Journal of Mathematical Research with Applications
Sept., 2021, Vol. 41, No. 5, pp. 461–472
DOI:10.3770/j.issn:2095-2651.2021.05.003
Http://jmre.dlut.edu.cn

Dynamics of Anti-Periodic Solutions for Inertial Quaternion-Valued Hopfield Neural Networks with Time-Varying Delays

Ailing LI[1,*], **Mengting LÜ**[2]

1. *College of Science, Hebei North University, Hebei* 075000*, P. R. China;*
2. *School of Mathematics, Hunan University, Hunan* 410082*, P. R. China*

Abstract Periodicity, anti-periodicity and almost periodicity are significant dynamic behaviors of time-varying neural networks. This paper researches the dynamics of anti-periodic solutions for a kind of inertial Quaternion-valued Hopfield neural networks with varying-time delays. Without resolving the explored neural networks into real-valued systems, in the light of a continuation theorem of coincidence degree theory and inequality skills, by constructing different Lyapunov functions from those constructed in the existing research of the stability of equilibrium point, periodic solutions and anti-periodic solutions for neural networks, a newfangled sufficient condition insuring the existence of periodic solutions for above neural networks is gained. By constructing the same Lyapunov functions as those constructed in the proof of the existence of anti-periodic solutions, the newfangled asymptotic stability of anti-periodic solutions for above networks is acquired.

Keywords anti-periodic solution; asymptotic stability; continuation theorem; Quaternion-valued inertial neural networks

MR(2020) Subject Classification 34K13

1. Introduction

Quaternion algebra was introduced into mathematics by Hamilton in 1843. The expression of skew field of quaternions is

$$\mathbf{H} = \{u | u = u^R + iu^I + ju^J + ku^K\},$$

where $u^R, u^I, u^J, u^K \in R, i, j, k$ are the fundamental quaternion units, which meet the Hamilton's multiplication rules: $ij = -ji = k, jk = -kj = i, ki = -ik = j, i^2 = j^2 = k^2 = -1$. For each $u \in \mathbf{H}$, the conjugate of u is $u^* = u^R - iu^I - ju^J - ku^K$ and the norm of u is defined as $\|u\| = \sqrt{uu^*} = \sqrt{(u^R)^2 + (u^I)^2 + (u^J)^2 + (u^K)^2}$.

Quaternion-valued neural networks is involved in various fields such as attitude control, quantum mechanics and computer graphics. It can use multistate activation functions particularly to process multi-level information, the research of periodic solutions, stability and synchronization have become a very topical issues. Lately, by resolving a quaternion-valued neural networks

Received July 29, 2020; Accepted April 6, 2021
Supported by the Basic Research Expenses for Provincial Colleges and Universities (Grant No. JYT2020030).
* Corresponding author
E-mail address: liailing@hnu.edu.cn (Ailing LI); lvmengting@hnu.edu.cn (Mengting LÜ)

into real-valued neural networks, some authors get a number of sufficient conditions insuring the existence and exponential stability of the equilibrium point, periodic solutions, anti-periodic solutions for quaternion-valued neural networks, in this regard, we can see the references [1–6]. In [7], so as to avoid the complexity of computation of resolving a quaternion-valued neural networks into real-valued neural networks, the authors studied the stability of a class of quaternion-valued neural networks by the use of non-decomposition ways, that is, without resolving a quaternion-valued neural networks into real-valued neural networks.

In many researches on neural networks, physical or natural phenomenon is applied to the periodic motion control, and can be converted into transaction cycle mode expression recognition, especially the latter is used for image processing and other complex static pattern recognition is expected to bring computing superiority, and improve the efficiency of information processing, so as to realize more intelligent information processing system, to create a possible way to achieve intelligent computer. These applications rely massively on the dynamic behaviors of the neural networks. Thus, the qualitative analysis of these dynamical behavior is prerequisite step in practical designing of neural networks, for example [8–12].

Furthermore, due to rich periodic and anti-periodic significant dynamic behaviors of neural networks, it becomes very useful to analyze the periodicity, anti-periodicity and almost periodicity of neural networks. It has been observed that many physical or natural phenomena do not show exact periodic behaviour, rather they show anti-periodic behaviour. Especially, neural network signal transmission courses can often be represented as anti-periodic courses. As a special case of periodic solution, the anti-periodicity of a variety of forms neural networks has been studied extensively, it plays an important role in the dynamic behavior of nonlinear differential equations. For example [4, 13–15]. Thus, it is essential to research the periodicity and anti-periodicity of neural networks. We have improved and generalized some known results.

Inertial neural network is a significant neural network, till now, the stability of equilibrium point and periodic solutions for inertial neural networks has been discriminated clearly, which we can refer to [16–18].

In [19], lately, the authors researched the existence and exponential stability of anti-periodic solutions for a class of inertial Quaternion-valued high-order Hopfield neural networks with state-dependent delays. Without resolving the considered neural networks into real-valued systems, in the light of a continuation theorem of coincidence degree theory and the wirtinger inequality, an adequate qualification on the existence of anti-periodic solutions for above neural networks is got. Then by constructing a Lyapunov functional, an adequate qualification insuring the global exponential stability of the anti-periodic solution of the neural networks is gained.

In this paper, we are going to investigate the existence and global asymptotic stability of anti-periodic solutions for a kind of inertial Quaternion-valued Hopfield neural networks with time-varying delays. Without resolving the researchful neural networks into eight real-valued systems, such as [19], putting the continuation theorem of coincidence degree theory into use as before, but averting the prior estate method of periodic solutions used in [19], by wielding new approach of studying the existence and global asymptotic stability, we will explore newfangled adequate qualification for the considered neural networks. In this paper, we investigate the following inertial quaternion-valued Hopfield neural networks with time-varying delays:

$$u_p''(t) = -\alpha_p(t)u_p'(t) - \gamma_p(t)u_p(t) + \sum_{q=1}^{n} c_{pq}(t)F_q u_q(t) + \sum_{q=1}^{n} d_{pq}(t)F_q(u_q(t-\tau(t)) + I_p(t), \quad (1.1)$$

thereinto, $p = 1, 2, \ldots, n$ and n is the number of units in the neural network, $u_p(t) \in \mathbf{Q}$ corresponds to the state vector of the pth unit at time t; $\alpha_p(t) \geq 0, \gamma_p(t) \geq 0; c_{pq}(t), d_{pq}(t) \in \mathbf{Q}$ are the first and second-order connection weights of the neural network at time $t, \tau(t) \geq 0$ are the transmission delays, $I_p(t) \in \mathbf{Q}$ denotes the external input at time t, and $F_q : \mathbf{Q} \to \mathbf{Q}$ is the activation function of signal transmission.

