Journal of Mathematical Research with Applications
Mar., 2021, Vol. 41, No. 2, pp. 161–170
DOI:10.3770/j.issn:2095-2651.2021.02.005
Http://jmre.dlut.edu.cn

Higher Order Estimates for Boundary Blow-Up Solutions of Elliptic Equations with Gradient Term

Yajie ZHANG, Feiyao MA*, Weifeng WO
Department of Mathematics, Ningbo University, Zhejiang 315000, P. R. China

Abstract In this paper, the higher order asymptotic behaviors of boundary blow-up solutions to the equation $\Delta u = u^p \pm |\nabla u|^q$ in bounded smooth domain $\Omega \subset R^N$ are systematically investigated for p and q. The second and third order boundary behaviours of the equation are derived. The results show the role of the mean curvature of the boundary $\partial\Omega$ and its gradient in the high order asymptotic expansions of the solutions.

Keywords second order estimates; third order estimates; semilinear elliptic equations

MR(2020) Subject Classification 35J25; 35J61; 35B40

1. Introduction

Consider the boundary blow-up problem

$$\begin{cases} \Delta u = u^p \pm |\nabla u|^q, & x \in \Omega \\ u(x) = \infty, & x \in \partial\Omega, \end{cases} \tag{$P_\pm$}$$

where $\Omega \subset R^N$ is a bounded smooth domain, $N \geq 2$. We are interested in the high order asymptotic behavior of the boundary blow-up solutions to $(P_\pm)$. For the special case $q = \frac{2p}{p+1}$, the boundary behavior has been studied in [1]. The aim of this essay is to explore the relationship between higher order (higher than the first order) asymptotic behavior of the boundary blow-up solutions and the geometry of the domain Ω for more general p, q. Second order estimates (approximation) of the blow-up solutions to the equation $\Delta u = u^p$ have been investigated in many papers [2–4]. For $1 < p < 3$, Bandle [2] obtained the second order estimates:

$$u(x) = \Big(\frac{\sqrt{2p+2}}{p-1}\Big)^{\frac{2}{p-1}} \delta(x)^{\frac{2}{1-p}} \Big(1 + \frac{(N-1)H(\bar{x})\delta(x)}{p+3} + o(\delta(x))\Big),$$

where $\bar{x} \in \partial\Omega$ is the point such that $\delta(x) = |x - \bar{x}|$ and $H(\bar{x})$ denotes the mean curvature of $\partial\Omega$. High order estimates of the equation $\Delta u = u^p + u^q$ has been studied in [5]. More results about behavior of the boundary blow-up solutions can be found in [6–10].

Received March 24, 2020; Accepted October 24, 2020
Supported by the Zhejiang Provincial Natural Science Foundation of China (Grant Nos. LY20A010010; LY20A010011), the National Natural Science Foundation of China (Grant No. 11971251) and K. C. Wong Magna Fund in Ningbo University.
* Corresponding author
E-mail address: zhangyajie315@qq.com (Yajie ZHANG); mafeiyao@nbu.edu.cn (Feiyao MA); woweifeng@nbu.edu.cn (Weifeng WO)

Recently, researchers have shown an increased interest in large solutions of equations with gradient term. Second order estimates of the blow-up solutions for the equations with gradient term are discussed in [1, 11–16]. In [11] the equation $\Delta u = u^p|\nabla u|^q$ for $0 \le p \le \frac{p+3}{p+2}$ has been studied. These results show how the gradient term affects the asymptotic behaviour of u. Giarrusso and Porru [1] studied an equation with a special gradient term $\Delta u = u^p + \lambda|\nabla u|^{\frac{2p}{p+1}}$, $0 \le \lambda \le 1$, $p > 1$, and derived that

$$u(x) = L(\delta(x))^{\frac{2}{1-p}}\Big(1 + \frac{(N-1)H(\bar{x})(p+1)\delta(x)}{(3-p)(p+1) + L^{p-1}p(p-1)^2} + o(\delta(x))\Big),$$

where L is the solution of the equation

$$(\frac{2}{p-1})^{\frac{2p}{p+1}} L^{\frac{p-1}{p+1}}\lambda + L^{p-1} = \frac{2(p+1)}{(p-1)^2}.$$

This paper considers general case of p and q in $(P_\pm)$. Our estimates also include the critical case of the exponent in gradient term, where the factors of logarithmic function rise in the estimates.

