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One-pth Riordan Arrays in the Construction of Identities

Tianxiao HE
Department of Mathematics, Illinois Wesleyan University, Bloomington, Illinois 61702, *USA*

In Memory of Professor Leetsch C. Hsu

Abstract For an integer $p \geq 2$ we construct vertical and horizontal one-pth Riordan arrays from a Riordan array. When $p = 2$ one-pth Riordan arrays are reduced to well known half Riordan arrays. The generating functions of the A-sequences of vertical and horizontal one-pth Riordan arrays are found. The vertical and horizontal one-pth Riordan arrays provide an approach to construct many identities. They can also be used to verify some well known identities readily.

Keywords Riordan array; one-pth Riordan arrays; A-sequence; generating function; identities

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1. Introduction

The Riordan group is a group of infinite lower triangular matrices defined by two generating functions. Let $g(z) = g_0 + g_1 z + g_2 z^2 + \cdots$ and $f(z) = f_1 z + f_2 z^2 + \cdots$ with g_0 and f_1 nonzero. Without much loss of generality we will also set $g_0 = 1$. Given $g(z)$ and $f(z)$, the matrix they define is $D = (d_{n,k})_{n,k\geq 0}$, where $d_{n,k} = [z^n]g(z)f(z)^k$. For the sake of readability we often shorten $g(z)$ and $f(z)$ to g and f and we will denote D as (g, f). Essentially the columns of the matrix can be thought of as a geometric sequence with g as the leading term and f as the multiplier term. Two examples are the identity matrix

$$(1, z) = \begin{bmatrix} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \\ & & \cdots & & \ddots \end{bmatrix}$$

and the Pascal matrix

$$(\frac{1}{1-z}, \frac{z}{1-z}) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \\ 1 & 2 & 1 & 0 & \\ 1 & 3 & 3 & 1 & \\ & & \cdots & & \ddots \end{bmatrix}.$$

Here is a list of six important subgroups of the Riordan group [1].

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E-mail address: the@iwu.edu

• the Appell subgroup $\{(g(z),\ z)\}$.

• the Lagrange (associated) subgroup $\{(1,\ f(z))\}$.

• the k-Bell subgroup $\{(g(z),\ z(g(z))^k)\}$, where k is a fixed positive integer.

• the hitting-time subgroup $\{(zf'(z)/f(z),\ f(z))\}$.

• the derivative subgroup $\{(f'(z),\ f(z))\}$.

• the checkerboard subgroup $\{(g(z),\ f(z))\}$, where g is an even function and f is an odd function.

The 1-Bell subgroup is referred to as the Bell subgroup for short, and the Appell subgroup can be considered as the 0-Bell subgroup if we allow $k = 0$ to be included in the definition of the k-Bell subgroup.

The Riordan group acts on the set of column vectors by matrix multiplication. In terms of generating functions we let $d(z) = d_0 + d_1 z + d_2 z^2 + \cdots$ and $h(z) = h_0 + h_1 z + h_2 z^2 + \cdots$. If $[d_0, d_1, d_2, \ldots]^T$ and $[h_0, h_1, h_2, \ldots]^T$ are the corresponding column vectors we observe that

$$(g, f)[d_0, d_1, d_2, \ldots]^T = [h_0, h_1, h_2, \ldots]^T$$

translates to

$$d_0 g(z) + d_1 g(z) f(z) + d_2 g(z) f(z)^2 + \cdots = g(z) \cdot d(f(z)) = h(z).$$

This simple observation is called the Fundamental theorem of Riordan Arrays and is abbreviated as FTRA.

The first application of the fundamental theorem is to set $d(z) = \hat{g}(z)\hat{f}(z)^k$ so that

$$h(z) = g(z) \cdot \hat{g}(f(z))\hat{f}(f(z))^k.$$

As k ranges over $0, 1, 2, \ldots$ the multiplication rule for Riordan arrays emerges.

We define the Riordan group as the set of all pairs (g, f) as above together with the multiplication operation

$$(g, f)(\hat{g}, \hat{f}) = (g \cdot (\hat{g} \circ f), \hat{f} \circ f).$$

The identity element for this group is $(1, z)$. If we denote the compositional inverse of f as $\bar{f}$, then

$$(g, f)^{-1} = (\frac{1}{g \circ \bar{f}}\ \bar{f}).$$

As an example we return to the Pascal matrix where $f = \frac{z}{1-z}$. The inverse is $\overline{f} = \frac{z}{1+z}$, $g(\overline{f}) = \frac{1}{1-(\frac{z}{1+z})} = 1 + z$ and the inverse matrix starts

$$(\frac{1}{1+z}, \frac{z}{1+z}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}.$$

Both Pascal matrix and $(1/(1+z), z/(1+z))$ are pseudo-involution Riordan array due to the fact that their multiplications with $(1, -z)$ are involutions.

Riordan arrays play an important unifying role in enumerative combinatorics, especially in proving combinatorial identities, for instance, some results presented in [2–4], etc. This paper will define a new type of Riordan arrays and study their applications in the construction of identities.

For more information about the Riordan group see Shapiro, Getu, Woan and Woodson [1], Shapiro [5], Barry [6], and Zeleke [7]. Shapiro and the author presented palindromes of pseudo-involutions in a recent paper [8]. For general information about such items as Catalan numbers, Motzkin numbers, generating functions and the like there are many excellent sources including Stanley [9,10] and Aigner [11]. A short survey and an extension of Catalan numbers and Catalan matrices can be seen in [12,13]. Fundamental papers by Sprugnoli [2,14] investigated the Riordan arrays and showed that they constitute a practical device for solving combinatorial sums by means of the generating functions and the Lagrange inversion formula.

For a function f as above, there is a sequence $a_0, a_1, a_2, \dots$ called the A sequence such that

$$f = z(a_0 + a_1 f + a_2 f^2 + a_3 f^3 + \cdots).$$

The corresponding generating function is $A(z) = \sum_{n\geq 0} a_n z^n$ so we have, in terms of generating functions, $f = zA(f)$. See Merlini, Rogers, Sprugnoli, and Verri, [15] for a proof and Sprugnoli and the author [16] and the author [17] for further results. The A sequence enables us to inductively compute the next row of a Riordan matrix since

$$d_{n+1,k} = a_0 d_{n,k-1} + a_1 d_{n,k} + a_2 d_{n,k+1} + \cdots .$$

The missing item is for the left most, i.e., zeroth column and there is a second sequence, the Z sequence such that

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots .$$

The generating function $Z = \sum_{n\geq 0} z_n z^n$ is defined by the equation $g(z) = 1/(1 - zZ(f(z)))$.