The initial value conditions of system (1.1) is that:

$$u_p(s) = \phi_p^*(s),\ u_p'(s) = \psi_p^*(s), \quad s \in [-\rho, 0],\ p = 1, 2, \ldots, n, \quad (1.2)$$

where $\phi_p, \psi_p \in C([-\rho, 0], \mathbf{Q})$.

So far, in the research of the existence of periodic solutions (if $u(t)$ is a $\frac{\omega}{2}$ anti-periodic solution, then it is also a ω periodic solution) for delayed neural networks, a $V(t)$ function with

$$V(t) = \sum_{i=1}^{n} x_i^2(t) + \sum_{j=1}^{m} y_i^2(t) + \sum_{i=1}^{n} \int_{t-\tau}^{\tau} x_i^2(s)\mathrm{d}s + \sum_{j=1}^{m} \int_{t-\sigma}^{t} y_i^2(s)\mathrm{d}s + \cdots \text{ or}$$

$$V(t) = \sum_{i=1}^{n} |x_i(t)| + \sum_{j=1}^{m} |y_i(t)| + \sum_{i=1}^{n} \int_{t-\tau}^{\tau} |x_i(s)|\mathrm{d}s + \sum_{j=1}^{m} |y_i(s)|\mathrm{d}s + \cdots$$

has been constructed, for instance [14–16]. By then showing $V'(t) \leq -\mu\|(x, y)^T\| + k^*$ ($\mu > 0, k^* > 0$), the existence of at least one periodic solution was testified. In the research of the global stability of anti-periodic solutions and periodic solutions for delayed neural networks, a $V(t)$ function with

$$V(t) = \sum_{i=1}^{n}[x_i(t)-x_i^*(t)]^2+\sum_{j=1}^{m}[y_i(t)-y_i^*(t)]^2+\sum_{i=1}^{n}\int_{t-\tau}^{\tau}[x_i(s)-x_i^*(s)]^2\mathrm{d}s+\sum_{j=1}^{m}\int_{t-\sigma}^{t}[y_i(s)-y_i^*(s)]^2\mathrm{d}s+ \cdots \text{ or}$$

$$V(t) = \sum_{i=1}^{n}|x_i(t)-x_i^*(t)|+\sum_{j=1}^{m}|y_i(t)-y_i^*(t)|+\sum_{i=1}^{n}\int_{t-\tau}^{\tau}|x_i(s)-x_i^*(t)|\mathrm{d}s+\sum_{j=1}^{m}|y_i(s)-y_i^*(s)|\mathrm{d}s+\cdots$$

has been constructed, then by showing $V'(t) \leq 0$, the global stability of anti-periodic solutions and periodic solutions was testified. In this respect, we refer to [4, 5, 14, 15, 20–23].

Our purpose is constructing Lyapunov functions unlike those in the existing available literature and wielding a different research technique of anti-periodic solutions from that used in [19] to acquire newfangled adequate qualification of the existence and global asymptotic stability of periodic solutions for system (1.1). In our research, in the first place, a differential inequality group will be built, then by solving the differential inequality group, the adequate qualification of the existence of anti-periodic solutions is acquired. Furthermore, the exploring of global asymptotic stability is other than those in [19] and other existing papers. So, the efficiency of the paper is listed below: (1) A newfangled research technique of the existence and global stability of anti-periodic solution is introduced; (2) By putting differential inequality skills into use, newfangled adequate qualification insuring the existence and asymptotic stability of anti-periodic solutions for system (1.1) is received.

2. Preliminaries

Throughout this paper, we suppose that:

(v_1) For $p,q=1,2,\ldots,n,\alpha_p,\gamma_p,\tau\in BC(R\times\mathbf{Q},R),F_q\in C(\mathbf{Q},\mathbf{Q}),c_{pq},d_{pq},I_p\in C(R,\mathbf{Q})$ and there exists a constant $\omega>0$ such that for $t,u\in R$,

$$\alpha_p(t+\frac{\omega}{2})=\alpha_p(t),\quad \gamma_p(t+\frac{\omega}{2})=\gamma_p(t),$$

$$c_{pq}(t+\frac{\omega}{2})F_q(u)=-c_{pq}(t)F_q(-u),$$

$$\tau(t+\frac{\omega}{2})=\tau(t),\quad I_p(t+\frac{\omega}{2})=-I_p(t).$$

(v_2) There exist positive constants L_q such that for all $u,v\in\mathbf{Q}$,

$$\|F_q(u)-F_q(v)\|\le L_q\|u-v\|,$$

where $\|\cdot\|$ is the norm of $\mathbf{Q}$. We will take the following notations:

$$\alpha_p^-=\inf_{t\in[0,\omega]}\alpha_p(t),\gamma_p^+=\sup_{t\in[0,\omega]}\gamma_p(t),\tau^+=\max_{1\le p,q\le n}\{\sup_{t\in[0,\omega]}\tau(t)\},$$

$$c_{pq}^+=\sup_{t\in[0,\omega]}\|c_{pq}(t)\|,d_{pq}^+=\sup_{t\in[0,\omega]}\|d_{pq}(t)\|,I_p^+=\sup_{t\in[0,\omega]}\|I_p(t)\|.$$