Theorem 1.1 *Assume $p > 0$, $q > 0$, and $\partial\Omega \in C^4$. If $u(x)$ is a solution to problem $(P_\pm)$ in Ω. Then $u(x)$ satisfies the following second order estimates on a sufficiently small neighborhood of $\partial\Omega$ as in the following cases.*

(i) $1 < p < 3$ and $0 < q < 1$ for $(P_\pm)$:

$$u(x) = (\frac{p-1}{\sqrt{2p+2}})^{\frac{2}{1-p}}\delta(x)^{\frac{2}{1-p}}\Big(1 + \frac{(N-1)H(\bar{x})}{p+3} + o(\delta(x))\Big);$$

(i)′ $1 < p < 3$ and $q = 1$ for $(P_\pm)$:

$$u(x) = (\frac{p-1}{\sqrt{2p+2}})^{\frac{2}{1-p}}\delta(x)^{\frac{2}{1-p}}\Big(1 + \frac{(N-1)H(\bar{x})\delta(x)}{p+3} \mp \frac{2^{-\frac{p}{p+1}}}{(p+3)(p+1)^{\frac{1}{p+1}}}(p-1)^{\frac{p+5}{p+1}}\delta(x) + o(\delta(x))\Big);$$

(ii) $p > 3$ and $\frac{2p}{p+1} < q < \frac{p}{2}$ for (P_-):

$$u(x) = (\frac{(p-q)\delta(x)}{q})^{-\frac{q}{p-q}}\Big(1 + \frac{(N-1)H(\bar{x})(p-q)\delta(x)}{p-2q} + o(\delta(x))\Big);$$

(ii)′ $p > 3$ and $q = \frac{p}{2}$ for (P_-):

$$u(x) = \frac{1}{\delta(x)}\Big(1 + (N-1)H(\bar{x})\delta(x)\log\delta(x) + o(\delta(x)\log\frac{1}{\delta(x)})\Big);$$

(iii) $\frac{2p-1}{p} < q < \frac{3}{2}$ and $1 < p < 2$ for (P_+):

$$u(x) = \frac{[(q-1)\delta(x)]^{\frac{q-2}{q-1}}}{2-q}\Big(1 + \frac{(N-1)H(\bar{x})(q-2)\delta(x)}{4q-6} + o(\delta(x))\Big);$$

(iii)′ $q = \frac{3}{2}$ and $1 < p < 2$ for (P_+):

$$u(x) = \frac{4}{\delta(x)}\Big(1 - \frac{(N-1)H(\bar{x})\delta\log\delta(x)}{2} + o(\delta(x)\log\frac{1}{\delta(x)})\Big);$$

(iii)'' $q = \frac{2p-1}{p}$ *and* $1 < p < 2$ *for* (P_+)*:*

$$u(x) = \Big(\frac{p}{(p-1)\,\delta(x)}\Big)^{\frac{1}{p-1}}\Big(1 + \frac{(N-1)\,H\,(\bar{x})}{2-2\,p}\delta(x) - \frac{(2-p)^{-p-1}}{2}\delta(x) + o(\delta(x))\Big);$$

(iv) $q = 2$ *and* $p > 0$ *for* (P_+)*:*

$$u(x) = \frac{1}{\log \delta(x)}\Big(1 + \frac{(N-1)\,H\,(\bar{x})\,\delta(x)}{3} + o(\delta(x))\Big).$$

Remark 1.2 The existence theory of $(P_\pm)$ can be found in [17]. Related to the case (iii), $q = \frac{2p}{p+1}$ and $p > 1$, the second order estimates (see (1)) have been obtained in [1].

We also get the third order estimates of the $(P_\pm)$, where mean curvature $H(x)$ of the surface $\{y \in \Omega, \delta(y) = \delta(x)\}$ at the x and $\nabla H(x)$ are included in the estimates.

Theorem 1.3 *Assume* $1 < p < 3$, $0 < q \le \frac{2}{p+1}$, *and* $\partial\Omega \in C^5$. *If* $u\,(x)$ *is a solution to the problem* $(P_\pm)$*, then we have the third order estimates as the following cases.*

(i) $1 < p < 3$ *and* $0 < q < \frac{2}{p+1}$*:*

$$u(x) = \Big(\frac{p-1}{\sqrt{2\,p+2}}\Big)^{\frac{2}{1-p}}\delta^{\frac{2}{1-p}}(x)\Big(1 + \frac{(N-1)\,H\,(x)}{p+3}\delta(x) + \frac{(3-p)\,\big((N-1)\,H^2(x) - 2\,\nabla H(x)\nabla\delta(x)\big)\,(N-1)}{12\,(p+3)}\delta^2(x) + o(\delta^2(x))\Big); \tag{1.1}$$