By substituting $z = \overline{f}$ into the equation $f = z(A(f))$, we may have $z = \overline{f}(z)A(z) = \overline{f}A$. Similarly, applying $\overline{f}$ gives us a useful alternate form of $g(z) = 1/(1 - zZ(f(z)))$ as $Z = (g(\bar{f}) - 1)/(\bar{f}g(\bar{f}))$. We call $A(z)$ and $Z(z)$ the A and Z functions of the Riordan array (g, f).

We now consider an extension of Riordan arrays called half Riordan arrays, which will be extended to one-pth Riordan arrays in the next section.

The entries of a Riordan array have a multitude of interesting combinatorial explanations. The central entries play a significant role. For instance, the central entries of the Pascal matrix $(1/(1-z), z/(1-z))$ are the central binomial coefficients $\binom{2n}{n}$ (see the sequence A000984 in OEIS [18]) that can be explained as the number of ordered trees with a distinguished point. In addition, its exponential generating function is a modified Bessel function of the first kind. Similarly, the central entries of the Delannoy matrix $(1/(1-z), z(1+z)/(1-z))$, called the Pascal-like Riordan array, are the central Delannoy numbers $\sum_{k=0}^{n} \binom{n}{k}^2 2^k$ (see the sequence A001850 in OEIS [18]). The central Delannoy numbers can be explained as the number of paths from $(0,0)$ to (n,n) in an $n \times n$ grid using only steps north, northeast and east (i.e., steps $(1,0)$, $(1,1)$, and $(0,1)$). In addition, the nth central Delannoy number is the nth Legendre polynomial's value at

3. It is interesting, therefore, to be able to give generating functions of such central terms in a systematic way. In recent papers [19–24] (cf. also the references of [22]), it has been shown how to find generating functions of the central entries of some Riordan arrays.

Yang, Zheng, Yuan, and the author [24] gave the following definition of half Riordan arrays (HRAs), which are called vertical half Riordan arrays in Barry [20] and in [17].

Definition 1.1 *Let $(g, f) = (d_{n,k})_{n,k\geq 0}$ be a Riordan array. Its related half Riordan array $(v_{n,k})_{n,k\geq 0}$, called the vertical half Riordan array (VHRA), is defined by*

$$v_{n,k} = d_{2n-k,n}. \tag{1.1}$$

Denote $\phi = \overline{t^2/f}$. A direct approach is used in [22] to show that $(v_{n,k})_{n,k\geq 0} = (t\phi'(t)g(\phi)/\phi, \phi)$ based on the Lagrange inversion formula.

In [22], a decomposition of $(v_{n,k})_{n,k\geq 0}$ is presented as

$$(\frac{t\phi'(t)g(\phi)}{\phi}, \phi) = (\frac{t\phi'}{\phi}, \phi)(g, t). \tag{1.2}$$

Decomposition (1.2) suggests a more general type of half of Riordan array (g, f) defined by

$$(\frac{t\phi'(t)g(\phi)}{\phi}, f(\phi)) = (\frac{t\phi'}{\phi}, \phi)(g, f), \tag{1.3}$$

which is called the horizontal half of Riordan array (HHRA) in [17,20], in order to distinguish it from VHRA. A similar approach can be used to show that the entries of the HHRA $(h_{n,k})_{n,k\geq 0}$ of $(g, f) = (d_{n,k})_{n,k\geq 0}$ are

$$h_{n,k} = d_{2n,n+k}, \tag{1.4}$$

while a constructive approach is presented in [20] and an (m, r) extension can be seen in [23].

In the next section the VHRA and the HHRA of a given Riordan array will be extended to the one-pth vertical and the one-pth horizontal Riordan arrays of the Riordan array. Then the one-pth vertical transformation operators and the one-pth horizontal Riordan array transformation operators will be defined. We will present the relationship between the two types of one-pth Riordan arrays by using their matrix factorization and the Lagrange inversion formula. In Section 3, the sequence characterizations of the two types of one-pth Riordan arrays and several illustrating examples are given. In Section 4, we study transformations among Riordan arrays by using the one-pth Riordan array operators. The conditions for transforming Riordan arrays to pseudo-involution Riordan arrays by using the one-pth Riordan arrays are given. The condition for preserving the elements of a certain subgroup of the Riordan group under the one-pth Riordan array transformation is shown. Other properties of the halves of Riordan arrays and their entries such as related recurrence relations, double variable generating functions, combinatorial explanations are also studied in the section. In the last section, we will show the construction of identities and summation formulae by using one-pth Riordan arrays.

2. One-pth Riordan arrays

The vertical and horizontal one-pth Riordan arrays of a Riordan array (g, f) will be defined and constructed in the following two theorems.

Theorem 2.1 *Given a Riordan array $(d_{n,k})_{n,k\geq 0} = (g, f)$, for any integers $p \geq 1$ and $r \geq 0$, $(\widehat{d}_{n,k} = d_{pn+r-k,(p-1)n+r})_{n,k\geq 0}$ defines a new Riordan array, called the one-pth or (p, r) vertical Riordan array of (g, f), which can be written as*

$$(\frac{t\phi'(t)g(\phi)f(\phi)^r}{\phi^{r+1}}, \phi), \text{ where } \phi(t) = \overline{\frac{t^p}{f(t)^{p-1}}}, \tag{2.1}$$

and $\bar{h}(t)$ is the compositional inverse of $h(t)$ ($h(0) = 0$ and $h'(0) \neq 0$). Particularly, if $p = 1$ and $r = 0$, then $(\widehat{d}_{n,k} = d_{n-k,0})_{n,k\geq 0}$ is the Toeplitz matrix (or diagonal-constant matrix) of the 0th column of $(d_{n,k})_{n,k\geq 0}$, and if $p = 2$ and $r = 0$, then $(\widehat{d}_{n,k} = d_{2n-k,n})_{n,k\geq 0}$ is the VHRA of the Riordan array $(d_{n,k})_{n,k\geq 0}$.