Let $(u_1,u_2,\ldots,u_n)^T$ be a solution of system (1.1) with initial value (1.2). Making variable substitution:

$$v_p(t)=u_p'(t)+u_p,\quad p=1,2,\ldots,n,$$

then for $p=1,2,\ldots,n$, system (1.1) is converted to

$$u_p'(t)=-u_p(t)+v_p(t)=\pi_p(u,v,t),$$

$$\begin{aligned}v_p'(t)=&-[1+\gamma_p(t)-\alpha_p(t)]u_p(t)-(\alpha_p(t)-1)v_p(t)+\sum_{q=1}^n d_{pq}(t)F_q(u_q(t-\tau(t)))+\\&\sum_{q=1}^n c_{pq}(t)F_q(u_q(t))+I_p(t)=\Gamma_p(u,v,t),\end{aligned}\tag{2.1}$$

with the initial values:

$$u_p(s)=\phi_p^*(s),\ v_p(s)=\psi_p^*(s),\quad s\in[\rho,0],$$

where $p=1,2,\ldots,n$.

It is clear that the existence and global asymptotic stability of anti-periodic solutions of system (1.1) are equivalent to the existence and global asymptotic stability of anti-periodic solutions of system (2.1). Therefore, we only need to discuss the existence and global asymptotic stability of anti-periodic solutions of system (2.1).

Lemma 2.1([17]) *Let $\hat{X}$ and $\hat{Y}$ be two Banach spaces, $\hat{L} : \mathrm{Dom}(\hat{L}) \subset \hat{X} \to \hat{Y}$ be linear, and $\hat{N} : \hat{X} \to \hat{Y}$ be continuous. Assume that $\hat{L}$ is one-to-one and $K := \hat{L}^{-1}\hat{N}$ is compact. Furthermore, assume that there exists a bounded and open subset $\hat{\Omega} \subset \hat{X}$ with $0 \in \hat{\Omega}$ such that equation $\hat{L}u = \lambda\hat{N}u$ has no solutions in $\partial\hat{\Omega} \cap \mathrm{Dom}\hat{L}$ for any $\lambda \in (0, 1)$. Then the problem $\hat{L}u = \hat{N}u$ has at least one solution in $\overline{\hat{\Omega}}$.*

Lemma 2.2 *For all $a, b \in \mathbf{Q}, a^*b + b^*a \le a^*a + b^*b$.*

3. The existence of anti-periodic solutions

In this section, we will acquire a sufficient condition on the existence of anti-periodic solutions for system (1.1) by applying Lemma 2.1. Let $p = 1, 2, \dots, n$,

$$\hat{X} = \big\{x : x = (u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n)^T \in C(R, \mathbf{Q}^{2n}), x(t + \frac{\omega}{2}) = -x(t), \quad \forall t \in R\big\},$$

$$\|x\|_{\hat{X}} = \sum_{p=1}^{n}(|u_p|_0 + |v_p|_0),$$

$$|u_p|_0 = \sup_{t\in[0,\omega]}\sqrt{u_p(t)u_p^*(t)}, \quad |v_p|_0 = \sup_{t\in[0,\omega]}\sqrt{v_p(t)v_p^*(t)},$$

then $\hat{X}$ is a Banach space with the norm $\|\cdot\|_{\hat{X}}$.

Theorem 3.1 *Assume that (v_1) and (v_2) hold. Furthermore assume that*

(v_3) $\alpha_p^- > \gamma_p^+ + 1$;

(v_4) $\mu_1 > \mu_2 + \hat{\mu}_1 + \hat{\mu}_2$, where

$$\mu_1 = \min_{1\le p\le n}\{\alpha_p^- + \gamma_p^+ - 2\}, \quad \mu_2 = \max_{1\le p\le n}\{\alpha_p^- - \gamma_p^+ - 1\},$$

$$\hat{\mu}_1 = n\max_{1\le q\le n}\Big\{\sum_{p=1}^{n}[L_q^2c_{pq}^+(c_{pq}^+ + d_{pq}^+)]\Big\}, \quad \hat{\mu}_2 = n\max_{1\le q\le n}\Big\{\sum_{p=1}^{n}[L_q^2d_{pq}^+(d_{pq}^+ + c_{pq}^+)]\Big\}.$$

Then system (1.1) has at least $\frac{\omega}{2}$ anti-periodic solution.

Proof On the basis of (v_4), it follows that there exists a positive constant ε such that

(v_5) $\mu_1 > \mu_2 + \mu_3 + \mu_4$, where

$$\mu_3 = n\max_{1\le p\le n}\Big\{\sum_{p=1}^{n}[L_q^2c_{pq}^+(c_{pq}^+ + d_{pq}^+) + L_qc_{pq}^+\varepsilon]\Big\},$$

$$\mu_4 = n\max_{1\le p\le n}\Big\{\sum_{p=1}^{n}[L_q^2d_{pq}^+(d_{pq}^+ + c_{pq}^+) + L_qd_{pq}^+\varepsilon]\Big\}.$$

Define a linear operator $\hat{L} : \mathrm{Dom}\,\hat{L} \subset \hat{X} \to \hat{X}$ by $\hat{L}x = x'$, where $\mathrm{Dom}\,\hat{L} = \{x : x \in X, x' \in \hat{X}\}$ and a continuous operator $\hat{N} : \hat{X} \to \hat{X}$ by

$$(\hat{N})(t) = \big(\pi_1(u, v, t), \pi_2(u, v, t), \dots, \pi_n(u, v, t), \Gamma_1(u, v, t), \Gamma_2(u, v, t), \dots, \Gamma_n(u, v, t)\big).$$

It is easy to know that $\operatorname{Ker}\hat{L}=\{0\}$ and $\operatorname{Im}\hat{L}=\{z\in\hat{X},\int_0^\omega z(t)\mathrm{d}t=0\}=\hat{X}$. Consequently, $\hat{L}:\operatorname{Dom}\hat{L}\to\hat{X}$ is one-to-one. Denote by $(\hat{L})^{-1}$ the inverse of $\hat{L}$ and take $\hat{K}=(\hat{L})^{-1}\hat{N}$, then by applying Arzela-Ascoli theorem, it can be tested that $\hat{K}$ is compact.