(ii) $1 < p < 3$ *and* $q = \frac{2}{p+1}$*:*

$$u(x) = \Big(\frac{p-1}{\sqrt{2\,p+2}}\Big)^{\frac{2}{1-p}}\delta^{\frac{2}{1-p}}(x)\Big(1 + \frac{(N-1)\,H\,(x)}{p+3}\delta(x) + \frac{(3-p)\,\big((N-1)\,H^2(x) - 2\,\nabla H(x)\nabla\delta(x)\big)\,(N-1)}{12\,(p+3)}\delta^2(x) \mp \frac{2^{-\frac{2p+1}{p+1}}}{3\,(p+1)^{\frac{1}{p+1}}}\,(p-1)^{\frac{p+5}{p+1}}\delta^2(x) + o(\delta^2(x))\Big). \tag{1.2}$$

This theorem shows how $\nabla H(x)$ appears in the third order estimates. The proof is based on the construction of upper and lower solutions near the boundary. The construction is inspired by [1]. The paper is organized as follows. In Section 2, we introduce the first order estimates in the boundary asymptotic behavior and some notations. In Section 3, the upper and lower solutions are constructed. In Section 4, we prove Theorem 1.1. Section 5 is devoted to the third order estimates (see Theorem 1.3).

2. Preliminaries

In this section, we state some first order estimates, which will be used in the proof of our higher order estimates in the following sections. The first order approximation of the large solution to $\Delta\, u = u^p + |\nabla u|^q$ has been obtained in [17]. That is if $\max\{\frac{2p}{p+1}, 1\} < q < 2$ and

$p>0$, then

$$u\,(x)=\frac{1}{2-q}[(q-1)\delta(x)]^{\frac{q-2}{q-1}}(1+o(1)). \tag{2.1}$$

In Theorem 1.1, p and q are divided into 5 cases. We need some notations. In case (iii) $\frac{2p-1}{p}<q<\frac{3}{2}$, $1<p<2$, we define

$$\psi(\delta(x))=\frac{1}{2-q}[(q-1)\delta(x)]^{\frac{q-2}{q-1}}. \tag{2.2}$$

To simplify notation, we denote $\delta(x)$ by δ and $\psi(\delta)$ by ψ. We have $\psi\to\infty$ as $\delta\to 0$ and $\psi'<0$. And we get

$$\psi=\psi'\frac{q-1}{2-q}\delta, -\psi'=\psi''(q-1)\delta, -\psi=\psi''\frac{(q-1)^2}{q-2}\delta^2. \tag{2.3}$$

Then $\frac{(-\psi')^q}{\psi''}=(\frac{1}{q-1}(q-1)^{\frac{q-2}{q-1}})^q(q-1)^2(q-1)^{\frac{2-q}{q-1}}=1$. Set

$$G(\delta)=\frac{(\psi')^p}{\psi''}=\frac{[(q-1)\delta]^{\frac{q(p+1)-2p}{q-1}}}{(2-q)^p}. \tag{2.4}$$

Define similar functions ψ_1, ψ_2, and ψ_4, for case (i)–(iii), respectively, as

$$\psi_1=(\frac{p-1}{\sqrt{2p+2}})^{\frac{2}{1-p}}\delta^{\frac{2}{1-p}},\ \psi_2=(\frac{(p-q)\delta}{q})^{-\frac{q}{p-q}},\ \psi_4=\log\delta^{-1}.$$

3. Second order estimates

To derive second order estimates, we need the following lemma, which will be proved in the section.

Lemma 3.1 *Assume $\frac{2p-1}{p}<q<\frac{3}{2}$ and $1<p<2$. ψ is defined as (2.2). If $u\,(x)$ is a solution to the problem (P_+), then there exists a positive constant C such that for $x\in\Omega$,*

$$(1-C\delta)\,\psi\,(\delta)<u\,(x)<(1+C\delta)\,\psi\,(\delta). \tag{3.1}$$

Furthermore, for $q=3/2$, $1<p<2$ and $\delta<1$,

$$\left(1-C\delta\log\delta^{-1}\right)\psi\,(\delta)<u\,(x)<\left(1+C\delta\log\delta^{-1}\right)\psi\,(\delta). \tag{3.2}$$