Moreover, the generating function of the A-sequence of the new array is $(A(f))^{p-1} = (f/t)^{p-1}$, where $A(t)$ is the generating function of the A-sequence of the given Riordan array.

The Lagrange Inverse Formula (LIF) will be used in the proof. Let $F(t)$ be any formal power series, and let $\phi(t)$ and $u(t) = f(t)/t$ satisfy $\phi = tu(\phi)$. Then the following LIF holds (see, for example, $K6'$ in Merlini, Sprugnoli, and Verri [25]).

$$[t^n]F(\phi(t)) = [t^n]F(t)u(t)^{n-1}(u(t) - tu'(t)). \tag{2.2}$$

Proof From $\phi(t) = \overline{t^p/f(t)^{p-1}}$ we have $\bar{\phi}(t) = t^p/f(t)^{p-1}$ and consequently $t = \phi(t)^p/f(\phi(t))^{p-1}$. Hence, we may write

$$\phi = tu(\phi) \quad \text{where} \quad u(t) = (\frac{f(t)}{t})^{p-1}.$$

Taking derivative on the both sides of the equation $\phi = tu(\phi)$ and noting the definition of $u(t)$, we obtain

$$\phi'(t) = (\frac{f(\phi)}{\phi})^{p-1} + t(p-1)(\frac{f(\phi)}{\phi})^{p-2}\frac{f'(\phi)\phi'(t)\phi - \phi'(t)f(\phi)}{\phi^2},$$

which yields

$$\phi'(t) = (\frac{f(\phi)}{\phi})^{p-1}/(1 - t(p-1)(\frac{f(\phi)}{\phi})^{p-2}\frac{f'(\phi)\phi - f(\phi)}{\phi^2}).$$

Noting $t = \phi/u(\phi) = \phi^p/f(\phi)^{p-1}$, the last expression devotes

$$\begin{aligned}\phi'(t) =&(\frac{f(\phi)}{\phi})^{p-1}/(1 - \frac{p-1}{f(\phi)}(f'(\phi)\phi - f(\phi))) \\ =&\frac{(f(\phi))^p}{\phi^{p-1}(f(\phi) - (p-1)(\phi f'(\phi) - f(\phi)))}\end{aligned} \tag{2.3}$$

We now use (2.3), $t = \phi^p/f(\phi)^{p-1}$, and the LIF shown in (2.2) to calculate $\widehat{d}_{n,k}$ for $n, k \geq 0$

$$\begin{aligned}\widehat{d}_{n,k} =&[t^n]\frac{t\phi'(t)g(\phi)f(\phi)^r}{\phi^{r+1}}(\phi)^k \\ =&[t^n]\frac{\phi^p}{(f(\phi))^{p-1}}\frac{\phi^k(f(\phi))^{p+r}g(\phi)}{\phi^{p+r}(f(\phi) - (p-1)(\phi f'(\phi) - f(\phi)))} \\ =&[t^n]\frac{(f(\phi))^{r+1}g(\phi)}{\phi^{r-k}(f(\phi) - (p-1)(\phi f'(\phi) - f(\phi)))}\end{aligned}$$

$$=[t^n]\frac{(f(t))^{r+1}g(t)}{t^{r-k}(f(t)-(p-1)(tf'(t)-f(t)))}u(t)^{n-1}(u(t)-tu'(t)),$$

where $u(t)=(\frac{f(t)}{t})^{p-1}$ and

$$u'(t)=(p-1)(\frac{f(t)}{t})^{p-2}\frac{tf'(t)-f(t)}{t^2}.$$

Substituting the expressions of $u(t)$ and $u'(t)$ into the rightmost expression of $\widehat{d}_{n,k}$, we have

$$\begin{aligned}\widehat{d}_{n,k}=&[t^n]\frac{(f(t))^{r+1}g(t)}{t^{r-k}(f(t)-(p-1)(tf'(t)-f(t)))}\frac{(f(t))^{(p-1)(n-1)}}{t^{(p-1)(n-1)}}\times\\&(\frac{(f(t)^{p-1}}{t^{p-1}}-t(p-1)\frac{(f(t))^{p-2}}{t^{p-2}}\frac{tf'(t)-f(t)}{t^2})\\=&[t^n]\frac{(f(t)^{(p-1)(n-1)+r+1}g(t)}{t^{(p-1)(n-1)+r-k}(f(t)-(p-1)(tf'(t)-f(t)))}\times\\&\frac{(f(t))^{p-2}}{t^{p-1}}(f(t)-(p-1)(tf'(t)-f(t)))\\=&[t^n]g(t)\frac{(f(t))^{(p-1)n+r}}{t^{(p-1)n+r-k}}=[t^{pn+r-k}]g(t)(f(t))^{(p-1)n+r}=d_{pn+r-k,(p-1)n+r}.\end{aligned}$$

Particularly, if $p=1$ and $r=0$, then $(\widehat{d}_{n,k}=d_{n-k,0})_{n,k}$ is the Toeplitz matrix of the 0th column of (g,f). If $p=2$ and $r=0$, then $(\widehat{d}_{n,k}=d_{2n-k,n})_{n,k\geq 0}$ is the VHRA of (g,f).

As for the $\widehat{A}_p$, the generating function of the A-sequence of $(\widehat{d}_{n,k})_{n,k\geq 0}$, we have $t\widehat{A}_p(\phi)=\phi$, which implies $\widehat{A}_p(t)=t/(t^p/f^{p-1})$, or equivalently,

$$\widehat{A}_p(\bar{f})=(\frac{t}{\bar{f}})^{p-1}=(A(t))^{p-1}.$$

Hence, $\widehat{A}_p(t)=(A(f))^{p-1}=(f/t)^{p-1}$ because $tA(f)=f$, completing the proof of the theorem. □

Theorem 2.2 *Given a Riordan array $(d_{n,k})_{n,k\geq 0}=(g,f)$, for any integers $p\geq 1$ and $r\geq 0$, $(\tilde{d}_{n,k}=d_{pn+r,(p-1)n+r+k})_{n,k\geq 0}$ defines a new Riordan array, called the one-pth or (p,r) horizontal Riordan array of (g,f), which can be written as*

$$(\frac{t\phi'(t)g(\phi)f(\phi)^r}{\phi^{r+1}},f(\phi)),\ \text{where }\phi(t)=\overline{\frac{t^p}{f(t)^{p-1}}},\tag{2.4}$$

and $\bar{h}(t)$ is the compositional inverse of $h(t)$ ($h(0)=0$ and $h'(0)\neq 0$). Particularly, if $p=1$ and $r=0$, the one-pth Riordan array reduces to the given Riordan array, and if $p=2$ and $r=0$, the one-pth Riordan array is the HHRA of the given Riordan array.