Let $x\in\hat{X}$ be an arbitrary solution of $\hat{L}x=\lambda_1\hat{N}x$ for a certain $\lambda\in(0,1)$. Then one has

$$u_p'(t)=\lambda_1\pi_p(u,v,t),\ v_p'(t)=\lambda_1\Gamma_p(u,v,t),\quad p=1,2,\ldots,n. \tag{3.1}$$

Define two Lyapunov functions as follows:

$$M_1(t)=\sum_{p=1}^n u_p^*(t)u_p(t),\quad M_2(t)=\sum_{p=1}^n v_p^*(t)v_p(t).$$

Applying Lemma 2.2, we have on the basis of (3.1),

$$\begin{aligned}\frac{\mathrm{d}M_1(t)}{\mathrm{d}t}=&\lambda_1\sum_{p=1}^n[(u_p^*)'(t)u_p(t)+u_p^*(t)u_p'(t)]\\
=&\lambda_1\sum_{p=1}^n\big(u_p^*(t)[-u_p(t)+v_p(t)]+[-u_p^*(t)+v_p^*(t)]u_p(t)\big)\\
=&\lambda_1\Big\{-2M_1(t)+\sum_{p=1}^n[u^*(t)v_p(t)+v_p^*(t)u_p(t)]\Big\}\\
\le&\lambda_1\Big\{-2M_1(t)+\sum_{p=1}^n[u_p^*(t)u_p(t)+v_p^*(t)v_p(t)]\Big\}\\
=&\lambda_1[-M_1(t)+M_2(t)]\end{aligned}\tag{3.2}$$

$$\begin{aligned}\frac{\mathrm{d}M_2(t)}{\mathrm{d}t}=&\sum_{p=1}^n[(v_p^*)'(t)v_p(t)+v_p^*(t)v_p'(t)]\\
=&\lambda_1\sum_{p=1}^n\Big\{\Big[-(1+\gamma_p(t)-\alpha_p(t))u_p^*(t)-(\alpha_p(t)-1)v_p^*(t)+\sum_{q=1}^n d_{pq}(t)F_q(u_q^*(t-\tau(t)))+\\
&\sum_{q=1}^n c_{pq}(t)F_q(u_q^*(t))+I_p(t)\Big]v_p(t)+v_p^*(t)\Big[-(1+\gamma_p(t)-\alpha_p(t))u_p(t)-(\alpha_p(t)-1)v_p(t)+\\
&\sum_{q=1}^n d_{pq}(t)F_q(u_q(t-\tau_{pq}(t)))+\sum_{q=1}^n c_{pq}(t)F_q(u_q(t))+I_p(t)\Big]\Big\}\\
\le&\lambda_1\sum_{p=1}^n\Big\{-2(\alpha_p^--1)v_p^*(t)v_p(t)-[1+\gamma_p(t)-\alpha_p(t)][u_p^*(t)v_p(t)+v_p^*(t)u_p(t)]+\Big[\sum_{q=1}^n d_{pq}(t)\times\\
&F_q(u_q^*(t-\tau(t)))+\sum_{q=1}^n c_{pq}(t)F_q(u_q^*(t))+I_p(t)\Big]v_p(t)+v_p^*(t)\Big[\sum_{q=1}^n F_q(u_q(t-\tau(t)))\times\\
&d_{pq}(t)+\sum_{q=1}^n c_{pq}(t)F_q(u_q(t))+I_p(t)\Big]\Big\}\\
\le&\lambda_1\sum_{p=1}^n\Big\{-2(\alpha_p^--1)v_p^*(t)v_p(t)+[\alpha_p^--\gamma_p^+-1][u_p^*(t)u_p(t)+v_p^*(t)v_p(t)]+\Big[\sum_{q=1}^n d_{pq}(t)\times\end{aligned}$$

$$F_q(u_q^*(t-\tau(t))) + \sum_{q=1}^{n} c_{pq}(t)F_q(u_q^*(t)) + I_p(t)\Big]v_p(t) + v_p^*(t)\Big[\sum_{q=1}^{n} F_q(u_q(t-\tau(t)))\times$$
$$d_{pq}(t) + \sum_{q=1}^{n} c_{pq}(t)F_q(u_q(t)) + I_p(t)\Big]\Big\}. \tag{3.3}$$

By Lemma 2.2, one has

$$\begin{aligned}
&\Big[\sum_{q=1}^{n} d_{pq}(t)F_q(u_q^*(t-\tau(t))) + \sum_{q=1}^{n} c_{pq}(t)F_q(u_q^*(t)) + I_p(t)\Big]v_p(t) + v_p^*(t)\times\\
&\quad\Big[\sum_{q=1}^{n} d_{pq}(t)F_q(u_q(t-\tau(t))) + \sum_{q=1}^{n} c_{pq}(t)F_q(u_q(t)) + I_p(t)\Big]\\
&\le v_p^*(t)v_p(t) + \Big(\sum_{q=1}^{n} d_{pq}(t)F_q(u_q^*(t-\tau(t))) + \sum_{q=1}^{n} c_{pq}(t)F_q(u_q^*(t)) + I_p(t)\Big)\times\\
&\quad\Big(\sum_{q=1}^{n} d_{pq}(t)F_q(u_q(t-\tau(t))) + \sum_{q=1}^{n} c_{pq}(t)F_q(u_q(t)) + I_p(t)\Big)\\
&\le v_p^*(t)v_p(t) + \Big(\sum_{q=1}^{n} d_{pq}^+[L_q\|u_q^*(t-\tau(t))\| + \|F_q(0)\|] + \sum_{q=1}^{n} c_{pq}^+[\|L_q\|u_q^*(t)\| + \|F_q(0)\|]+\\
&\quad I_p^+\Big)\Big(\sum_{q=1}^{n} d_{pq}^+[L_q\|u_q(t-\tau(t))\| + \|F_q(0)\|] + \sum_{q=1}^{n} c_{pq}^+[\|L_q\|u_q(t)\| + \|F_q(0)\|] + I_p^+\Big)\\
&= v_p^*(t)v_p(t) + \Big\{\sum_{q=1}^{n}[A_{pq} + L_q(d_{pq}^+\|u_q(t-\tau(t)\| + c_{pq}^+\|u_q(t)\|)]\Big\}^2\\
&\le v_p^*(t)v_p(t) + n\sum_{q=1}^{n}[A_{pq} + L_q(d_{pq}^+\|u_q(t-\tau(t))\| + c_{pq}^+\|u_q\|)]^2\\
&\le v_p^*(t)v_p(t) + n\sum_{q=1}^{n}\big\{A_{pq}^2 + L_q^2(c_{pq}^+)^2u_q^*(t)u_q(t) + L_q^2(d_{pq}^+)^2u_q^*(t-\tau(t))u_q(t-\tau(t))+\\
&\quad 2A_{pq}L_qd_{pq}^+\|u_q(t-\tau(t))\| + 2A_{pq}L_qc_{pq}^+\|u_q(t)\| + 2L_q^2d_{pq}^+c_{pq}^+\|u_q(t-\tau(t))\|\times\\
&\quad \|u_q(t)\|\big\},
\end{aligned} \tag{3.4}$$