Proof Define $\Omega_\eta=\{x\in\Omega;\delta(x)<\eta,\eta>0\}$. For $x\in\Omega_\eta$, we know [18]

$$|\nabla\delta|=1,\quad \Delta\delta=-\,(N-1)\,H(x). \tag{3.3}$$

First, consider the case $\frac{2p-1}{p}<q<\frac{3}{2}$, $1<p<2$. Let

$$v=(1-\alpha\delta)\,\psi\,(\delta). \tag{3.4}$$

By Taylor expansion, we have

$$v^p<(1-p\alpha\delta+\tilde{C}_1(\alpha\delta)^2)\psi^p, \tag{3.5}$$

where $\tilde{C}_i$ $(i\geq 1, i\in N)$ is a positive constant independent of α. From (2.3) and (3.4),

$$\nabla v=\left(1-\frac{2q-3}{q-2}\alpha\delta\right)\psi'\nabla\delta. \tag{3.6}$$

Noticing that $1 - \frac{2q-3}{q-2}\alpha\delta \geq \frac{q-1}{2-q} > 0$, we derive

$$|\nabla v|^q < \big(1 - \frac{(2q-3)q}{q-2}\alpha\delta + \tilde{C}_2(\alpha\delta)^2\big)|\psi'|^q.$$

From (3.3) and (3.5),

$$\Delta v = \big(1 - \frac{2q-3}{q-2}\alpha\delta\big)(\psi'' + \psi'\Delta\delta) - \frac{2q-3}{q-2}\alpha\psi'.$$

Consequently,

$$\Delta v = \big[1 - (q-1)\Delta\delta\delta + (2q-3)\alpha\delta + \frac{(2q-3)(q-1)}{q-2}\alpha\delta^2\Delta\delta\big]\psi''.$$

Consider the domain $\Omega_\mu = \{x \in \Omega : \delta(x) < \mu\}$ for μ small. Claim v is a lower solution of (P_-) in Ω_μ, i.e.,

$$\Delta v > |\nabla v|^q + v^p, \quad x \in \Omega_\mu. \tag{3.7}$$

In order to prove (3.7), we need

$$\begin{aligned}&\big[1 - (q-1)\delta\Delta\delta + (2q-3)\alpha\delta + \frac{(2q-3)(q-1)}{q-2}\alpha\delta^2\Delta\delta\big]\psi'' \\ &\quad > \big(1 - \frac{(2q-3)q}{q-2}\alpha\delta + \tilde{C}_2(\alpha\delta)^2\big)|\psi'|^q + (1 - p\alpha\delta + \tilde{C}_1(\alpha\delta)^2)\psi^p.\end{aligned}$$

Applying (2.4) to the inequality, we can rewrite the inequality as

$$\begin{aligned}&\big[1 - (q-1)\Delta\delta\delta + (2q-3)\alpha\delta + \frac{(2q-3)(q-1)}{q-2}\alpha\delta^2\Delta\delta\big] \\ &\quad > B(1 - \frac{(2q-3)q}{q-2}\alpha\delta + \tilde{C}_2(\alpha\delta)^2) + G(\delta)(1 - p\alpha\delta + \tilde{C}_1(\alpha\delta)^2),\end{aligned}$$

which can be simplified as

$$\begin{aligned}&(q-1)\Delta\delta - \frac{(2q-3)(q-1)}{q-2}\alpha\delta\Delta\delta \\ &\quad < \alpha\frac{(2q-3)(q-1)2}{q-2} - \frac{G(\delta)(1 - p\alpha\delta + \tilde{C}_1(\alpha\delta)^2)}{\delta} - \tilde{C}_3(\alpha)^2\delta.\end{aligned}$$

Since $q > \frac{2p-1}{p}$, we obtain

$$G(\delta) = \frac{((q-1)\delta)^{\frac{q(p+1)-2p}{q-1}}}{(2-q)^p} = o(\delta).$$

Thus if $\Delta\delta < \alpha\frac{2(2q-3)}{q-2}$, then (3.6) can be derived straightly. Since $q < 3/2$, taking α_0 large and δ_0 small enough, we have (3.6) holds for $0 < \delta \leq \delta_0$ and $\alpha \geq \alpha_0$. Then, choosing δ_1 such that $\delta_1 < \delta_0$, by (2.1) we have

$$u > \psi(\delta)(1 - \alpha_0\delta_0) \text{ for } \delta < \delta_1.$$

Choose α_1 such that $\alpha_0\delta_0 = \alpha_1\delta_1$. Then we have

$$u(x) > \psi(\delta_1)(1 - \alpha_1\delta_1) \text{ for } \delta = \delta_1.$$

By (2.1), $\theta u(x) > \psi(\delta)\,(1 - \alpha_1\delta)$ for any $\theta > 1$ near $\partial\Omega$, and

$$\theta\Delta u < (\theta u)^p + |\nabla(\theta u)|^q\,.$$

Using the comparison principle and (3.7), we obtain $\theta u > \psi(\delta)(1-\alpha_1\delta)$ for $\delta < \delta_1$. Let $\theta \to 1$, $u > \psi(\delta)(1-\alpha_1\delta)$ for $\delta < \delta_1$. Increasing α_1, we obtain $u > \psi(\delta)(1-\alpha_1\delta)$ on Ω. The left side of inequality (3.1) is proved. The proof of right side is similar and omitted. Thus (3.1) holds. Now we consider $\Delta u = |\nabla u|^{3/2} + u^p$, where $1 < p < 2$ and $\delta(x) < 1$. Let $v = (1 - \alpha\delta\log\delta^{-1})\psi(\delta)$. Using the arguments similar to that in the proof of (3.1) gives the desired result (3.2). $\square$

