{1}{2}})$, and

$$[u]_{C^\alpha(B_{\frac{1}{2}})} = [(-\Delta)^{1-\sigma} \circ (-\Delta)^{-1} w]_{C^\alpha(B_{\frac{1}{2}})} \le C[(-\Delta)^{-1} w]_{C^{1,1-(2\sigma-\alpha)}(B_1)}.$$

And then, together with the definition of $C^\alpha$-norm, we get that

$$\|u\|_{C^\alpha(B_{\frac{1}{2}})} = [u]_{C^\alpha(B_{\frac{1}{2}})} + \|u\|_{L^\infty(B_{\frac{1}{2}})} \le C(\|u\|_{L^\infty(B_1)} + \|w\|_{L^\infty(B_1)}).$$

The proof is completed. □

By the preceding proposition and the Dirichlet problem (1.3) we get the following estimate for the regional fractional Laplacian.

**Proposition 2.4** *Let $u \in L^\infty(B_1)$, and let $w \in L^p(B_1)$ be a weak solution of $w = (-\Delta)^\sigma u$ in $B_1$ with $\sigma > \frac{1}{2}$. Then*

*(1) If $\sigma \in (\frac{1}{2}, 1)$ and $p > N$, then $u \in C^\alpha(B_{\frac{1}{2}})$ for every $\alpha \in (0, 2\sigma - 1)$. Moreover*

$$\|u\|_{C^\alpha(B_{\frac{1}{2}})} \le C(\|u\|_{L^\infty(B_1)} + \|w\|_{L^p(B_1)}),$$

*where $C$ is a constant number depending on $N, \alpha$ and $\sigma$, and $B_1, B_{\frac{1}{2}}$ are similar as in Proposition 2.4.*

*(2) If $\sigma \in [1, \frac{3}{2})$ and $p \in (\frac{N}{2}, N-1)$, then $u \in C^\alpha(B_{\frac{1}{2}})$ for every $\alpha \in (0, 2(\sigma - 1))$. Moreover*

$$\|u\|_{C^\alpha(B_{\frac{1}{2}})} \le C(\|u\|_{L^\infty(B_1)} + \|w\|_{L^p(B_1)}),$$

*where $C$ is a constant number depending on $N, \alpha$ and $\sigma$.*

*(3) If $\sigma \in [\frac{3}{2}, \infty)$ and $p \in (\frac{N}{2}, N-1)$, then $u \in C^{1,\alpha}(B_{\frac{1}{2}})$ for every $\alpha \in (0, 2(\sigma - 1) - 1)$. Moreover*

$$\|u\|_{C^{1,\alpha}(B_{\frac{1}{2}})} \le C(\|u\|_{L^\infty(B_1)} + \|w\|_{L^p(B_1)}),$$

*where $C$ is a constant number depending only on $N, \alpha$ and $\sigma$.*

**Proof** The proof is identically of the one of Proposition 2.4. □

**Lemma 2.5** ([7]) *Let $p > \frac{N}{2-2\alpha}$ with $\frac{1}{2} < \alpha < 1$. Assume that $u \in C^{2\alpha-1+\varepsilon}(\overline{B_1})$ with $\varepsilon > 0$ satisfying*

$$(-\Delta)^\alpha u = h, \text{ in } B_1,$$

*where $h \in L^p(\overline{B_1})$ and $\overline{B_1} = \{x : |x - 0| \le 1\}$. Then, for every $\beta \in (0, 2\alpha - 1)$, there exists $C > 0$, such that*

$$\|u\|_{C^\beta(\overline{B_{\frac{1}{4}}})} \le C(\|u\|_{L^\infty(B_1)} + \|h\|_{L^p(B_1)} + \|(1 + |z|)^{-N-2\alpha} u(z)\|_{L^1(\mathbb{R}^N)}). \tag{2.2}$$

**Proposition 2.6** ([12]) *Let $f \in L^p(\Omega)$ with $p > \frac{N}{2-2\alpha}$. Assume that $\omega \in C^{2\alpha-1+\varepsilon}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ is a weak solution of the Dirichlet problem (1.5) with $\varepsilon > 0$. Then for all $\theta \in (0, 2\alpha - 1)$, and each open set $\mathcal{O} \subset \Omega$ with $d_{\mathcal{O}} = \operatorname{dist}(\mathcal{O}, \partial\Omega) > 0$. There exists $C > 0$ independent of $d_{\mathcal{O}}$ and $\theta$, such that*

$$\|\omega\|_{C^\theta(\mathcal{O})} \le C d_{\mathcal{O}}^{-1} \|f\|_{L^p(\Omega)}. \tag{2.3}$$

**Proof** Let $\widetilde{\omega} = \omega$ in $\Omega$, $\widetilde{\omega} = 0$ on $\mathbb{R}^N \backslash \Omega$. By (2.2)

$$(-\Delta)^\alpha \widetilde{\omega}(x) = (-\Delta)^\alpha_\Omega \omega(x) + \omega(x) V_\Omega(x) = f(x) + \widetilde{\omega} V_\Omega(x), \quad \forall\, x \in \Omega,$$

where $V_\Omega(x) = \int_{\mathbb{R}^N \backslash \Omega} \frac{1}{|x-y|^{N+2\alpha}} \mathrm{d}y$. It follows from Lemma 2.2 that $V_\Omega \in C^{0,1}_{\text{loc}}(\Omega)$.

Let $\mathcal{O}_1 (\subset \Omega)$ be a $C^2$ open set, satisfying

$$\mathcal{O} \subset \mathcal{O}_1, \quad \operatorname{dist}(\mathcal{O}_1, \partial\Omega) = d_{\mathcal{O}}/2, \quad \operatorname{dist}(\mathcal{O}, \partial\mathcal{O}_1) = d_{\mathcal{O}}/2.$$

By Lemma 2.5, for every $\theta \in (0, 2\alpha - 1)$, we have that

$$\|\widetilde{\omega}\|_{C^\theta(\mathcal{O})} \le C(\|\widetilde{\omega}\|_{L^\infty(\mathcal{O}_1)} + \|\widetilde{\omega}\|_{L^1(\Omega)} + \|f + \widetilde{\omega} V_\Omega\|_{L^p(\mathcal{O}_1)}).$$