where $A_{pq} = I_p^+ + (d_{pq}^+ + c_{pq}^+)\|F_q(0)\|$. Because

$$2L_qd_{pq}^+A_{pq}\|u_q(t-\tau(t))\| \le L_qd_{pq}^+[\varepsilon u_q^*(t-\tau(t))u_q(t-\tau(t)) + \frac{A_{pq}}{\varepsilon}], \tag{3.5}$$

$$2A_{pq}L_qc_{pq}^+\|u_q\| \le L_qc_{pq}^+[\varepsilon u_q^*(t)u_q(t) + \frac{A_{pq}}{\varepsilon}] \tag{3.6}$$

and

$$2L_q^2d_{pq}^+c_{pq}^+\|u_q(t-\tau(t))\|\|u_q(t)\| \le L_q^2d_{pq}^+c_{pq}^+[u_q^*(t-\tau(t))u_q(t-\tau(t)) + u_q^*(t)u_q(t)], \tag{3.7}$$

Substituting (3.5)–(3.7) into (3.4) leads to

$$\Big[\sum_{q=1}^{n} d_{pq}(t)F_q(u_q^*(t-\tau(t))) + \sum_{q=1}^{n} c_{pq}(t)F_q(u_q^*(t)) + I_p(t)\Big]v_p(t) + v_p^*(t)\times$$

$$\Big[\sum_{q=1}^{n} d_{pq}(t)F_q(u_q(t-\tau(t))) + \sum_{q=1}^{n} c_{pq}(t)F_q(u_q(t)) + I_p(t)\Big]$$
$$\le v_p^*(t)v_p(t) + n\sum_{q=1}^{n}\Big\{\Big[L_q^2 c_{pq}^+(c_{pq}^+ + d_{pq}^+) + L_q c_{pq}^+\varepsilon\Big]u_q^*(t)u_q(t) + \Big[L_q^2(d_{pq}^+)^2 + L_q^2 d_{pq}^+ c_{pq}^+ +$$
$$L_q d_{pq}^+\varepsilon\Big]u_q^*(t-\tau(t))u_q(t-\tau_{pq}(t)) + B_0\Big\}, \tag{3.8}$$

where B_0 is a positive constant. Substituting (3.8) into (3.3) leads to

$$\begin{aligned}\frac{\mathrm{d}M_2(t)}{\mathrm{d}t} \le&\lambda_1\sum_{p=1}^{n}\Big\{-(\alpha_p^- + \gamma_p^+ - 2)v_p^*(t)v_p(t) + (\alpha_p^- - \gamma_p^+ - 1)u_p^*(t)u_p(t)+\\ &n\sum_{q=1}^{n}([L_q^2 c_{pq}^+(c_{pq}^+ + d_{pq}^+) + L_q c_{pq}^+\varepsilon]u_q^*(t)u_q(t)+\\ &[L_q^2(d_{pq}^+)^2 + L_q^2 d_{pq}^+ c_{pq}^+ + L_q d_{pq}^+\varepsilon]u_q^*(t-\tau(t))u_q(t-\tau(t)) + B_0)\Big\}\\ \le&\lambda_1[-\mu_1 M_2(t) + (\mu_2+\mu_3)M_1(t) + \mu_4 M_1(t-\tau(t)) + n^3 B_0].\end{aligned} \tag{3.9}$$

Let $M_1(\xi_1) = \max_{t\in[0,\omega]}\{M_1(t)\}$, $M_2(\xi_2) = \max_{t\in[0,\omega]}\{M_2(t)\}$, $\xi_1,\xi_2\in[0,\omega]$. Then

$$M_1'(\xi_1) = 0, \quad M_2'(\xi_2) = 0. \tag{3.10}$$

On the basis of (3.10), (3.2) and (3.9), since $\mu_1>0,\mu_2>0,\mu_3>0,\mu_4>0,B_0>0$, one has

$$M_1(\xi_1) \le M_2(\xi_1), \tag{3.11}$$

$$\mu_1 M_2(\xi_2) \le (\mu_2+\mu_3)M_1(\xi_2) + \mu_4 M_1(\xi_2 - \tau(\xi_2)(\xi_2)) + n^3 B_0 = 0. \tag{3.12}$$

In view of (3.12) and (3.11), one has

$$\mu_1 M_2(\xi_2) \le (\mu_2+\mu_3+\mu_4)M_1(\xi_1) + n^3 B_0 \le (\mu_2+\mu_3+\mu_4)M_2(\xi_2) + n^3 B_0.$$

Consequently,

$$M_2(\xi_2) \le \frac{n^3 B_0}{\mu_1 - (\mu_2+\mu_3+\mu_4)}, \quad M_1(\xi_1) \le \frac{n^3 B_0}{\mu_1 - (\mu_2+\mu_3+\mu_4)}.$$