Lemma 3.2 *Assume $\frac{2p-1}{p} < q < \frac{3}{2}$ and $1 < p < 2$. ψ is defined as (2.2). If $u(x)$ is a solution to the problem (P_+), then there exists a positive constant C such that for $x \in \Omega$,*

$$\Big|\frac{u(x)}{\psi(\delta)} - \Big(1 + \frac{(N-1)\,H(x)\,(q-2)\,\delta}{4q-6}\Big)\Big| < C\delta^{\sigma}, \tag{3.8}$$

where $G(\delta)$ is defined in (2.4) and $1 < \sigma < \frac{q(p+1)-2p}{q-1}$ is a suitable constant.

Furthermore, for $q = 3/2$, $1 < p < 2$, and $\delta < 1$.

$$\Big|\frac{u(x)}{\psi(\delta)} - \Big(1 - \frac{(N-1)\,H(x)\,\delta\log\delta}{2}\Big)\Big| < C\delta(-\log\delta)^{\sigma'}, \tag{3.9}$$

where $0 < \sigma' < 1$ is a constant.

Proof To prove (3.8) it suffices to show that

$$\begin{aligned}&\Big(1 + \frac{(N-1)\,H(x)\,(q-2)\,\delta}{4\,q-6} - \alpha\delta^{\sigma}\Big)\psi(\delta) < u(x)\\ &\quad< \Big(1 + \frac{(N-1)\,H(x)\,(q-2)\,\delta}{4\,q-6} + \alpha\delta^{\sigma}\Big)\psi(\delta),\end{aligned} \tag{3.10}$$

where α is a positive constant. Consider the left side of (3.10). Let

$$v = (1 + A\delta - \alpha\delta^{\sigma})\psi(\delta), \tag{3.11}$$

where $A = \frac{-\Delta\delta(q-2)}{4\,q-6}$. From (2.3),

$$\nabla v = \Big[(1 + \frac{2q-3}{q-2}A\delta - (1 + \frac{q-1}{q-2}\sigma)\alpha\delta^{\sigma})\nabla\delta + \frac{q-1}{q-2}\nabla A\delta\Big]\psi'. \tag{3.12}$$

Taking α large enough and δ small enough such that

$$1 + \frac{2q-3}{q-2}A\delta - (1 + \frac{q-1}{q-2}\sigma)\alpha\delta^{\sigma} > 0, \tag{3.13}$$

we obtain

$$|\nabla v|^q < \big(1 + \frac{2q-3}{q-2}Aq\delta - (1 + \frac{q-1}{q-2}\sigma)q\alpha\delta^{\sigma} + \bar{C}_1\delta^2 + \bar{C}_1(\alpha\delta^{\sigma})^2\big)|\psi'|^q,$$

where $\bar{C}_i$ $(i \geq 1, i \in N)$ is a positive constant independent of α. Using (3.12) yields

$$\begin{aligned}\Delta v =& (1 + A\delta - \alpha\delta^{\sigma})\,(\psi'' + \psi'\Delta\delta) + (\nabla A\nabla\delta\delta + A - \alpha\sigma\delta^{\sigma-1})2\psi' +\\ &(\Delta A\delta + 2\nabla A\nabla\delta + A\Delta\delta - \alpha\sigma(\sigma-1)\delta^{\sigma-2} - \alpha\sigma\delta^{\sigma-1}\Delta\delta)\psi.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\Delta v >& \{1 + [(3-2q)A - (q-1)\Delta\delta]\delta - \alpha\delta^{\sigma} + (q-1)2\alpha\sigma\delta^{\sigma} +\\ &\frac{(q-1)^2}{q-2}\alpha\sigma(\sigma-1)\delta^{\sigma} - \bar{C}_2\delta^2 - \bar{C}_2\alpha\delta^{(\sigma+1)}\}\psi''.\end{aligned}$$