Considering Lemma 2.1, we obtain that

$$\|\widetilde{\omega}\|_{L^\infty(\mathcal{O}_1)} = \|\omega\|_{L^\infty(\mathcal{O}_1)} \le \|\omega\|_{L^\infty(\Omega)} \le C\|f\|_{L^p(\Omega)}$$

and

$$\|\widetilde{\omega}\|_{L^1(\Omega)} = \|\omega\|_{L^1(\Omega)} \le \|\omega\|_{L^\infty(\Omega)} \le C\|f\|_{L^p(\Omega)},$$

where we use the fact that $\widetilde{\omega} = \omega$ in $\Omega$, $\widetilde{\omega} = 0$ on $\mathbb{R}^N\backslash\Omega$, and $\omega$ is a solution of the Dirichlet problem (1.5). Taking into account of Lemmas 2.1 and 2.2, we obtain that

$$|\widetilde{\omega}|V_\Omega(x) \le C\rho^{2\alpha-1}\|f\|_{L^p(\Omega)}\rho^{-2\alpha} = C\rho^{-1}\|f\|_{L^p(\Omega)}, \quad \forall\, x \in \Omega,$$

since $\operatorname{dist}(\mathcal{O}_1, \partial\Omega) = \operatorname{dist}(\mathcal{O}, \partial\mathcal{O}_1) = d_{\mathcal{O}}/2$. If $x \in \mathcal{O}_1$, then $\rho \ge \frac{d_{\mathcal{O}}}{2}$ and

$$\|f + \widetilde{\omega}V_\Omega\|_{L^p(\mathcal{O}_1)} \le \|f\|_{L^p(\mathcal{O}_1)} + \|\widetilde{\omega}V_\Omega\|_{L^p(\mathcal{O}_1)} \le Cd_{\mathcal{O}}^{-1}\|f\|_{L^p(\Omega)},$$

for some $C > 0$. Together with the preceding estimates, we obtain that $\|\widetilde{\omega}\|_{C^\theta(\mathcal{O})} \le Cd_{\mathcal{O}}^{-1}\|f\|_{L^p(\Omega)}$. Considering of the fact that $\widetilde{\omega} = \omega$ in $\Omega$, Inequality (2.3) is obvious. □

**Proof of Proposition 1.5** We first prove Inequality (1.6). Assume $f \in L^p(\Omega)$ with $p > \frac{N}{2-2\alpha}$. By Lemma 2.1, $\mathbb{G}_{\Omega,\alpha}[f]$ is the unique weak solution of the Dirichlet problem (1.5). Then $\mathbb{G}_{\Omega,\alpha}[f_+]$ and $\mathbb{G}_{\Omega\alpha}[f_-]$ are weak solutions of the Dirichlet problem (1.5) replaced $f$ by $f_+$ and $f_-$, respectively. Since $f_+ \ge f \ge -f_-$, together with Lemma 1.8, we get that $\mathbb{G}_{\Omega,\alpha}[f_+] \ge \mathbb{G}_{\Omega,\alpha}[f] \ge -\mathbb{G}_{\Omega,\alpha}[f_-]$. Furthermore, by using Lemma 2.1, we obtain that $\mathbb{G}_{\Omega,\alpha}[f_+] \le C_1\|f_+\|_{L^p}\rho^{2\alpha-1}$ and $\mathbb{G}_{\Omega,\alpha}[f_-] \le C_1\|f_-\|_{L^p}\rho^{2\alpha-1}$. Then $-C_1\|f_-\|_{L^p}\rho^{2\alpha-1} \le \mathbb{G}_{\Omega,\alpha}[f] \le C_1\|f_+\|_{L^p}\rho^{2\alpha-1}$ for a constant $C_1 > 0$. We denote $u_f = \mathbb{G}_{\Omega,\alpha}[f]$, then Inequality (1.6) is obvious.

Now we show Inequality (1.7). For every open set $\mathcal{O} \subset \Omega$, it is easy to show that $u_f \in C^{2\alpha-1+\varepsilon}(\mathcal{O}) \cap L^\infty(\mathcal{O})$ for $\varepsilon > 0$. By using Proposition 2.6, we obtain that $u_f \in C^\theta(\mathcal{O})$ with $\theta \in (0, 2\alpha - 1)$. Moreover

$$\|u_f\|_{C^\theta(\mathcal{O})} \le Cd_{\mathcal{O}}^{-1}\|f\|_{L^p(\Omega)} \le C_2\|f\|_{L^p(\Omega)},$$

where $C$ and $C_2$ are two constants. We complete our proof. □

To prove the $C^\theta$ $(0 < \theta < 2\alpha - 1)$ regularity of weak solution. By using the scaling property, we first give the following uniform estimate.

**Lemma 2.7** *Let $f \in L^p(\Omega)$ with $p > \frac{N}{2-2\alpha}$. Assume $u_f$ is the weak solution of the Dirichlet problem (1.5). Then there exists a constant $C > 0$ independent of $\rho(x_0)$, such that*

$$\|u_f\|_{C^\theta(B_{\rho_0}(x_0))} \le C\rho_0^{2\alpha-1-\theta}\|f\|_{L^p(\Omega)}, \tag{2.4}$$

*for every $x_0 \in \Omega$ and $\theta \in (0, 2\alpha - 1)$, where $\rho_0 := \frac{\rho(x_0)}{3} = \frac{1}{3}\operatorname{dist}(x_0, \partial\Omega)$.*

**Proof** Given $x_0 \in \Omega$, $\Omega_0 := \{y \in \mathbb{R}^N : x_0 + \rho_0 y \in \Omega\}$ and

$$v_f(x) := u_f(x_0 + \rho_0 x), \quad x \in \mathbb{R}^N,$$

where $u_f$ is the weak solution of the Dirichlet problem (1.3). Extending $u_f$ by 0 on $\Omega^c$, then $v_f(x) = 0$ on $\Omega_0^c$. By Lemma 2.1, we get that

$$\|v_f\|_{L^\infty(B_2(0))} = \|u_f\|_{L^\infty(B_{2\rho_0}(x_0))} \le \|u_f\|_{L^\infty(\Omega)} \le C\rho_0^{2\alpha-1}\|f\|_{L^p(\Omega)}. \tag{2.5}$$

If $x \in B_2(0)$, then

$$\begin{aligned}(-\Delta)^\alpha_{\Omega_0} v_f(x) =& P.V. C_{N,\alpha} \int_{\Omega_0} \frac{u_f(x_0+\rho_0 x) - u_f(x_0+\rho_0 y)}{|x-y|^{N+2\alpha}} \mathrm{d}y \\ =& \rho_0^{2\alpha} P.V. C_{N,\alpha} \int_{\Omega_0} \frac{u_f(x_0+\rho_0 x) - u_f(x_0+\rho_0 y)}{|(x_0+\rho_0 x) - (x_0+\rho_0 y)|^{N+2\alpha}} d(\rho_0 y) \\ =& \rho_0^{2\alpha} (-\Delta)^\alpha_{\Omega_0} u_f(x_0+\rho_0 x) = \rho_0^{2\alpha} f(x_0+\rho_0 x)\end{aligned}$$

and

$$(-\Delta)^\alpha v_f(x) = (-\Delta)^\alpha_{\Omega_0} v_f(x) + v_f V_{\Omega_0}(x) = \rho_0^{2\alpha} f(x_0+\rho_0 x) + v_f V_{\Omega_0}(x), \tag{2.6}$$

where $V_{\Omega_0} = C_{N,\alpha} \int_{\mathbb{R}^N \setminus \Omega_0} \frac{1}{|x-y|^{N+2\alpha}} \mathrm{d}y$. By the definition of $\Omega_0$, it is easy to show that $B_3(0) \subset \Omega_0$ then

$$V_{\Omega_0} \le C_{N,\alpha}, \quad \forall\, x \in B_2(0).$$

Let $\theta \in (0, 2\alpha - 1)$. By the preceding inequality, Equalities (2.6) and Lemma 2.4, we have that

$$\begin{aligned}\|v_f\|_{C^\theta(B_1(0))} \le& C(\|\rho_0^{2\alpha} f(x_0+\rho_0 \cdot) + v_f V_{\Omega_0}\|_{L^p(B_2(0))} + \|v_f\|_{L^\infty(B_2(0))}) \\ \le& C(\|\rho_0^{2\alpha} f\|_{L^p(B_2(0))} + \|v_f V_{\Omega_0}\|_{L^p(B_2(0))} + \|v_f\|_{L^\infty(B_2(0))}) \\ \le& C(\rho_0^{2\alpha} \|f\|_{L^p(\Omega)} + \|v_f\|_{L^\infty(B_2)}).\end{aligned}$$