Moreover, the generating function of the A-sequence of the new array is $(A(t))^p$, where $A(t)$ is the generating function of the A-sequence of the given Riordan array.

Proof We now use (2.3) above, $t=\phi^p/f(\phi)^{p-1}$, and the LIF shown in (2.2) to calculate $\tilde{d}_{n,k}$ for $n,k\geq 0$

$$\begin{aligned}\tilde{d}_{n,k}=&[t^n]\frac{t\phi'(t)g(\phi)f(\phi)^r}{\phi^{r+1}}(f(\phi))^k\\=&[t^n]\frac{\phi^p}{(f(\phi))^{p-1}}\frac{(f(\phi))^{p+r+k}g(\phi)}{\phi^{p+r}(f(\phi)-(p-1)(\phi f'(\phi)-f(\phi)))}\end{aligned}$$

$$
\begin{aligned}
=&[t^n]\frac{(f(\phi))^{r+k+1}g(\phi)}{\phi^r(f(\phi)-(p-1)(\phi f'(\phi)-f(\phi)))}\\
=&[t^n]\frac{(f(t))^{r+k+1}g(t)}{t^r(f(t)-(p-1)(tf'(t)-f(t)))}u(t)^{n-1}(u(t)-tu'(t)),
\end{aligned}
$$

where $u(t)=(\frac{f(t)}{t})^{p-1}$ and from the proof of Theorem 2.1

$$u'(t)=(p-1)(\frac{f(t)}{t})^{p-2}\frac{tf'(t)-f(t)}{t^2}.$$

Substituting the expressions of $u(t)$ and $u'(t)$ into the rightmost expression of $\tilde{d}_{n,k}$, we have

$$
\begin{aligned}
\tilde{d}_{n,k}=&[t^n]\frac{(f(t))^{r+k+1}g(t)}{t^r(f(t)-(p-1)(tf'(t)-f(t)))}\frac{(f(t))^{(p-1)(n-1)}}{t^{(p-1)(n-1)}}\times\\
&(\frac{(f(t))^{p-1}}{t^{p-1}}-t(p-1)\frac{(f(t))^{p-2}}{t^{p-2}}\frac{tf'(t)-f(t)}{t^2})\\
=&[t^n]\frac{(f(t))^{(p-1)(n-1)+r+k+1}g(t)}{t^{(p-1)(n-1)+r}(f(t)-(p-1)(tf'(t)-f(t)))}\times\\
&\frac{(f(t))^{p-2}}{t^{p-1}}(f(t)-(p-1)(tf'(t)-f(t)))\\
=&[t^n]g(t)\frac{(f(t))^{(p-1)n+r+k}}{t^{(p-1)n+r}}=[t^{pn+r}]g(t)(f(t))^{(p-1)n+r+k}=d_{pn+r,(p-1)n+r+k}.
\end{aligned}
$$

Particularly, if $p=1$ and $r=0$, then $\tilde{d}_{n,k}=d_{n,k}$, while $p=2$ and $r=0$ yields $\tilde{d}_{n,k}=d_{2n,n+k}$, the (n,k) entry of the HHRA of (g,f).

Let $A(t)$ be the generating function of the A-sequence of the given Riordan array (g,f). Then $A(f(t))=f(t)/t$. Let $A_p(t)$ be the generating function of the A-sequence of the Riordan array shown in (2.1). Then $A_p(f(\phi))=\frac{f(\phi)}{t}$. Substituting $t=\overline{\phi}(t)$ into the last equation yields

$$A_p(f)=\frac{f(t)}{\bar{\phi}(t)}=\frac{f(t)}{t^p/(f(t))^{p-1}}=(\frac{f(t)}{t})^p=(A(f))^p,$$

i.e., $A_p(t)=(A(t))^p$ completing the proof. □

3. Identities related to one-pth Riordan arrays

We may use Theorems 2.1 and 2.2 and the Faà di Bruno formula to establish a class of summation formulae.

Let $h(t)=\sum_{n=0}^{\infty}\alpha_n t^n$ be a given formal power series with the case $h(0)=\alpha_0\neq 0$. Assume that $f(a+t)$ has a formal power series expansion in t with $a\in\mathbb{R}$, real numbers, and let $\bar{f}$ denote the compositional inverse of f so that $(\bar{f}\circ f)(t)=(f\circ\bar{f})(t)=t$. Then the composition of f and h in the case of $h(0)=a$ still possesses a formal series expansion in t, namely,

$$
\begin{aligned}
(f\circ h)(t)=&\sum_{n=0}^{\infty}([t^n](f\circ h)(t))t^n=f(a+\sum_{n=1}^{\infty}\alpha_n t^n)\\
=&f(a)+\sum_{n=1}^{\infty}([t^n](f\circ h)(t))t^n. \qquad (3.1)
\end{aligned}
$$

Let $f^{(k)}(a)$ denote the kth derivative of $f(t)$ at $t=a$, i.e.,

$$f^{(k)}(a)=(d^k/dt^k)f(t)|_{t=a}.$$

Recall the Faà di Brumo's formula when applied to $(f\circ h)(t)$ may be written in the form [26, Section 3.4]

$$[t^n](f\circ\phi)=\sum_{\sigma(n)}f^{(k)}(\phi(0))\Pi_{j=1}^n\frac{1}{k_j!}([t^i]\phi)^{k_j}, \tag{3.2}$$

where the summation ranges over the set $\sigma(n)$ of all partitions of n, that is, over the set of all nonnegative integral solutions $(k_1,k_2,\ldots,k_n)$ of the equations $k_1+2k_2+\cdots+nk_n=n$ and $k_1+k_2+\cdots+k_n=k$, $k=1,2,\ldots,n$. Each solution $(k_1,k_2,\ldots,k_n)$ of the equations is called a partition of n with k parts and is denoted by $\sigma(n,k)$. Of course, the set $\sigma(n)$ is the union of all subsets $\sigma(n,k)$, $k=1,2,\ldots,n$.