Choose α and δ such that

$$-1 < A\delta - \alpha\delta^{\sigma} < 1. \tag{3.14}$$

By (3.11), we derive

$$v^p < (1 + pA\delta - p\alpha\delta^{\sigma} + \bar{C}_3\delta^2 + \bar{C}_3(\alpha\delta^{\sigma})^2)\psi^p.$$

Claim v is a lower solution of (P_+) in Ω_μ, i.e.,

$$\Delta v > |\nabla v|^q + v^p, \quad x \in \Omega_\mu. \tag{3.15}$$

That is

$$\begin{aligned}
&\{1 + [(3-2q)A - (q-1)\Delta\delta]\delta - \alpha\delta^{\sigma} + \frac{(q-1)^2}{q-2}\alpha\sigma(\sigma-1)\delta^{\sigma} + \\
&2(q-1)\alpha\sigma\delta^{\sigma} - \bar{C}_2\delta^2 - \bar{C}_2\alpha\delta^{(\sigma+1)}\}\psi'' > (1 + \frac{2q-3}{q-2}Aq\delta - \\
&q(1 + \frac{q-1}{q-2}\sigma)\alpha\delta^{\sigma} + \bar{C}_1\delta^2 + \bar{C}_1(\alpha\delta^{\sigma})^2)|\psi'|^q + (1 + pA\delta - \\
&p\alpha\delta^{\sigma} + \bar{C}_3\delta^2 + \bar{C}_3(\alpha\delta^{\sigma})^2)\psi^p.
\end{aligned}$$

Applying (2.4) to the inequality, we can rewrite the inequality as

$$\begin{aligned}
&1 + [(3-2q)A - (q-1)\Delta\delta]\delta - \alpha\delta^{\sigma} + \frac{(q-1)^2}{q-2}\alpha\sigma(\sigma-1)\delta^{\sigma} + \\
&2(q-1)\alpha\sigma\delta^{\sigma} - \bar{C}_2\delta^2 - \bar{C}_2\alpha\delta^{(\sigma+1)} > (1 + \frac{2q-3}{q-2}Aq\delta - \\
&q(1 + \frac{q-1}{q-2}\sigma)\alpha\delta^{\sigma} + \bar{C}_1\delta^2 + \bar{C}_1(\alpha\delta^{\sigma})^2) + (1 + pA\delta - \\
&p\alpha\delta^{\sigma} + \bar{C}_3\delta^2 + \bar{C}_3(\alpha\delta^{\sigma})^2)G(\delta).
\end{aligned} \tag{3.16}$$

Since $A = \frac{-\Delta\delta(q-2)}{4q-6}$, we have

$$(3-2q)A - (q-1)\Delta\delta = \frac{2q-3}{q-2}Aq. \tag{3.17}$$

Hence (3.16) becomes

$$\begin{aligned}
\bar{C}_4\delta^2 + G(\delta)(1 + pA\delta + \bar{C}_3\delta^2) <&\alpha\delta^{\sigma}[(q-1)(1+2\sigma) + \frac{q-1}{q-2}(\sigma q - \sigma + 1)\sigma - \\
&\bar{C}_1\alpha\delta^{\sigma} + G(\delta)p - G(\delta)\bar{C}_3\alpha\delta^{\sigma}].
\end{aligned}$$

From $1 < \sigma < \frac{q(p+1)-2p}{q-1}$, we get

$$G(\delta) = \frac{(q-1)^{\frac{q(p+1)-2p}{q-1}}(\delta)^{\frac{q(p+1)-2p}{q-1}-\sigma}}{(2-p)^p} = o(\delta^{\sigma}).$$

By a direct calculation, (3.16) can be derived if we have

$$(q-1)(1+2\sigma) + \frac{q-1}{q-2}\sigma(\sigma q - \sigma + 1) > 0.$$

Since $1 < q < \frac{3}{2}$ and $1 < \sigma < \frac{q(p+1)-2p}{q-1}$, the inequality holds.

Taking α_0 large and δ_0 small enough, we have (3.13) and (3.14). Decrease δ_0 until (3.15) holds. Hence, we have inequality (3.15) for $\delta \leq \delta_0$ and $\alpha \geq \alpha_0$. Choose δ_1 such that $\delta_1 \leq \delta_0$. Using (3.1), we obtain $u(x) > (1 - C\delta)\psi(\delta)$ for $\delta \leq \delta_1$. From (3.11),

$$v(x) - u(x) < \psi(\delta)\left((A + C)\delta - \alpha\delta^\sigma\right).$$

Choosing α_1 such that $\alpha_1\delta_1^\sigma = \alpha_0\delta_0^\sigma$, we obtain $(A + C)\delta_1 - \alpha_1\delta_1^\sigma < 0$. Then $u(x) \geq v(x)$ for $\delta(x) = \delta_1$. For any $\theta > 1$, we have $v(x) < \theta u(x)$ for $\delta(x) = \delta_1$. Hence, by (2.1) we obtain $v(x) < \theta u(x)$ near $\partial\Omega$ and derive

$$\theta\Delta u < (\theta u)^p + |\nabla(\theta u)|^q.$$

Using comparison principle and (3.15), we have proved that $v(x) < \theta u(x)$ for $\delta \leq \delta_1$.