Namely,

$$M_1(t) = \sum_{p=1} u_p^*(t)u_p(t) \le \frac{n^3 B_0}{\mu_1 - (\mu_3+\mu_2+\mu_4)},$$

$$M_2(t) = \sum_{p=1} u_p^*(t)u_p(t) \le \frac{n^3 B_0}{\mu_1 - (\mu_3+\mu_2+\mu_4)}.$$

As a result

$$\begin{aligned}\|x\|_{\hat{X}}^2 \le& \Big[\sum_{p=1}^{n}(|u_p|_0 + |v_p|_0)\Big]^2 \le 2n\sum_{p=1}^{n}(|u_p|_0^2 + |v_p|_0^2)\\ \le& \frac{4n^4 B_0}{\mu_1 - (\mu_2+\mu_3+\mu_4)}.\end{aligned}$$

Take $\hat{\Omega} = \{x \in \hat{X} : \|x\|_X < \frac{4n^4 B_0}{\mu_1 - (\mu_2+\mu_3+\mu_4)} + 1\}$, then $\hat{\Omega} \in \hat{X}$ with $0 \in \hat{\Omega}$ such that equation $\hat{L}x = \lambda_1 \hat{N}x$ has no solutions in $\partial\hat{\Omega} \cap \mathrm{Dom}\,\hat{L}$ for any $\lambda_1 \in (0,1)$. Then by Lemma 2.1, the system (1.1) has at least one $\frac{\omega}{2}$ anti-periodic solution. □

4. Global asymptotic stability of anti-periodic solution

In the section, by constructing two Lyapunov functions and applying integral inequality techniques, we study the global asymptotic stability of anti-periodic solutions of system (1.1).

Theorem 4.1 *Let* (v_1)–(v_4) *hold. Furthermore, assume that* $\tau'(t) \le \tau^* < 1$ *and*

(v_6) $\hat{\mu}_1 + \mu_2 < 1$;

(v_7) $\mu_1 > 1 + \frac{\hat{\mu}_2}{1-\tau^*}$.

Then system (1.1) has a unique $\frac{\omega}{2}$ *anti-periodic solution which is globally asymptotically stable.*

Proof By Theorem 3.1, we know that system (1.1) or system (2.1) has an $\frac{\omega}{2}$-anti-periodic solution, say, $\hat{x}(t) = (\hat{u}_1(t), \hat{u}_2(t), \dots, \hat{u}_n(t), \hat{v}_1(t), \hat{v}_2(t), \dots, \hat{v}_n(t))^T$. Let $x(t) = (u_1(t), u_2(t), \dots, u_n(t), v_1(t), v_2(t), \dots, v_n(t))^T$ be an arbitrary solution of system (2.1). By system (2.1), we have for $p = 1, 2, \dots, n$,

$$\begin{aligned}\frac{d[u_p(t) - \hat{u}_p(t)]}{\mathrm{d}t} =& -[u_p(t) - \hat{u}_p(t)] + v_p(t) - \hat{v}_p(t)\\ \frac{d[v_p(t) - \hat{v}_p(t)]}{\mathrm{d}t} =& -[1 + \gamma_p(t) - \alpha_p(t)][u_p(t) - \hat{u}_p(t)] - [\alpha_p(t) - 1][v_p(t) - \hat{v}_p(t)] + \sum_{q=1}^{n} d_{pq}(t) \times \\ & [F_q(u_q(t - \tau(t))) - F_q(\hat{u}(t - \tau(t)))] + \sum_{q=1}^{n} c_{pq}(t)[F_q(u_q(t) - F_q(\hat{u}(t)].\end{aligned} \tag{4.1}$$

Two Lyapunov functions are constructed as follows:

$$F_1(t) = \sum_{p=1}^{n} [u_p(t) - \hat{u}_p(t)]^* [u_p(t) - \hat{u}_p(t)], \ F_2(t) = \sum_{p=1}^{n} [v_p(t) - \hat{v}_p(t)]^* [v_p(t) - \hat{v}_p(t)].$$

By system (4.1), by the similar argument to those of (3.2)–(3.9) in the proof of the existence of anti-periodic solution of system (2.1) (namely, the $u_p(t)$, $v_p(t)$, $M_1(t)$, $M_2(t)$, $\mu_3, \mu_4, A_{pq}, B_0, \lambda$ respectively correspondingly change into $u_p(t) - \hat{u}_p(t)$, $v_p(t) - \hat{v}_p(t)$, $F_1(t)$, $F_2(t)$, $\hat{\mu}_1$, $\hat{\mu}_2, 0, 0, 1$), we obtain

$$\frac{\mathrm{d}F_1(t)}{\mathrm{d}t} \le -F_1(t) + F_2(t), \tag{4.2}$$

$$\frac{\mathrm{d}F_2(t)}{\mathrm{d}t} \le -\mu_1 F_2(t) + (\mu_2 + \hat{\mu}_1) F_1(t) + \hat{\mu}_2 F_2(t - \tau(t)). \tag{4.3}$$

Define

$$F(t) = F_1(t) + F_2(t) + \frac{\hat{\mu}_2}{1 - \tau^*} \int_{t-\tau(t)}^{t} F_2(s)\mathrm{d}s.$$

Then by (4.2) and (4.3), one has

$$F'(t) \le (\hat{\mu}_1 + \mu_2 - 1) F_1(t) + (1 - \mu_1 + \frac{\hat{\mu}_2}{1 - \tau'(t)}) F_2(t)$$

$$\leq (\hat{\mu}_1+\mu_2-1)F_1(t)+(1-\mu_1+\frac{\hat{\mu}_2}{1-\tau^*})F_2(t)<0.$$

This accomplishes the proof of Theorem 4.1. □

Claim 4.2 In [19], by putting the priori method of periodic solutions into use, the sufficient criterion to guarantee the existence of anti-periodic solutions for the discussed Quaternion-Valued inertial Hopfield neural networks was built. In this paper, by constructing two Lyapunov functions and the behaviors of periodic solutions, the boundary of periodic solutions is estimated. As a consequence, the way to prove periodic solutions or anti-periodic solutions for neural networks in our paper is different from those in the available literature and our means of exploring anti-periodic solutions or periodic solutions is newfangled. Moreover, the exploration of global asymptotic stability of periodic solutions is different from those in [19] and the available literature.