Together with Inequalities (2.5), we get that

$$\|u_f\|_{C^\theta(B_{\rho_0}(x_0))} \le C \rho_0^{2\alpha-1-\theta} \|f\|_{L^p(\Omega)}.$$

The proof is completed. □

Using Lemma 2.6, we obtain the $C^\theta$ regularity up to the boundary.

**Proof of Theorem 1.6** Taking $\theta = 2\alpha - 1$ in Lemma 2.7, we have that

$$\frac{|u(x) - u(y)|}{|x-y|^\theta} \le C \|f\|_{L^p(\Omega)}, \tag{2.7}$$

for all $x, y$ such that $y \in B_R(x)$ with $R = \frac{\rho(x)}{3}$. Inequality (2.7) holds for all $x, y \in \overline{\Omega}$ with some renewed constant.

Define a Lipschitz function: $\phi : \Omega \to \mathbb{R}^N$, then $\phi$ is differentiable almost everywhere and rectifiable. From [16, Theorem 4.1], we have that

$$\|f \circ \phi^{-1}\|_{L^p(\phi(\Omega))} \le (\operatorname*{ess\,sup}_{\Omega} |\det \phi'|)^{\frac{1}{p}} \|f\|_{L^p(\Omega)}.$$

That is after a Lipschitz change of coordinates, the bound of Inequality (2.7) remains the same except for the value of the constant $C$. Hence, we can flatten the boundary near $x_0 \in \partial\Omega$ to assume that $\Omega \cap B_{\rho_0}(x_0) = \{x_n > 0\} \cap B_1(0)$. Thus, Inequality (2.7) holds for all $x, y$ satisfying $|x - y| \le \gamma x_n$ for some $\gamma = \gamma(\Omega) \in (0, 1)$ depending on the Lipschitz mapping.

Next, let $z = (z', z_n)$ and $\omega = (\omega', \omega_n)$ be two points in $\{x_n > 0\} \cap B_{\frac{1}{4}}(0)$, and $r = |z - \omega|$. Denote that $\overline{z} = (z', z_n + r)$, $\overline{\omega} = (\omega', \omega_n + r)$, $z_k = (1-\gamma^k) z + \gamma^k \overline{z}$ and $\omega_k = \gamma^k \omega + (1-\gamma^k)\overline{\omega}$, $k \ge 0$. Then, by using the fact that the bound of Inequality (2.7) holds whenever $|x - y| \le \gamma x_n$, we obtain that

$$|u(z_{k+1})-u(z_k)|\le C|z_{k+1}-z_k|^{\theta}\|f\|_{L^p(\Omega)}=C|\gamma^k(z-\overline{z})(\gamma-1)|^{\theta}\|f\|_{L^p(\Omega)}$$
$$\le C(\gamma^k|z-\overline{z}|)^{\theta}\|f\|_{L^p(\Omega)}=C(\gamma^k r)^{\theta}\|f\|_{L^p(\Omega)}.$$

Moreover, since $x_n>r$ in all the segment joining $\overline{z}$ and $\overline{\omega}$, splitting this segment into a finite number of segments of length less than $\gamma r$, we get that

$$|u(\overline{z})-u(\overline{\omega})|\le C|\overline{z}-\overline{\omega}|^{\theta}\|f\|_{L^p(\Omega)}\le C(\gamma r)^{\theta}\|f\|_{L^p(\Omega)}.$$

Therefore,

$$\begin{aligned}|u(z)-u(\omega)|&\le\sum_{k\ge0}|u(z_{k+1})-u(z_k)|+|u(\overline{z})-u(\overline{\omega})|+\sum_{k\ge0}|u(\omega_{k+1})-u(\omega_k)|\\&\le\Big(C\sum_{k\ge0}(\gamma^k r)^{\theta}+C(\gamma r)^{\theta}\Big)\|f\|_{L^p(\Omega)}\\&\le C\|f\|_{L^p(\Omega)}|z-\omega|^{\theta}.\end{aligned}$$

So $\|u\|_{C^{\theta}(\Omega)}\le C\|f\|_{L^p(\Omega)}$, $\theta\in(0,2\alpha-1)$.

Let $f\in L^p(\Omega)$ with $p>\frac{N}{2-2\alpha}$. By (A.2) we have that $W_0^{\alpha,2}(\Omega)\hookrightarrow L^{\frac{N}{N-2+2\alpha}}(\Omega)$, and then $L^p(\Omega)\hookrightarrow L^{\frac{N}{2-2\alpha}}(\Omega)\hookrightarrow W^{-\alpha,2}(\Omega)$. Then the Dirichlet problem (1.5) has a unique solution, and $f\in W^{-\alpha,2}(\Omega)$.

Since $\langle(-\Delta)^{\alpha}_{\Omega}u,u\rangle=\langle f,u\rangle$. From Remark A.1, we have that

$$\|u\|^2_{W_0^{\alpha,2}(\Omega)}=\langle(-\Delta)^{\alpha}_{\Omega}u,u\rangle=\frac{C_{N,\alpha}}{2}\int_{\Omega}\int_{\Omega}\frac{(u(x)-u(y))^2}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y.$$

By using Cauchy -Schwartz inequality, we get that $\|u\|^2_{W_0^{\alpha,2}(\Omega)}\le C\|f\|_{W^{-\alpha,2}(\Omega)}\|u\|_{W_0^{\alpha,2}(\Omega)}$. Then $\|u\|_{W_0^{\alpha,2}(\Omega)}\le C\|f\|_{W^{-\alpha,2}(\Omega)}\le C\|f\|_{L^p(\Omega)}$. That is $u\in W_0^{\alpha,2}(\Omega)$, which completes the proof. □

## 3. Höder regularity of solutions to the Dirichlet problem in a special case

In this section, we study the Hölder regularity of weak solutions to the following Dirichlet problem

$$\begin{cases}(-\Delta)^{\alpha}_{\Omega}u=0, & \text{in }\Omega\\ u=g, & \text{on }\partial\Omega,\end{cases}\tag{3.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded open set with $C^{1,1}$ boundary $\partial\Omega$. Here $g$ is a continuous function on the boundary $\partial\Omega$ and is denoted by $g\in C(\partial\Omega)$.

First, we introduce some preliminary results that will be useful for the proof of Theorem 1.7.