Let $\theta \to 1$. We get $u(x) \geq v(x)$ for $\delta \leq \delta_1$. Increasing α_1, we obtain $u(x) \geq v(x)$ on Ω. Hence, the left side of Eq.(3.10) is proved. The proof of the right side of Eq.(3.10) is similar and is omitted. Hence, we prove (3.10).

By a direct calculation, (3.9) can be derived if we have

$$\begin{aligned}&(1 - \frac{(N-1)\,H(x)\,\delta\log\delta}{2} - \alpha\delta(-\log\delta)^\sigma)\psi(\delta) < u(x)\\ &< (1 - \frac{(N-1)\,H(x)\,\delta\log\delta}{2} + \alpha\delta(-\log\delta)^\sigma)\psi(\delta(x)),\end{aligned}$$

where α is a positive constant. Consider the left side of the inequality. Let

$$v = (1 + \bar{A}\delta\log\delta - \alpha\delta(-\log\delta)^\sigma)\psi(\delta),$$

where $\bar{A} = \frac{\Delta\delta}{2}$. Similarly, one can prove (3.10). This completes our proof of the lemma. □

4. Proof of Theorem 1.1

In this section, we give a proof for all the cases in Theorem 1.1.

Proof First, we consider the case (iii)$''$. By replacing (3.17) in the proof of Lemma 3.2 with

$$(3 - 2q)A - (q - 1)\Delta\delta = \frac{2q - 3}{q - 2}Aq + G(\delta),$$

we can prove

$$u(x) = \big(\frac{p}{(p-1)\,\delta}\big)^{\frac{1}{p-1}}\Big(1 + \frac{(N-1)\,H(x)}{2 - 2p}\delta - \frac{(2-p)^{-p-1}}{2}\delta + o(\delta)\Big).$$

Similarly, $H(x)$ can be replaced by $H(\bar{x})$ in the inequality, where $\bar{x}$ is the point such that $\delta(x) = |x - \bar{x}|$. $\Delta\delta(x)$ has eigenvalues given by

$$\frac{-\kappa_1(\bar{x})}{1 - \kappa_1(\bar{x})\delta}, \dots, \frac{-\kappa_{N-1}(\bar{x})}{1 - \kappa_{N-1}(\bar{x})\delta},$$

where $\kappa_1(\bar{x}), \dots, \kappa_{N-1}(\bar{x})$ are the principal curvatures of $\partial\Omega$, N is the dimension of Ω. We know [18]

$$\Delta\delta = \sum_{i=1}^{N-1}\frac{-\kappa_i(\bar{x})}{1 - \kappa_i(\bar{x})\delta} = -\sum_{i=1}^{N-1}\kappa_i(\bar{x})[1 + \kappa_i(\bar{x})\delta + \cdots + (\kappa_i(\bar{x})\delta)^n + o((\kappa_i(\bar{x})\delta)^n)].$$

Let $n=1$. We get

$$\begin{aligned}\Delta\delta &= \sum_{i=1}^{N-1}\frac{-\kappa_i(\bar{x})}{1-\kappa_i(\bar{x})\delta} = -\sum_{i=1}^{N-1}\kappa_i(\bar{x}) - \sum_{i=1}^{N-1}\kappa_i(\bar{x})(\kappa_i(\bar{x})\delta+o(\delta))\\ &= -(N-1)H(\bar{x})+o(\delta).\end{aligned}$$

Hence, replace $H(x)$ by $H(\bar{x})$. For the other cases (i), (ii) and (iv), using the same arguments as (iii), one can prove the second order estimates. The proof for the cases (i)$'$ and (ii)$'$ follow the ones of (iii)$''$ and (iii)$'$, respectively. □

5. Proof of Theorem 1.3

In this section, we prove Theorem 1.3.