5. Numerical test

In this section, a simulation is conducted to illustrate the effectiveness and superiority of our theoretical results.

Example 5.1 Take into account a Quaternion-valued inertial neural network as follows:

$$u_p''(t)=-\alpha_p(t)u_p'(t)-\gamma_p(t)u_p(t)+\sum_{q=1}^{n}c_{pq}(t)F_qu_q(t)+\sum_{q=1}^{n}d_{pq}(t)F_q(u_q(t-\tau(t)))+I_p(t). \quad (5.1)$$

Transform it into the following one-order differential system:

$$\begin{aligned}
u_p'(t)=&-u_p(t)+v_p(t)=\pi_p(u,v,t),\\
v_p'(t)=&-[1+\gamma_p(t)-\alpha_p(t)]u_p(t)-(\alpha_p(t)-1)v_p(t)+\sum_{q=1}^{n}d_{pq}(t)F_q(u_q(t-\tau(t)))+\\
&\sum_{q=1}^{n}c_{pq}(t)F_q(u_q(t))+I_p(t)=\Gamma_p(u,v,t),
\end{aligned} \quad (5.2)$$

where $n=2$, $p=q=2$, $\omega=\frac{\pi}{3}$,

$$\begin{aligned}
&\alpha_p(t)=\cos 12t+6,\quad \gamma_p(t)=\sin 12t+2.8,\\
&c_{11}(t)=c_{21}(t)=0.3\cos 6t-i0.4\sin 6t+j0.5\cos 6t+k0.63\cos 6t,\\
&c_{12}(t)=c_{22}(t)=0.5\sin 6t-i0.3\cos 6t+j0.3\sin 6t+k0.2\cos 6t,\\
&d_{11}(t)=d_{21}(t)=1.3-i\cos 12t+j0.5\sin 12t-k\cos 12t,\\
&d_{21}(t)=d_{22}(t)=0.6-i0.7\sin 12t-j\cos 12t-k0.4\cos 12t,\\
&I_p(t)=0.4\sin 6t+i0.7\cos 6t+j\sin 6t+k2\cos 6t,
\end{aligned}$$

and for any $u\in H$, $u=u^R+u^I+u^J+u^K$, $F_1(u)=F_2(u)=\frac{1}{4}|u^R|+i\frac{1}{5}|u^I|+j\frac{1}{6}|u^J|+k\frac{1}{7}|u^K|$, $\tau(t)=1+\frac{1}{24}\sin 12t$. Thus in Theorem 4.1, $\tau^*=0.5$, $L_1=L_2=\frac{1}{4}$,

$$\alpha_p^-=5,\ \gamma_p^+=3.8,\ \mu_1=6.8,\ \mu_2=0.2,$$

$$c_{11}^{+}=c_{21}^{+}=0.9470,\ c_{12}^{+}=c_{22}^{+}=0.6856,\ d_{11}^{+}=d_{21}^{+}=1.9849,\ d_{12}^{+}=d_{22}^{+}=1.4177,$$

thus $\hat{\mu_1}=2\max\{2L_1^2c_{11}^{+}(c_{11}^{+}+d_{11}^{+}),2L_2^2c_{12}^{+}(c_{12}^{+}+d_{12}^{+})\}=2\max\{0.3471,0.1802\}=0.6942$, $\hat{\mu_2}=2\max\{0.7275,0.3727\}=1.4550$. Consequently, $\mu_1>\mu_2+\hat{\mu_1}+\hat{\mu_2}=2.3491, \hat{\mu_1}+\mu_2<1$. Hence ($\mathrm{v}_3$) and ($\mathrm{v}_4$) are met. We also check and testify that (v_6) and (v_7) are parameters of satisfaction. The graphs of variables $u_1^R(t),u_2^R(t),v_1(t)^R$ and $v_2^R(t)$ are presented in Figure 1, The graphs of variables $u_1^I(t),u_2^I(t),v_1(t)^I$ and $v_2^I(t)$ are presented in Figure 2, The graphs of variables $u_1^R(t),u_2^R(t),v_1(t)^R$ and $v_2^R(t)$ are presented in Figure 3. The graphs of variables $u_1^R(t),u_2^R(t),v_1(t)^R$ and $v_2^R(t)$ are presented in Figure 4,

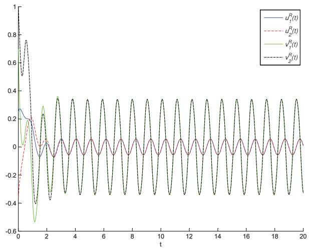

Figure 1 Curves of the $u_1^R(t)$, $u_2^R(t)$, $v_1^R(t)$, $v_2^R(t)$

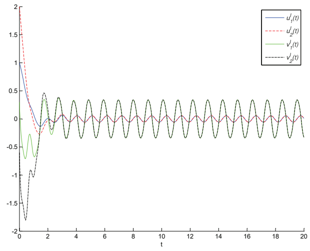

Figure 2 Curves of the $u_1^I(t)$, $u_2^I(t)$, $v_1^I(t)$, $v_2^I(t)$

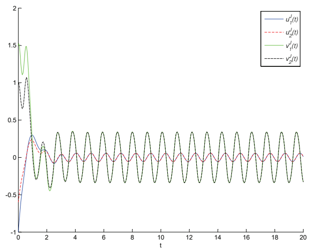

Figure 3 Curves of the $u_1^J(t)$, $u_2^J(t)$, $v_1^J(t)$, $v_2^J(t)$

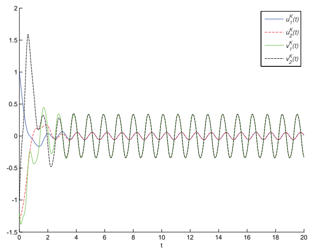

Figure 4 Curves of the $u_1^K(t)$, $u_2^K(t)$, $v_1^K(t)$, $v_2^K(t)$

6. Conclusion

Without resolving the Quaternion-valued inertial neural networks into eight real-valued neural networks, in the light of combining a continuation theorem of coincidence degree theory with two newfangled Lyapunov functions, the existence of anti-periodic solutions for above Quaternion-valued neural networks is acquired. By constructing a Lyapunov functional, the adequate criterion to ensure the asymptotic stability of periodic solutions of above networks is acquired.