Let $\beta_n=[t^n](f\circ h)(t)$ and $h(0)=\alpha_0=a$. Then there exists a pair of reciprocal relations

$$\beta_n=\sum_{\sigma(n)}f^{(k)}(a)\frac{\alpha_1^{k_1}\cdots\alpha_n^{k_n}}{k_1!\cdots k_n!}, \tag{3.3}$$

$$\alpha_n=\sum_{\sigma(n)}\bar{f}^{(k)}(f(a))\frac{\beta_1^{k_1}\cdots\beta_n^{k_n}}{k_1!\cdots k_n!}, \tag{3.4}$$

where the summation ranges the set $\sigma(n)$ of all partitions of n. In fact, from (3.1) the given conditions ensure that there holds a pair of formal series expansions

$$f(a+\sum_{n\geq 1}\alpha_n t^n)=f(a)+\sum_{n\geq 1}\beta_n t^n, \tag{3.5}$$

$$\bar{f}(f(a)+\sum_{n\geq 1}\beta_n t^n)=a+\sum_{n\geq 1}\alpha_n t^n. \tag{3.6}$$

Thus, an application of the Faà di Bruno formula (3.2) to $(f\circ\phi)(t)$, on the LHS of (3.5) yields the expression (3.3) with $[t^i]\phi=\alpha_i$, $[t^n](f\circ\phi)=\beta_n$, and $\phi(0)=a$. Note that the LHS of (3.6) may be expressed as $\phi(t)=((\bar{f}\circ f)\circ\phi)(t)=(\bar{f}\circ(f\circ\phi))(t)$, so that in a like manner and application of the Faà di Bruno formula to the LHS of (3.6) gives precisely the equality (3.4).

Replacing α_n by $x_n/n!$ and β_n by $y_n/n!$, we see that (3.3) and (3.4) may be expressed in terms of the exponential Bell polynomials, namely,

$$y_n=\sum_{k=1}^n f^{(k)}(a)B_{n,k}(x_1,x_2,\ldots,x_{n-k+1}), \tag{3.7}$$

$$x_n=\sum_{k=1}^n \bar{f}^{(k)}(a)B_{n,k}(y_1,y_2,\ldots,y_{n-k+1}), \tag{3.8}$$

where $B_{n,k}(\ldots)$ is defined by [26, Section 3.3])

$$B_{n,k}(x_1,x_2,\ldots,x_{n-k+1})=\sum_{\sigma(n,k)}\frac{n!}{k_1!k_2!\cdots}(\frac{x_1}{1!})^{k_1}(\frac{x_2}{2!})^{k_2}\cdots$$

and $\sigma(n,k)$ as shown above is the set of the solutions of the partition equations for a given k $(1\leq k\leq n)$. $B_{n,k}=B_{n,k}(f_1,f_2,\ldots)$ is the Bell polynomial with respect to $(n!)_{n\in\mathbb{N}}$, defined as follows:

$$\frac{1}{k!}(f(z))^k = \sum_{n=k}^{\infty} B_{n,k}\frac{z^n}{n!}. \tag{3.9}$$

Therefore, $B_{n,k} = [z^n/n!](f(z))^k/k!$, which implies that the iteration matrix $B(f(z))$ is the Riordan array $(1, f(z))$. Now, the following important property of the iteration matrix (see Comtet [26, Theorem A on p.145], Roman [27], and Roman and Rota [28])

$$B(f(g(z))) = B(g(z))B(f(z))$$

is trivial in the context of the theory of Riordan arrays, i.e.,

$$(1, f(g(z))) = (1, g(z))(1, f(z))\,;$$

and the Faà di Bruno formula derived from the above property is an application of the FTRA.

Let $f(x) = x^p$ $(p \neq 0)$. Then $\bar{f}(x) = x^{1/p}$ with $f^{(k)}(1) = (p)_k$ and $\bar{f}^{(k)}(1) = (1/p)_k$. Hence, we obtain the special cases of (3.3) and (3.4):

$$\beta_n = \sum_{\sigma(n)} (\alpha)_k \frac{\alpha_1^{k_1}\cdots\alpha_n^{k_n}}{k_1!\cdots k_n!}, \tag{3.10}$$

$$\alpha_n = \sum_{\sigma(n)} (1/\alpha)_k \frac{\beta_1^{k_1}\cdots\beta_n^{k_n}}{k_1!\cdots k_n!}, \tag{3.11}$$

where $(p)_k = p(p-1)\cdots(p-k+1)$ and $(\alpha)_0 = 1$. The above Faà di Bruno's relations have the associated expressions

$$\Big(1 + \sum_{n=1}^{\infty}\alpha_n t^n\Big)^p = 1 + \sum_{n=1}^{\infty}\beta_n t^n, \tag{3.12}$$

$$\Big(1 + \sum_{n=1}^{\infty}\beta_n t^n\Big)^{1/p} = 1 + \sum_{n=1}^{\infty}\alpha_n t^n. \tag{3.13}$$

As an example, if $h = a_0 + a_1 t$ and $f(t) = t^p$, then $f(h(t)) = a_0^p(1+\alpha_1 t)^p$, where $\alpha_1 = a_1/a_0$. From (3.12) we have

$$(a_0 + a_1 t)^p = a_0^p(1+\alpha_1 t)^p = a_0^p(1 + \sum_{j=1}^{\infty}\beta_j t^j),$$

where

$$\beta_j = \sum_{\sigma(j)} (\alpha)_k \frac{\alpha_1^{k_1}\cdots\alpha_n^{k_n}}{k_1!\cdots k_n!} = (p)_j\frac{\alpha_1^j}{j!} = \binom{p}{j}\alpha_1^j,$$

which presents the obvious expression $(a_0 + a_1 t)^p = a_0^p + \sum_{j=1}^{p}\binom{p}{j}a_0^{p-j}a_1^j t^j$.