Proof It suffices to prove that there exists a positive constant C such that

$$\left|\frac{u(x)}{\psi_1(\delta)} - \left(1+A_1\delta+A_2\delta^2\right)\right| < C\delta^\sigma,$$

where

$$A_1 = -\frac{\Delta\delta}{p+3},\ A_2 = \frac{(-A_1\Delta\delta - 2\nabla A_1\nabla\delta)(3-p)}{12},$$

and $2<\sigma<3$ is a constant. First, consider the case $1<p<3$, $0<q<\frac{2}{p+1}$ for (P_+). Let

$$v = \left(1+A_1\delta+A_2\delta^2-\alpha\delta^\sigma\right)\psi_1(\delta),$$

where

$$A_1 = -\frac{\Delta\delta}{p+3},\quad A_2 = \frac{(-A_1\Delta\delta - 2\nabla A_1\nabla\delta)(3-p)}{12},$$

and $2<\sigma<3$ is a constant. The left part is the same as Lemma 3.2. The proof is left to the reader. Hence, we have (1.1) and (1.2). □

Acknowledgements We thank the referees for their time and comments.

References

[1] E. GIARRUSSO, G. PORRU. *Second order estimates for boundary blow-up solutions of elliptic equations with an additive gradient term*. Nonlinear Anal., 2015, **129**: 160–172.

[2] C. BANDLE, M. MARCUS. *On second-order effects in the boundary behaviour of large solutions of semilinear elliptic problems*. Differential Integral Equations, 1998, **11**(1): 23–34.

[3] C. BANDLE. *Asymptotic behaviour of large solutions of quasilinear elliptic problems*. Z. Angew. Math. Phys., 2003, **54**(5): 731–738.

[4] M. DEL PINO, R. LETELIER. *The influence of domain geometry in boundary blow-up elliptic problems*. Nonlinear Anal., Ser. A, 2002, **48**(6): 897–904.

[5] C. ANEDDA, G. PORRU. *Higher order boundary estimates for blow-up solutions of elliptic equations*. Differential Integral Equations, 2006, **19(3)**: 345–360.

[6] J. GARCÍA-MELIÁN, R. LETELIER-ALBORNOZ, J. SABINA DE LIS. *Uniqueness and asymptotic behaviour for solutions of semilinear problems with boundary blow-up*. Proc. Amer. Math. Soc., 2001, **129**(12): 3593–3602.

[7] C. ANEDDA, G. PORRU. *Second Order Estimates for Boundary Blow-Up Solutions of Elliptic Equations*. Conference Publications, 2007.

[8] C. ANEDDA, G. PORRU. *Boundary behaviour for solutions of boundary blow-up problems in a borderline case*. J. Math. Anal. Appl., 2009, **352**(1): 35–47.

[9] M. MUSSO. *Sign-changing blowing-up solutions for a non-homogeneous elliptic equation at the critical exponent*. J. Fixed Point Theory Appl., 2017, **19**(1): 345–361.

[10] C. BANDLE, M. MARCUS. *Dependence of blowup rate of large solutions of semilinear elliptic equations, on the curvature of the boundary*. Complex Var. Theory Appl., 2004, **49**(7-9): 555–570.

[11] E. GIARRUSSO, M. MARRAS, G. PORRU. *Second order estimates for boundary blowup solutions of quasilinear elliptic equations*. J. Math. Anal. Appl., 2015, **424**(1): 444–459.

[12] E. GIARRUSSO, G. PORRU. *Asymptotic behavior up to the second order term of solutions to quasilinear elliptic singular problems*. Nonlinear Anal., 2016, **145**: 49–67.

[13] Shibo HUANG, Qiaoyu TIAN. *Asymptotic behavior of large solution for boundary blowup problems with non-linear gradient terms*. Appl. Math. Comput., 2009, **215**(8): 3091–3097.

[14] S. ALARCÓN, J. GARCÍA-MELIÁN, A. QUAAS. *Keller-Osserman type conditions for some elliptic problems with gradient terms*. J. Differential Equations, 2012, **252**(2): 886–914.

[15] Zhijun ZHANG. *Large solutions of semilinear elliptic equations with a gradient term: existence and boundary behavior*. Commun. Pure Appl. Anal., 2013, **12**(3): 1381–1392.

[16] Bo LI, Haitao WAN. *Blow-up rates and uniqueness of entire large solutions to a semilinear elliptic equation with nonlinear convection term*. Bound. Value Probl. 2018, Paper No. 179, 14 pp.

[17] C. BANDLE, E. GIARRUSSO. *Boundary blow up for semilinear elliptic equations with nonlinear gradient terms*. Adv. Differential Equations, 1996, **1**(1): 133–150.

[18] D. GILBARG, N. S. TRUDINGER. *Elliptic Partial Differential Equations of Second Order*. Springer Verlag, Berlin, 1997.