**Lemma 3.1** *There exists a constant $C>0$ and a function $\omega\in L^{\infty}(\Omega)$, such that*

$$\begin{cases}(-\Delta)^{\alpha}_{\Omega}\omega\ge0, & \textit{in }\Omega\\ \omega=1, & \textit{on }\partial\Omega\\ \omega\le C, & \textit{in }\Omega.\end{cases}\tag{3.2}$$

**Proof** Let $f \in L^p(\Omega)$ with $p > \frac{N}{2-2\alpha}$, and $\alpha \in (\frac{1}{2}, 1)$. Assume $\Omega \subset \mathbb{R}^N$ is a bounded open set with $C^{1,1}$ boundary $\partial\Omega$. By Proposition 1.5, the Dirichlet problem (1.5) has a unique weak solution $u_f$ such that

$$-C_1\|f_-\|_{L^p(\Omega)}\rho(x)^{2\alpha-1} \leq u_f(x) \leq C_1\|f_+\|_{L^p(\Omega)}\rho(x)^{2\alpha-1}, \quad x \in \Omega$$

holds for some $C_1 > 0$. Now replace $f$ by $f_0$, which satisfies: $f_0 \geq 0$ in $\Omega$ and $f_0 \in L^p(\Omega)$, and the corresponding equations have a unique solution $u_0$, satisfying

$$\begin{cases} (-\Delta)^\alpha_\Omega u_0 = f_0 \geq 0, & \text{in } \Omega \\ u_0 = 0, & \text{on } \partial\Omega \\ u_0 \leq C', & \text{in } \Omega, \end{cases}$$

for some $C' > 0$. Choose $\omega = u_0 + 1$ and $C > C' + 1$ to get our desired result. □

Base on Lemma 3.1, we get the following maximum principle for the Dirichlet problem (3.1).

**Property 3.2** *Assume that $u$ is a weak solution of the Dirichlet problem (3.1). Then*

$$\sup_\Omega |u| \leq C \sup_{\partial\Omega} |g|,$$

*where $C$ is the same constant as in Lemma 3.1.*

**Proof** Denote $v(x) := \sup_{\partial\Omega} |g| \cdot \omega(x)$, where $\omega$ is the same as in Lemma 3.1. Then $(-\Delta)^\alpha_\Omega v = \sup_{\partial\Omega} |g| \cdot (-\Delta)^\alpha_\Omega \omega \geq 0$ in $\Omega$. Assume $u$ is a weak solution of the Dirichlet problem (3.1), then $(-\Delta)^\alpha_\Omega u = 0$ in $\Omega$. Therefore, $(-\Delta)^\alpha_\Omega u \leq (-\Delta)^\alpha_\Omega v$ in $\Omega$, and $u = g \leq \sup_{\partial\Omega} |g| = v$ on $\partial\Omega$, since $\omega = 1$ on $\partial\Omega$.

By using the comparison principle (Lemma 1.8), we get $u \leq v$ in $\Omega$. Since $v(x) = \sup_{\partial\Omega} |g| \cdot \omega(x) \leq C \sup_{\partial\Omega} |g|$ in $\Omega$, then we have that $u \leq C \sup_{\partial\Omega} |g|$ in $\Omega$, where $C$ is the same as in Lemma 3.1. Applying the same argument to $(-u)$, we have that $-u \leq C \sup_{\partial\Omega} |g|$ in $\Omega$, and the result follows. □

In what follows we will construct a sequence of $C^\infty$ functions, which uniformly converge to $g$ on the boundary of $\Omega$. We change the Dirichlet problem (3.1) into the form of the Dirichlet problem (1.5), then we get the desired results.

In the following part, when we mention a cube, we mean a closed cube in $\mathbb{R}^N$ with sides parallel to axes, and two cubes will be said to be disjoint if their interiors are disjoint.

**Proposition 3.3** ([20]) *Let $g \in C(\partial\Omega)$. Then there is a sequence of $C^\infty(\overline{\Omega})$ functions $\{h_n\}_{n=1}^\infty$, such that, $h_n$ converges to $g$ uniformly in $\overline{\Omega}$ as $n \to \infty$.*

**Proof** Construct a series of $C^\infty(\overline{\Omega})$ functions $\{h_n\}_{n=1}^\infty$ as follows:

(1) We write $\Omega = \bigcup_j Q_{n_j}$, where $Q_{n_j}$s are disjoint;

(2) Pick a point $x_{n_j} \in \partial\Omega$ that realizes the distance $\mathrm{dist}(Q_{n_j}, \partial\Omega)$;

(3) If $\mathrm{dist}(Q_{n_j}, \partial\Omega) > \frac{1}{n}$, define $\varphi_{n_j} = 0$. If $\mathrm{dist}(Q_{n_j}, \partial\Omega) \leq \frac{1}{n}$, construct a $C^\infty$ function $\psi_{n_j}$ with properties: (a) $0 \leq \psi_{n_j} \leq 1$; (b) $\psi_{n_j}(x) = 1$, if $x \in Q_{n_j}$; (c) $\psi_{n_j}(x) = 0$, if $x \notin \frac{3}{2}Q_{n_j}$. Define $\varphi_{n_j}(x) := \frac{\psi_{n_j}(x)}{\phi_{n_j}(x)}$, where $\phi_{n_j}(x) = \sum_j \psi_{n_j}(x)$. Then $\varphi_{n_j}$ is the partition of unity subordinate to

the cover $\frac{3}{2}Q_{n_j}$;

(4) Define $h_n = g$ on $\partial\Omega$, and $h_n = \sum_j g(x_{n_j})\varphi_{n_j}$ in $\Omega$. Then, $\{h_n\}_{n=1}^{\infty}$ has the desired properties.

Indeed, $h_n$ is $C^\infty$ smooth, being a locally finite sum of $C^\infty$ functions. As $x \to \zeta \in \partial\Omega$, the values of $g$ used in the construction of $h_n$ are taken from progressively smaller neighborhoods of $\zeta$. Hence, $h_n(\zeta)$ converge to $g(\zeta)$ uniformly. By the definition of $h_n$ we get $h_n = g$ on $\partial\Omega$.