Similarly, if $h = a_0 + a_1 t + a_2 t^2$, $a_0 \neq 0$, then

$$(a_0 + a_1 t + a_2 t^2)^p = a_0^p(1 + \frac{a_1}{a_0}t + \frac{a_2}{a_0}t^2)^p = a_0^p(1 + \sum_{j=1}^{p}\beta_j t^j),$$

where

$$\beta_j = \sum_{\sigma(j)} (p)_j \frac{1}{j_i! j_2!}(\frac{a_1}{a_0})^{j_1}(\frac{a_2}{a_0})^{j_2} = \sum_{j_i=0}^{j}\binom{p}{j}\binom{j}{j_1}(\frac{a_1}{a_0})^{j_1}(\frac{a_2}{a_0})^{j-j_1}.$$

Theorem 3.1 *Let $A(t)=\sum_{n\geq 0} a_n t^n$ ($a_0\neq 0$) be the generating function of the A-sequence of the given Riordan array $(d_{n,k})_{n,k\geq 0}=(g,f)$, and let $(\tilde{d}_{n,k}=d_{pn+r,(p-1)n+r+k})_{n,k\geq 0}$ be the (p,r) Riordan array of (g,f). Then there exists the following summation formula:*

$$d_{p(n+1)+r,(p-1)(n+1)+r+k+1}=\sum_{j=0}^{n-k}\beta_j d_{pn+r,(p-1)n+r+k+j}, \tag{3.14}$$

where by denoting $(p)_j=p(p-1)\cdots(p-j+1)$, $\beta_0=a_0^p$, and for $n\geq 1$ and $\alpha_i=a_i/a_0$,

$$\begin{aligned}\beta_j=&a_0^p[t^j](A(t))^p=\sum_{\sigma(j)}(p)_j\frac{\alpha_1^{k_1}\cdots\alpha_j^{k_j}}{k_1!\cdots k_j!}\\ =&\sum_{i=1}^{j}\sum_{\sigma(j,i)}\binom{p}{j}\frac{j!}{k_1!k_2!\ldots}(\alpha_1)^{k_1}(\alpha_2)^{k_2}\ldots.\end{aligned} \tag{3.15}$$

Particularly, for $A(t)=a_0+a_1t$ and $A(t)=a_0+a_1t+a_2t^2$, we have

$$\beta_j=\binom{p}{j}a_0^{p-j}a_1^j \text{ and}$$
$$\beta_j=\sum_{i=0}^{j}\binom{p}{j}\binom{j}{i}a_0^{p-j}a_1^{j-i}a_2^i,$$

respectively.

Proof Since $(f(t))^p$ is the generating function of the A-sequence of $(\tilde{d}_{n,k})$ and

$$\tilde{d}_{n,k}=d_{pn+r,(p-1)n+r+k},$$

we obtain (3.14) from the definition of A-sequence, where β_j can be found from (3.1) and (3.10). □

Using (3.14) in Theorem 3.1, one may obtain many identities.

Example 3.2 Consider Pascal matrix $(1/(1-t),t/(1-t))$, its A-sequence generating function is $A(t)=1+t$. Applying (3.14), we have

$$\binom{p(n+1)+r}{(p-1)(n+1)+r+k+1}=\sum_{j=0}^{\min\{p,n-k\}}\binom{p}{j}\binom{pn+r}{(p-1)n+r+k+j}. \tag{3.16}$$

If $p=1$ and $r=0$, the above identity reduces to the well-known identity $\binom{n+1}{k+1}=\binom{n}{k}+\binom{n}{k+1}$.

The Riordan array $(1/(1-t-t^2),tC(t))$ is considered, where

$$C(t)=\sum_{n=0}^{\infty}\binom{2n}{n}t^n/(n+1)=(1-\sqrt{1-4t})/(2t)$$

is the Catalan function. It can be found that the A-sequence of the Riordan array $(1/(1-t-t^2),tC(t))$ is $(1,1,1,\ldots)$, i.e., the A-sequence has the generating function $A(t)=1/(1-t)$. From [13,29] we have

$$C(t)^k=\sum_{n=0}^{\infty}\frac{k}{2n+k}\binom{2n+k}{n}t^n. \tag{3.17}$$

Thus, the (n,k) entry of the Riordan array $(1/(1-t-t^2), tC(t))$ is

$$\begin{aligned} d_{n,k} =&[t^n]\frac{1}{1-t-t^2}(tC(t))^k \\ =&[t^{n-k}]\Big(\sum_{i\geq 0}F_i t^i\Big)\Big(\sum_{j\geq 0}\frac{k}{2j+k}\binom{2j+k}{j}t^j\Big) \\ =&[t^{n-k}]\sum_{i\geq 0}\Big(\sum_{j=0}^{i}F_{i-j}\frac{k}{2j+k}\binom{2j+k}{j}\Big)t^i \\ =&\sum_{j=0}^{n-k}F_{n-k-j}\frac{k}{2j+k}\binom{2j+k}{j}. \end{aligned}$$

Since

$$(A(t))^p=(1-t)^{-p}=\sum_{i\geq 0}\binom{-p}{i}(-t)^i=\sum_{i\geq 0}\binom{p+i-1}{i}t^i,$$

from (3.14) there holds an identity

$$\begin{aligned} &\sum_{j=0}^{n-k}F_{n-k-j}\frac{(p-1)(n+1)+r+k+1}{2j+(p-1)(n+1)+r+k+1}\binom{2j+(p-1)(n+1)+r+k+1}{j} \\ =&\sum_{i\geq 0}\binom{p+i-1}{i}\sum_{j=0}^{n-k-i}F_{n-k-i-j}\frac{(p-1)n+r+k+i}{2j+(p-1)n+r+k+i}\binom{2j+(p-1)n+r+k+i}{j}. \end{aligned}$$

Similarly, for the Riordan array $(C(t), tC(t))$, its (n,k) entry is

$$\begin{aligned} d_{n,k} =&[t^n]t^k(C(t))^{k+1} \\ =&[t^{n-k}]\sum_{j\geq 0}\frac{k+1}{2j+k+1}\binom{2j+k+1}{j}t^j \\ =&\frac{k+1}{2n-k+1}\binom{2n-k+1}{n-k}. \end{aligned}$$