Acknowledgements We are thankful to the reviewers for their constructive comments which help us to improve the manuscript.

References

[1] Xujun YANG, Chuandong LI, Qiankun SONG, et al. *Global mittag-leffler stability and synchronization analysis of fractional-order quaternion-valued neural networks with linear threshold neurons*. Neural Netw., 2018, **105**: 88–103.

[2] Yongkun LI, Jiali QIN, Bing LI. *Periodic solutions for quaternion-valued fuzzy cellular neural networks with time-varying delays*. Adv. Difference Equ., 2019, Paper No. 63, 23 pp.

[3] Jingwen ZHU, Jitao SUN. *Stability of quaternion-valued neural networks with mixed delay*. Neural Processing Lett., 2019, **49**(2): 819–833.

[4] Yongkun LI, Jiali QIN, Bing LI. *Anti-periodic solutions for quaternion-valued high-order Hopfield neural networks with time-varying delays*. Neural Processing Lett., 2019, **49**: 1217–1237.

[5] Yongkun LI, Jiali QIN. *Existence and global exponential stability of periodic solutions for quaternion-valued cellular neural networks with time-varying delays*. Neurocomputing, 2018, **292**: 91–103.

[6] Xingxing YOU, Qiankun SONG, Jing LIANG, et al. *Global μ-stability of quaternion-valued neural networks with mixed time-varying delays*. Neurocomputing, 2018, **290**: 12–25.

[7] Xiaofeng CHEN, Qiankun SONG, Zhongshan LI, et al. *Stability analysis of continuous-time and discrete-time quaternion-valued neural networks with linear threshold neurons*. IEEE Trans. Neural Netw. Learn. Syst., 2018, **29**(7): 2769–2781.

[8] S. ABBAS, Yonghui XIA. *Existence and attractivity of k-almost automorphic sequence solution of a model of cellular neural networks with delay*. Acta Math. Sci. Ser. B (Engl. Ed.), 2013, **33**(1): 290–302.

[9] S. TYAGI, S. ABBAS, M. HAFAYED. *Global Mittag-Leffler stability of complex valued fractional-order neural network with discrete and distributed delays*. Rend. Circ. Mat. Palermo (2), 2016, **65**(3): 485–505.

[10] S. ABBAS, Yonghui XIA. *Almost automorphic solutions of impulsive cellular neural networks with piecewise constant argument*. Neural Process Lett., 2015, **42**: 691–702.

[11] S. ABBAS. *Existence and attractivity of k-pseudo almost automorphic sequence solution of a model of bidirectional neural networks*. Acta Appl. Math., 2012, **119**: 57–74.

[12] H. M. SRIVASTAVA, S. ABBAS, S. TYAGI, et al. *Global exponential stability of fractional-order impulsive neural network with time-varying and distributed delay*. Math. Methods Appl. Sci., 2018, **41**(5): 2095–2104.

[13] Qianhong ZHANG, Fubiao LIN, Mingjun HU. *Stability and existence of anti-periodic solution for FCNNs with time-varying delays and impulsive impacts*. IEEE Access, 2019, **7**: 21734–21743.

[14] Yongkun LI, Jiali QIN, Bing LI. *Existence and global exponential stability of anti-periodic solutions for delayed quaternion-valued cellular neural networks with impulsive effects*. Math. Methods Appl. Sci., 2019, **42**(1): 5–23.

[15] Yongkun LI, Jianglian XIANG. *Existence and global exponential stability of anti-periodic solution for Clifford-valued inertial Cohen-Grossberg neural networks with delays*. Neurocomputing, 2019, **332**: 259–269.

[16] R. KRISHNASAMY, R. K. GEORGE. *Stochastic stability of mode-dependent Markovian jump inertial neural networks*. J. Anal., 2019, **27**(1): 179–196.

[17] Leimin WANG, Mingfeng GE, Junhao HU, et al. *Global stability and stabilization for inertial memristive neural networks with unbounded distributed delays*. Nonlinear Dynamics, 2019, **95**(2): 943–955.

[18] Deqiang OUYANG, Jie SHAO, Cheng HU. *Stability property of impulsive inertial neural networks with unbounded time delay and saturating actuators*. Neural Computing and Applications, 2020, **32**: 6571–6580.

[19] Nina HUO, Bing LI, Yongkun LI. *Existence and exponential stability of anti-periodic solutions for inertial Quaternion-valued high-order Hopfield neural networks with state-dependent delays*. IEEE ACCESS, 2019, **7**: 60010–60019.

[20] Zhengqiu ZHANG, Jinde CAO. *Periodic solutions for complex-valued neural networks of neutral type by combining graph theory with coincidence degree theory*. Adv. Difference Equ. 2018, Paper No. 261, 23 pp.

[21] Zhengqiu ZHANG, Ting ZHENG. *Global asymptotic stability of periodic solutions for delayed complex-valued Cohen-Grossberg by combining coincidence degree theory with LMI method*. Neurocomputing, 2018, **289**: 220–230.

[22] Zhengqiu ZHANG, Feng LIN. *Global asymptotic stability of periodic solutions for neutral-type delayed BAM neural networks by combining an abstract theorem of k-set contractive operator with LMI method*. Neural Processing Letters, 2019, **50**: 1571–1588.

[23] P. AMSTER. *Topological Methods in the Study of Boundary Valued Problems*. Springer, New York, 2014.