As $n \to \infty$, $\text{dist}(Q_{n_j}, \partial\Omega) \le \frac{1}{n} \to 0$. Then, we get $h_n \to g$. We complete our proof. □

In order to obtain the Hölder regularity for $h_n \in C^\infty(\overline{\Omega})$ with $n \in \mathbb{N}$, we give the following result taken from [21, Proposition 5.4], in a special case. We first introduce the following notations.

$$M(u) := \sup_{x\in\Omega} |\nabla u(x)|.$$

$$N(u) := \sup_{x\neq y\in\Omega} \sum_{i=1}^{N} \frac{|\partial_i u(x) - \partial_i u(y)|}{|x-y|}.$$

$$\rho(x) := \text{dist}(x, \partial\Omega) = \inf\{|x-y| : y \in \partial\Omega\}.$$

$$d_\Omega := \text{diameter of } \Omega = \sup\{|x-y| : x, y \in \Omega\}.$$

**Lemma 3.4** *If $u \in C^2(\Omega)$, then $(-\Delta)_\Omega^\alpha u$ is continuous in $\Omega$ and admits the following estimate*

$$|(-\Delta)_\Omega^\alpha u| \le C_{N,\alpha}(2\pi)^N \Big(\frac{M(u)}{2\alpha-1}\rho(x)^{1-2\alpha} + \frac{N(u)}{2-2\alpha} d_\Omega^{2-2\alpha}\Big).$$

**Proposition 3.5** *For each $n \in \mathbb{N}$, the following Dirichlet problem*

$$\begin{cases} (-\Delta)_\Omega^\alpha u_n = 0, & \text{in } \Omega \\ u_n = h_n, & \text{on } \partial\Omega \end{cases} \tag{3.3}$$

*has a unique weak solution $u_n \in C(\overline{\Omega}) \cap W^{\alpha,2}(\Omega)$. Moreover, there is a constant $C > 0$, such that $\|u_n\|_{L^\infty(\Omega)} \le C\|g\|_{L^\infty(\partial\Omega)}$ and $\|u_n\|_{W^{\alpha,2}(\Omega)} \le C\|g\|_{L^\infty(\partial\Omega)}$, where $h_n = \sum_j g(x_{n_j})\varphi_{n_j}$ is the same as we defined in Proposition 3.3, and $n = 1, 2, \ldots$.*

**Proof** From the construction of $h_n$ in Proposition 3.3, we know that $h_n \in C^\infty(\overline{\Omega})$. By using Proposition 3.4, we get that $(-\Delta)_\Omega^\alpha h_n$ is continuous in $\Omega$. Moreover

$$|(-\Delta)_\Omega^\alpha h_n(x)| \le C_{N,\alpha}(2\pi)^N \Big(\frac{M(h_n)}{2\alpha-1}\rho(x)^{1-2\alpha} + \frac{L(h_n)}{2-2\alpha} d_\Omega^{2-2\alpha}\Big), \quad x \in \Omega, \tag{3.4}$$

where $\rho(x) := \text{dist}(x, \partial\Omega) = \inf\{|x-y| : y \in \partial\Omega\}$, $d_\Omega :=$ diameter of $\Omega = \sup\{|x-y| : x, y \in \Omega\}$.

$$\begin{aligned} M(h_n) &:= \sup_{x\in\Omega} |\nabla h_n(x)| = \sup_{x\in\Omega} \Big|\nabla\Big(\sum_j g(x_{n_j})\varphi_{n_j}(x)\Big)\Big| \\ &\le \|g\|_{L^\infty(\partial\Omega)} \sup_{x\in\Omega} \Big|\nabla\Big(\sum_j \varphi_{n_j}(x)\Big)\Big| \le C\|g\|_{L^\infty(\partial\Omega)} \end{aligned}$$

and

$$L(h_n) := \sup_{x\neq y\in\Omega} \sum_{i=1}^{N} \frac{|\partial_i h_n(x) - \partial_i h_n(y)|}{|x-y|}$$

$$\leq \|g\|_{L^\infty(\partial\Omega)} \sup_{x\neq y\in\Omega} \sum_{i=1}^{N} \frac{|\partial_i\varphi_{n_j}(x) - \partial_i\varphi_{n_j}(y)|}{|x-y|} \leq C\|g\|_{L^\infty(\partial\Omega)}.$$

Note that $\varphi_{n_j} \in C^\infty(\overline{\Omega})$, which is a partition of unity. From the preceding estimates and Inequality (3.4), we get that

$$|(-\Delta)^\alpha_\Omega h_n(x)| \leq C\|g\|_{L^\infty(\partial\Omega)},$$

where $C$ is a constant depending on $N$, $\Omega$ and $\alpha$. Denote $w_n := u_n - h_n$, then we get that

$$\begin{cases} (-\Delta)^\alpha_\Omega w_n = -(-\Delta)^\alpha_\Omega h_n, & \text{in } \Omega \\ w_n = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

By Theorem 1.6, we know that the Dirichlet problem (3.5) has a unique weak solution $w_n \in C(\overline{\Omega}) \cap W^{\alpha,2}_0(\Omega)$ satisfying

$$\frac{C_{N,\alpha}}{2}\int_\Omega\int_\Omega \frac{(w_n(x)-w_n(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y = \langle -(-\Delta)^\alpha_\Omega h_n, v\rangle_{W^{-\alpha,2}(\Omega),W^{\alpha,2}_0(\Omega)}.$$

Moreover

$$\|w_n\|_{L^\infty(\Omega)} \leq C\|g\|_{L^\infty(\partial\Omega)}, \quad \|w_n\|_{W^{\alpha,2}_0(\Omega)} \leq C\|g\|_{L^\infty(\partial\Omega)}.$$

Then

$$\frac{C_{N,\alpha}}{2}\int_\Omega\int_\Omega \frac{(u_n(x)-u_n(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y = \langle 0, v\rangle_{W^{-\alpha,2}(\Omega),W^{\alpha,2}_0(\Omega)} = 0,$$

for every $v \in W^{\alpha,2}_0(\Omega)$, and $u_n \in K = \{v \in W^{\alpha,2}(\Omega) : v - \widetilde{g_n} \in W^{\alpha,2}_0(\Omega)\}$, where $\widetilde{g_n} \in W^{\alpha,2}(\Omega)$ and $\widetilde{g_n} = g_n$ on $\partial\Omega$. Then, the Dirichlet problem (3.3) has a unique weak solution $u_n \in C(\overline{\Omega}) \cap W^{\alpha,2}(\Omega)$, and

$$\|u_n\|_{L^\infty(\Omega)} = \|w_n + h_n\|_{L^\infty(\Omega)} \leq \|w_n\|_{L^\infty(\Omega)} + \|h_n\|_{L^\infty(\Omega)} \leq C\|g\|_{L^\infty(\partial\Omega)},$$

where we use the fact that

$$\|h_n\|_{L^\infty(\Omega)} = \Big\|\sum_j g(x_{n_j})\varphi_{n_j}\Big\|_{L^\infty(\Omega)} \leq C\|g\|_{L^\infty(\partial\Omega)}. \tag{3.6}$$