Hence, from (3.14) we may derive the identity

$$\begin{aligned} &\frac{(p-1)(n+1)+r+k+2}{(p+1)(n+1)+r-k}\binom{(p+1)(n+1)+r-k}{n-k} \\ =&\sum_{j=0}^{n-k}\frac{(p-1)n+r+k+j+1}{(p+1)n+r-k-j+1}\binom{p+j-1}{j}\binom{(p+1)n+r-k-j+1}{n-k-j}. \end{aligned}$$

4. More identities

The generating function $F_m(t)$ of the mth order Fuss-Catalan numbers $(F_m(n,1))_{n\geq 0}$ is called the generalized binomial series in [29], and it satisfies the function equation $F_m(t)=1+tF_m(t)^m$. Hence from Lambert's formula for the Taylor expansion of the powers of $F_m(t)$ (see [29, P. 201]), we have

$$F_m^r := F_m(t)^r=\sum_{n\geq 0}\frac{r}{mn+r}\binom{mn+r}{n}t^n \tag{4.1}$$

for all $r \in \mathbb{R}$, where $F_m(t)$ is defined by

$$F_m(t) = \sum_{k\geq 0} \frac{(mk)!}{((m-1)k+1)!} \frac{t^k}{k!} = \sum_{k\geq 0} \frac{1}{(m-1)k+1} \binom{mk}{k} t^k. \tag{4.2}$$

For instance,

$$\begin{aligned} F_0(t) &= 1+t, \\ F_1(t) &= \sum_{k\geq 0} t^k = \frac{1}{1-t}, \\ F_2(t) &= \sum_{k\geq 0} \frac{1}{k+1}\binom{2k}{k} t^k = C(t). \end{aligned}$$

The key case (4.1) leads the following formula for $F_m(t)$:

$$F_m(t) = 1 + tF_m^m(t). \tag{4.3}$$

Actually,

$$\begin{aligned} 1 + tF_m^m(t) &= 1 + \sum_{n\geq 0} \frac{m}{mn+m}\binom{mn+m}{n} t^{n+1} \\ &= 1 + \sum_{n\geq 1} \frac{m}{mn}\binom{mn}{n-1} t^n \\ &= \sum_{n\geq 0} \frac{1}{mn+1}\binom{mn+1}{n} t^n = F_m(t). \end{aligned}$$

For the cases $m = 1$ and 2, we have $F_1 = 1/(1-t)$ and $F_2 = C(t)$, respectively. When $m = 3$, the Fuss-Catalan numbers $(F_3)_n$ form the sequence $A001764$ (see [18]), $1, 1, 3, 12, 55, 273, 1428, \ldots$, which are the ternary numbers. The ternary numbers count the number of 3-Dyck paths or ternary paths. The generating function of the ternary numbers is denoted as $T(t) = \sum_{n=0}^{\infty} T_n t^n$ with $T_n = \frac{1}{3n+1}\binom{3n+1}{n}$, and is given equivalently by the equation $T(t) = 1 + tT(t)^3$.

We now give more examples of Theorem 2.1 related to Fuss-Catalan numbers. First, we establish the relation between the Fuss-Catalan numbers and the Riordan array $(\tilde{g}, \tilde{f}) = (\tilde{d}_{n,k})_{n,k\geq 0}$, where $\tilde{d}_{n,k} = d_{pn+r,(p-1)n+r+k}$ and $d_{n,k}$ is the (n,k) entry of the Pascal' triangle $(g, f) = (1/(1-t), t/(1-t))$.

Theorem 4.1 *Let $(d_{n,k})_{n,k\geq 0} = (1/(1-t), t/(1-t))$ be the Pascal triangle, for any integers $p \geq 2$ and $r \geq 0$ and a given Riordan array (g, f) let $(\tilde{d}_{n,k} = d_{pn+r,(p-1)n+r+k})_{n,k\geq 0} = (\tilde{g}, \tilde{f})$ be the one-pth or (p, r) Riordan array of (g, f). Then*

$$\tilde{g}(t) = \sum_{n\geq 0} \binom{pn+r}{n} t^n = \frac{(1+w)^{r+1}}{1-(p-1)w}\Big|_{w=t(1+w)^p} \tag{4.4}$$

$$\tilde{f}(t) = \sum_{n=1}^{\infty} \frac{1}{pn+1}\binom{pn+1}{n} t^n = F_p(t) - 1 = tF_p^p(t), \tag{4.5}$$

where $F_p(t)$ is the pth order Fuss-Catalan function satisfying

$$F_p(t(1-t)^{p-1}) = \frac{1}{1-t}. \tag{4.6}$$

Proof For expression (4.4), we find

$$\begin{aligned}
[t^n]\tilde{g} =& \tilde{d}_{n,0} = d_{pn+r,(p-1)n+r} = \binom{pn+r}{n} \\
=& [t^n](1+t)^{pn+r} = [t^n](1+t)^r((1+t)^p)^n \\
=& [t^n]\frac{(1+w)^r}{1-t(d/dw)((1+w)^p)}\Big|_{w=t(1+w)^p},
\end{aligned}$$

which implies (4.4).

From (2.4) of Theorem 2.2 we know that

$$(\tilde{g}, \tilde{f}) = (\frac{t\phi'(t)g(\phi)f(\phi)^r}{\phi^{r+1}}, f(\phi)), \tag{4.7}$$

where $\phi(t) = \overline{\frac{t^p}{(f(t))^{p-1}}}$, and $\bar{h}(t)$ is the compositional inverse of $h(t)$ ($h(0) = 0$ and $h'(0) \neq 0$). Moreover, the generating function of the A-sequence of the new array $(\tilde{g}, \tilde{f})$ is $(A(t))^p$, where $A(t)$ is the generating function of the A-sequence of the given Riordan array (g, f). By using the Lagrange Inverse Formula

$$[t^n](f(t))^k = \frac{k}{n}[t^{n-k}](A(t))^n,$$

we have

$$[t^n]\tilde{f} = \frac{1}{n}[t^{n-1}](A(t))^{pn} = \frac{1}{n}[t^{n-1}](1+t)^{pn} = \frac{1}{n}\binom{pn}{n-1}.$$