We also have that

$$\|h_n\|_{W^{\alpha,2}(\Omega)} = \Big(\int_\Omega |h_n(x)|^2\mathrm{d}x + \int_\Omega\int_\Omega \frac{|h_n(x)-h_n(y)|^2}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{2}} \tag{3.7}$$

and

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{|h_n(x)-h_n(y)|^2}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y \\
&\leq C\|g\|^2_{L^\infty(\partial\Omega)}\int_\Omega\int_\Omega \frac{|\sum_j\varphi_{n_j}(x) - \sum_j\varphi_{n_j}(y)|^2}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y \\
&= C\|g\|^2_{L^\infty(\partial\Omega)}\int_\Omega\int_\Omega \Big(\frac{|\sum_j\varphi_{n_j}(x) - \sum_j\varphi_{n_j}(y)|^2}{|x-y|^2}\frac{1}{|x-y|^{N+2\alpha-2}}\Big)\mathrm{d}x\mathrm{d}y \\
&\leq C\|g\|^2_{L^\infty(\partial\Omega)}\int_\Omega\Big(C\Big[\sup_{x\in\Omega}\Big|\nabla\Big(\sum_j\varphi_{n_j}(x)\Big)\Big|\Big]^2\int_\Omega\frac{1}{|x-y|^{N+2\alpha-2}}\mathrm{d}x\Big)\mathrm{d}y \\
&\leq C\|g\|^2_{L^\infty(\partial\Omega)},
\end{aligned} \tag{3.8}$$

where the partition of unity $\varphi_{n_j}(x) \in C^\infty(\overline{\Omega})$, and then $\sup_{x\in\Omega} |\nabla(\sum_j \varphi_{n_j}(x))| < \infty$. Since $N-1 < N+2\alpha-2 < N$, then we get $\int_\Omega \frac{1}{|x-y|^{N+2\alpha-2}}\mathrm{d}x < \infty$. Therefore,

$$\|h_n\|_{W^{\alpha,2}(\Omega)} \le C\|g\|_{L^\infty(\partial\Omega)}.$$

By Remark A.1, we notice that $\|w_n\|_{W^{\alpha,2}(\Omega)}$ and $\|w_n\|_{W_0^{\alpha,2}(\Omega)}$ are equivalent in $W_0^{\alpha,2}(\Omega)$. Then there is a constant $C > 0$, such that $\|w_n\|_{W^{\alpha,2}(\Omega)} \le C\|w_n\|_{W_0^{\alpha,2}(\Omega)}$.

From (3.5)–(3.8), we get that

$$\|u_n\|_{W^{\alpha,2}(\Omega)} = \|w_n + h_n\|_{W^{\alpha,2}(\Omega)} \le C(\|w_n\|_{W_0^{\alpha,2}(\Omega)} + \|h_n\|_{W^{\alpha,2}(\Omega)}) \le C\|g\|_{L^\infty(\partial\Omega)},$$

where $C > 0$ is a constant. We complete the proof. □

**Proof of Theorem 1.7** Assume $\Omega \subset \mathbb{R}^N$ is a bounded $C^{1,1}$ open set, $\alpha \in (\frac{1}{2}, 1)$ and $g \in C(\partial\Omega)$. We extend $g$ by 0 on $\mathbb{R}^N \setminus \Omega$.

Firstly, we show that the Dirichlet problem (3.1) has a unique weak solution $u \in C(\overline{\Omega}) \cap \in W^{\alpha,2}(\Omega)$.

Let $\{h_n\}_{n=1}^\infty$ be a sequence of $C^\infty(\overline{\Omega})$ functions as we defined in Proposition 3.3. We had proved that $h_n$ converge uniformly to $g$ in $\overline{\Omega}$. By Proposition 3.5, there is $u_n \in C(\overline{\Omega}) \cap W^{\alpha,2}(\Omega)$ such that

$$\begin{cases} (-\Delta)^\alpha_\Omega u_n = 0, & \text{in } \Omega \\ u_n = h_n, & \text{on } \partial\Omega, \end{cases}$$

for every $n \in \mathbb{N}$.

By using Theorem 3.2, we obtain that

$$\sup_\Omega |u_n - u_m| \le C \sup_{\partial\Omega} |h_n - h_m| \to 0 \text{ as } m, n \to \infty.$$

Therefore, $u_n$ converges uniformly to a function $u \in C(\overline{\Omega})$, and satisfying $u = g$ on $\partial\Omega$. Considering Proposition 3.5, we have that $\|u_n\|_{L^\infty(\Omega)} \le C\|g\|_{L^\infty(\partial\Omega)}$, and $\|u_n\|_{W^{\alpha,2}(\Omega)} \le C\|g\|_{L^\infty(\partial\Omega)}$, for each $n \in \mathbb{N}$. Then $u_n$ converges weakly to a function $w$ in $W^{\alpha,2}(\Omega)$. And then $u_n$ converges strongly to $w$ in $L^2(\Omega)$, since $W^{\alpha,2}(\Omega) \hookrightarrow L^2(\Omega)$. By the uniqueness of limit, we have that $u = w$, and then $u \in C(\overline{\Omega}) \cap W^{\alpha,2}(\Omega)$.

By the fact that $u_n$ converges weakly to $u$ in $W^{\alpha,2}(\Omega)$, that is

$$\lim_{n\to\infty} (u_n, \varphi) = (u, \varphi), \quad \text{for all } \varphi \in W^{\alpha,2}, \tag{3.9}$$

where

$$(u, \varphi) = \int_\Omega u\varphi \mathrm{d}x + \int_\Omega \int_\Omega \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2\alpha}} \mathrm{d}x\mathrm{d}y.$$

And $u_n$ converges to $u$ in $L^2(\Omega)$, that is

$$\lim_{n\to\infty} \int_\Omega u_n\varphi \mathrm{d}x = \int_\Omega u\varphi \mathrm{d}x. \tag{3.10}$$

By Equalities (3.9) and (3.10), we get that

$$\lim_{n\to\infty} \int_\Omega \int_\Omega \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2\alpha}} \mathrm{d}x\mathrm{d}y = \int_\Omega \int_\Omega \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2\alpha}} \mathrm{d}x\mathrm{d}y.$$

Then

$$\begin{aligned}
&\frac{C_{N,\alpha}}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y\\
&=\lim_{n\to\infty}\frac{C_{N,\alpha}}{2}\int_\Omega\int_\Omega \frac{(u_n(x)-u_n(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y\\
&=\langle 0,v\rangle_{W^{-\alpha,2}(\Omega),W_0^{\alpha,2}(\Omega)}=0,
\end{aligned}$$

for every $v\in W_0^{\alpha,2}(\Omega)$, and $u\in K=\{v\in W^{\alpha,2}(\Omega): v-\widetilde{g}\in W_0^{\alpha,2}(\Omega)\}$, where $\widetilde{g}\in W^{\alpha,2}(\Omega)$ and $\widetilde{g}=g$ on $\partial\Omega$. From Definition 1.3, we get that $u\in C(\overline{\Omega})\cap W^{\alpha,2}(\Omega)$ is the unique weak of the Dirichlet problem (3.1).

Secondly, we show that $u$ satisfies the inequalities in Theorem 1.7.

By Proposition 3.5, we get that $\|u_n\|_{L^\infty(\Omega)}\le C\|g\|_{L^\infty(\partial\Omega)}$, and $\|u_n\|_{W^{\alpha,2}(\Omega)}\le C\|g\|_{L^\infty(\partial\Omega)}$, for each $n\in\mathbb{N}$. Together with $u=\lim_{n\to\infty}u_n$, we get that $\|u\|_{L^\infty(\Omega)}\le C\|g\|_{L^\infty(\partial\Omega)}$, and $\|u\|_{W^{\alpha,2}(\Omega)}\le C\|g\|_{L^\infty(\partial\Omega)}$. We complete the proof. □

## 4. Hölder regularity of weak solution for the general Dirichlet problem

In this section, we use Theorems 1.5 and 1.6 to show the Hölder regularity of weak solutions to the general Dirichlet problem (1.1).

**Proof of Theorem 1.4** Firstly, we show existence.

Let $v$ be a weak solution of the Homogeneous Dirichlet problem (1.5). By Definition 1.1, for every $\varphi\in W_0^{\alpha,2}$, we have that

$$\frac{C_{N,\alpha}}{2}\int_\Omega\int_\Omega \frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y=\langle v,\varphi\rangle_{W^{-\alpha,2}(\Omega),W_0^{\alpha,2}(\Omega)}. \tag{4.1}$$

Together with Theorem 1.6, we obtain that $v\in W_0^{\alpha,2}(\Omega)\cap C(\overline{\Omega})$ is the unique weak solution of Problem (1.3). Moreover

$$\|v\|_{L^\infty(\Omega)}\le C\|f\|_{L^p(\Omega)} \tag{4.2}$$

and

$$\|v\|_{W_0^{\alpha,2}(\Omega)}\le C\|f\|_{L^p(\Omega)}. \tag{4.3}$$

Assume $w$ is a weak solution of the Inhomogeneous Dirichlet problem (3.1). By Definition 1.3, for each $w\in K=\{w\in W^{\alpha,2}(\Omega): w-\widetilde{g}\in W_0^{\alpha,2}(\Omega)\}$, and $\varphi\in W_0^{\alpha,2}$, the equality

$$\frac{C_{N,\alpha}}{2}\int_\Omega\int_\Omega \frac{(w(x)-w(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y=\langle w,\varphi\rangle_{W^{-\alpha,2}(\Omega),W_0^{\alpha,2}(\Omega)} \tag{4.4}$$

holds. Considering Theorem 1.7, we get that $w\in W^{\alpha,2}(\Omega)\cap C(\overline{\Omega})$ is the unique weak solution of Problem (3.1). Moreover

$$\|w\|_{L^\infty(\Omega)}\le C\|g\|_{L^\infty(\partial\Omega)} \tag{4.5}$$

and

$$\|w\|_{W^{\alpha,2}(\Omega)}\le C\|g\|_{L^\infty(\partial\Omega)}. \tag{4.6}$$

Denote $u := v + w$, then $u \in K = \{\varphi \in W^{\alpha,2}(\Omega) : \varphi - \widetilde{g} \in W_0^{\alpha,2}(\Omega)\}$. By using Inequalities (4.1) and (4.4), the equality

$$\frac{C_{N,\alpha}}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y = \langle u,\varphi\rangle_{W^{-\alpha,2}(\Omega),W_0^{\alpha,2}(\Omega)}$$

holds, for every $\varphi \in W_0^{\alpha,2}$. Then, by Definition 1.3 $u$ is a weak solution of the Dirichlet problem (1.1). Concerning Inequalities (4.2) and (4.5), we notice that

$$\|u\|_{L^\infty(\Omega)} \le \|v\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} \le C(\|f\|_{L^p(\Omega)} + \|g\|_{L^\infty(\partial\Omega)}),$$

where $C > 0$ is a constant.

By Remark A.1, it is natural to get that $\|v\|_{W^{\alpha,2}(\Omega)}$ and $\|v\|_{W_0^{\alpha,2}(\Omega)}$ are equivalent in $W_0^{\alpha,2}(\Omega)$. That is, there is a constant $C > 0$ such that $\|v\|_{W^{\alpha,2}(\Omega)} \le C\|v\|_{W_0^{\alpha,2}(\Omega)}$, together with (4.2) and (4.5), we obtain that

$$\begin{aligned}\|u\|_{W^{\alpha,2}(\Omega)} &\le \|v\|_{W^{\alpha,2}(\Omega)} + \|w\|_{W^{\alpha,2}(\Omega)}\\ &\le C(\|v\|_{W_0^{\alpha,2}(\Omega)} + \|w\|_{W^{\alpha,2}(\Omega)}) \le C(\|f\|_{L^p(\Omega)} + \|g\|_{L^\infty(\partial\Omega)}),\end{aligned}$$

where $C > 0$ is a constant.

Finally, we show uniqueness.

Suppose $u'$ and $u$ ($u' \ne u$) are two different weak solutions of the Dirichlet problem (1.1). By the preceding proof, we know that, $u'$ can be written as $u' = v' + w'$, where $v'$ and $w'$ are the weak solutions to the Dirichlet problems (1.5) and (3.1), respectively. By the uniqueness of $v'$ and $w'$, we get $v = v'$ and $w = w'$. Thus $u = u'$, which is a contradiction to the assumption. That is, the Dirichlet problem (1.1) has a unique weak solution. We complete the proof. □

## Appendix A.

For the sake of completeness, we first give a rigorous definition of the regional fractional Laplacian. This definition is similar to the one of the fractional Laplacian in [5]. Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set, $0 < \alpha < 1$, and

$$\mathcal{L}_\alpha^1(\Omega) := \Big\{u : \Omega \to \mathbb{R} \text{ measurable}, \ \int_\Omega \frac{|u(x)|}{(1+|x|)^{N+2\alpha}}\mathrm{d}x < \infty\Big\}.$$

We restrict the integral kernel of the fractional Laplacian to the open set $\Omega$. For $u \in \mathcal{L}_\alpha^1(\Omega)$, $x \in \Omega$ and $\varepsilon > 0$, we write

$$(-\Delta)_{\Omega,\varepsilon}^\alpha u(x) := C_{N,\alpha}\int_{\{y\in\Omega,|y-x|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2\alpha}}\mathrm{d}y, \quad x \in \Omega.$$

If $u \in \mathcal{L}_\alpha^1(\Omega)$, $(-\Delta)_{\Omega,\varepsilon}^\alpha u$ is well defined for every $\varepsilon > 0$, and it is continuous where $u$ is continuous. We define the operator

$$(-\Delta)_\Omega^\alpha u(x) := C_{N,\alpha}P.V.\int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2\alpha}}\mathrm{d}y = \lim_{\varepsilon\downarrow 0}(-\Delta)_{\Omega,\varepsilon}^\alpha u(x), \quad x \in \Omega, \tag{A.1}$$

provided that the limit exists. The operator $A_\Omega^\alpha$ is called the regional fractional Laplacian.

We assume that $u \in \mathcal{L}^1_\alpha(\Omega)$, when we consider the regional fractional Laplacian. Similar as the case of $\mathbb{R}^N$, if $\alpha \in (0, \frac{1}{2})$ and $u$ is smooth (for Example, $u \in L^\infty(\Omega) \cap C^{0,1}(\Omega)$), the integral in Equality (A.1) is not really singular near $x$ (see [2, Remark 3.1]).