Therefore,

$$\tilde{f} = \sum_{n=1}^{\infty} \frac{(pn)!}{((p-1)n+1)!n!}t^n = \sum_{n=1}^{\infty} \frac{1}{pn+1}\binom{pn+1}{n}t^n = F_p(t) - 1.$$

Since the key equation (4.3) of the Fuss-Catalan function F_p shows $F_p = 1 + tF_p^p$, we obtain (4.5). From (4.7),

$$f(\phi) = \tilde{f}(t) = tF_p^p(t).$$

Therefore, noting $f(t) = t/(1-t)$ we get

$$\frac{t}{1-t} = f(t) = \overline{\phi}F_p^p(\overline{\phi}) = \frac{t^p}{(f(t))^{p-1}}F_p^p(\frac{t^p}{(f(t))^{p-1}}) = t(1-t)^{p-1}F_p^p(t(1-t)^{p-1}),$$

and (4.6) follows from the comparison of the leftmost side and the rightmost side of the above equation. $\square$

For example, if $p = 2$ and $r \geq 0$, then

$$\tilde{f} = tF_2^2(t) = t(C(t))^2.$$

Since $w = t(1+w)^2$ has a solution

$$w = \frac{1-2t-\sqrt{1-4t}}{2t} = C(t) - 1,$$

we have

$$\tilde{g} = \frac{(1+w)^{r+1}}{1-w}\Big|_{w=t(1+w)^2} = \frac{(C(t))^{r+1}}{2-C(t)} = \frac{(C(t))^r}{\sqrt{1-4t}} = B(t)(C(t))^r,$$

where $B(t)$ is the generating function for the central binomial coefficients. Thus, $(\tilde{d}_{n,k})_{n,k\geq 0} = (d_{2n+r,n+r+k})_{n,k\geq 0}$ is the Riordan array

$$(\tilde{g}, \tilde{f}) = (B(t)C^r, t(C(t))^2).$$

We need one more property of Riordan arrays, which generalizes a well-known property of the Pascal triangle and is shown in Brietzke [30].

Theorem 4.2 *Let $(d_{n,k})_{n,k\geq 0} = (g, f)$ be a Riordan array. Then for any integers $k \geq s \geq 1$ we have*

$$d_{n,k} = \sum_{j=s}^{n} d_{n-j,k-s}[t^j](f(t))^s. \tag{4.8}$$

Particularly, for $s = 1$, $d_{n,k} = \sum_{j=1}^{n} f_j d_{n-j,k-1}$, where $f_j = [t^j]f(t)$.

Proof The (n, k) entry of the Riordan array (g, f) can be written as

$$\begin{aligned} d_{n,k} =& [t^n]g(t)(f(t))^k = [t^n]g(t)(f(t))^{k-s}((f(t))^s \\ =& \sum_{j=s}^{n}([t^{n-j}]g(t)(f(t))^{k-s})([t^j](f(t))^s) \\ =& \sum_{j=s}^{n} d_{n-j,k-s}[t^j]((f(t))^s. \quad \square \end{aligned}$$

Example 4.3 If $(g, f) = (1/(1-t), t/(1-t))$, then $f_j = [t^j](t/(1-t)) = 1$ for all $j \geq 1$. We have the well-known identity

$$\sum_{j=1}^{n} \binom{n-j}{k-1} = \binom{n}{k}. \tag{4.9}$$

More generally, for the Pascal triangle $(g, f) = (1/(1-t), t/(1-t))$, we have

$$[t^j](f(t))^s = [t^j]\frac{t^s}{(1-t)^s} = [t^{j-s}](1-t)^{-s} = [t^{j-s}]\sum_{i\geq 0}\binom{s+i-1}{i}t^i = \binom{j-1}{s-1}.$$

Consequently, (4.8) becomes the Chu-Vandermonde identity

$$\sum_{j=s}^{n} \binom{n-j}{k-s}\binom{j-1}{s-1} = \binom{n}{k},$$

which contains (4.9) as a special case.

Example 4.4 For fixed integers $p \geq 2$ and $r \geq 0$, starting with the Pascal triangle and using Theorem 2.1, we obtain the Riordan array $(\tilde{g}, \tilde{f})$ with its (n, k) entry as

$$\tilde{d}_{n,k} = \binom{pn+r}{(p-1)n+r+k} = \binom{pn+r}{n-k}$$

possesses the formal power series $\tilde{f}(t) = tF_p^p(t)$. Thus,

$$\begin{aligned}[t^j](\tilde{f}(t))^s =&[t^{j-s}]F_p^{ps}(t) = [t^{j-s}]\frac{ps}{pn+ps}\binom{pn+ps}{n}\\ =&\frac{ps}{p(j-s)+ps}\binom{p(j-s)+ps}{j-s} = \frac{s}{j}\binom{pj}{j-s}.\end{aligned}$$

From the expression (4.8) of Theorem 4.2 we obtain the identity

$$\sum_{j=s}^{n}\frac{s}{j}\binom{pj}{j-s}\binom{p(n-j)+r}{n-j-k+s} = \binom{pn+r}{n-k}. \tag{4.10}$$

Particularly, if $s=1$, then (4.10) becomes

$$\sum_{j=1}^{n}\frac{1}{pj+1}\binom{pj+1}{j}\binom{p(n-j)+r}{n-j-k+1} = \binom{pn+r}{n-k}$$

and, finally, adding to the both sides $\binom{pn+r}{n-k+1}$, we have

$$\sum_{j=0}^{n}\frac{1}{pj+1}\binom{pj+1}{j}\binom{p(n-j)+r}{n-j-k+1} = \binom{pn+r+1}{n-k+1}.$$

Setting $j=i+s$, $x=ps$, $y=pk-ps+r$, and replacing n by $n+k$, identity (4.10) becomes formula (5.62) of [29]:

$$\sum_{i=0}^{n}\frac{x}{x+pi}\binom{x+pi}{i}\binom{y+p(n-i)}{n-i} = \binom{x+y+pn}{n}.$$

Substituting $p=-q$, $x=r$, and $y+pn=p$, the above identity is equivalent to the *Gould identity*:

$$\sum_{i=0}^{n}\frac{r}{r-qi}\binom{r-qi}{i}\binom{p+qi}{n-i} = \binom{r+p}{n}.$$

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