Now we introduce some facts about the fractional Sobolav space. Let $\Omega \subset \mathbb{R}^N$ be an open set, $p \in [1, \infty)$ and $\alpha \in (0,1)$. The fractional order Sobolev space is denoted by

$$W^{\alpha,p}(\Omega) := \Big\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+p\alpha}} \mathrm{d}x\mathrm{d}y < \infty \Big\},$$

and we endow it with the norm

$$\|u\|_{W^{\alpha,p}(\Omega)} := \Big( \int_\Omega |u|^p \mathrm{d}x + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+p\alpha}} \mathrm{d}x\mathrm{d}y \Big)^{\frac{1}{p}}.$$

Let $W^{\alpha,p}_0(\Omega)$ denote the closure of $\mathcal{D}(\Omega)$ in the norm of $W^{\alpha,p}(\Omega)$ defined above, where $\mathcal{D}(\Omega) = C^\infty_0(\Omega)$ is the space of all continuously infinitely differentiable functions with compact support in $\Omega$. From [13], we have that

$$W^{\alpha,p}_0(\mathbb{R}^N) = W^{\alpha,p}(\mathbb{R}^N),$$

but in general, for $\Omega \subset \mathbb{R}^N$, $W^{\alpha,p}_0(\Omega) \neq W^{\alpha,p}(\Omega)$, i.e., $\mathcal{D}(\Omega)$is not always dense in $W^{s,p}(\Omega)$.

**Remark A.1** From [7], we know that

$$\|u\|_{W^{\alpha,2}_0(\Omega)} := \Big( \frac{C_{N,\alpha}}{2} \int_\Omega \int_\Omega \frac{[u(x) - u(y)]^2}{|x-y|^{N+2\alpha}} \mathrm{d}x\mathrm{d}y \Big)^{\frac{1}{2}}$$

defines an equivalent norm on $W^{\alpha,2}_0(\Omega)$ with $\frac{1}{2} < \alpha < 1$. This norm is equivalent to $\|u\|_{W^{\alpha,2}(\Omega)}$ in $W^{\alpha,2}_0(\Omega)$ and the associated scalar product of $\|\cdot\|_{W^{\alpha,2}_0(\Omega)}$ is

$$\mathcal{E}_\Omega(u,v) := \frac{C_{N,\alpha}}{2} \int_\Omega \int_\Omega \frac{[u(x) - u(y)][v(x) - v(y)]}{|x-y|^{N+2\alpha}} \mathrm{d}x\mathrm{d}y, \quad u, v \in W^{\alpha,2}_0(\Omega).$$

**Remark A.2** In case $p = 2$, the fractional order Sobolev spaces $W^{\alpha,2}(\Omega)$ and $W^{\alpha,2}_0(\Omega)$ turn out to Hilbert spaces. They are usually denoted by $H^\alpha(\Omega)$ and $H^\alpha_0(\Omega)$, respectively.

For $p \in (1, \infty)$ and $\alpha \geq 0$, we define the dual space of $W^{\alpha,p}_0(\Omega)$ as

$$W^{-\alpha,q}(\Omega) := (W^{\alpha,p}_0(\Omega))^\star, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

and equip it with the dual norm

$$\|f\|_{W^{-\alpha,q}(\Omega)} := \sup\{ |\langle f, v \rangle_\Omega| : v \in W^{\alpha,p}_0(\Omega), \|v\|_{W^{\alpha,p}_0(\Omega)} = 1 \}.$$

Finally, we mention a well known inequality. Let $A \subset \mathbb{R}^N$ be a bounded set and $B \subset \mathbb{R}^N$ an arbitrary set. Then there exists a constant $C > 0$ (depending on A and B) such that

$$|x - y| \geq C(1 + |y|), \quad \forall\, x \in A, \ \forall\, y \in \mathbb{R}^N \backslash B, \quad \mathrm{dist}(A, \mathbb{R}^N \backslash B) = \delta > 0. \tag{A.2}$$

**Acknowledgements** We thank the referees for their time and comments.

# References

[1] L. C. EVANS, R. F. GARIEPY. *Measure Theory and Fine Properties of Functions*. CRC Press, Champaign, 1992.

[2] E. D. NEZZA, G. PALATUCCI, E. VALDINOCI. *Hitchhikers guide to the fractional Sobolev spaces*. Bull. Sci. Math., 2012, **136**(5): 521–573.

[3] L. CAFFARELLI, L. SILVESTRE. *An extension problem related to the fractional Laplacian*. Comm. Partial Differential Equations, 2007, **32**(8): 1245–1260.

[4] X. ROS-OTON, J. SERRA. *The extremal solution for the fractional Laplacian*. Calc. Var. Partial Differential Equations, 2014, **50**(3-4): 723–750.

[5] U. BICCARI, M. WARMA, E. ZUAZUA. *Local elliptic regularity for the Dirichlet fractional Laplacian*. Adv. Nonlinear Stud., 2017, **17**(2): 387–409.

[6] C. G. GAL, M. WARMA. *Nonlocal transmission problems with fractional difusion and boundary conditions on non-smooth interfaces*. Comm. Partial Differential Equations, 2017, **42**(83): 579–625.

[7] Huyuan CHEN, L. VÉRON. *Weakly and strongly singular solutions of semilinear fractional elliptic equations*. Asymptot. Anal., 2014, **88**(3): 165–184.

[8] Huyuan CHEN. *The Dirichlet Eillpitic problem involving regional fractional Lapacian*. J. Math. Phys., 2015, **59**(7): 1509–1538.

[9] X. ROS-OTON, J. SERRA. *The Dirichlet problem for the fractional Laplacian: regualrity up to the boundary*. J. Math. Pures Appl., 2014, **101**(3): 275–302.

[10] L. CAFFARELLI, S. SALSA, L. SILVESTRE. *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian*. Invent. Math., 2008, **171**(2): 425–461.

[11] E. M. OUHABAZ. *Analysis of Heat Equations on Domains*. London Mathematical Society Monographs Series, 31. Princeton University Press, Princeton, 2005.

[12] A. VELEZ-SANTIAGO, M. WARMA. *A class of quasi-linear parabolic and elliptic equations with nonlocal Robin boundary conditions*. J. Math. Anal. Appl., 2010, **372**(1): 120–139.

[13] M. WARMA. *The p-Laplace operator with the nonlocal Robin boundary conditions on arbitrary open sets*. Ann. Mat. Pura Appl., 2014, **193**(4): 203–235.

[14] X. ROS-OTON, J. SERRA. *The Pohozaev identity for the fractional Laplacian*. Arch. Ration. Mech. Anal., 2014, **213**(2): 587–628.

[15] H. BREZIS. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, New York, 2011.

[16] L. SILIVER. *Regularity of obestacle problems for a fractional power of the Laplace operator*. Comm. Puer Appl. Math., 2007, **60**(1): 67–112.

[17] X. ROS-OTON. *Nonlocal elliptic equations in bounded domins: A survey*. Publ. Mat., 2016, **60**(1): 3–26.

[18] Qingyang GUAN. *Integration by parts formular for regional frctional Laplacian*. Comm. Math. Phys., 2016, **266**(2): 289–329.

[19] D. GILBARG, N. S. TRUDINGER. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, Berlin, 2001.

[20] E. STEIN. *Singular Integrals and Differentiability Properties of Functions*. Princeton Univ. Press, Princeton, 1970.

[21] Qingyang GUAN, Zhiming MA. *Reflect symmetric $\alpha$-stable processes and regional frational Laplacian*. Probab. Th. Rel. Fields, 2006, **134**(4): 649–694.