Unicyclic Graphs with Five Laplacian Eigenvalues Different from 0 and 1

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Abstract Let $U$ be a unicyclic graph of order $n$, and $m_U(1)$ the multiplicity of Laplacian eigenvalue 1 of $U$. It is well-known that 0 is a simple Laplacian eigenvalue of connected graph. This means that if $U$ has five Laplacian eigenvalues different from 0 and 1, then $m_U(1) = n - 6$. In this paper, we completely characterize all the unicyclic graphs with $m_U(1) = n - 6$.

Keywords unicyclic graph; Laplacian eigenvalue; multiplicity

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1. Introduction

Throughout this paper we consider finite undirected simple graphs of order $n$. Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. Let $A(G)$ be the adjacency matrix of $G$. We denote by $d(v_i)$ the degree of $v_i$ in $G$. Let $D(G)$ be the diagonal matrix of the degrees of $G$. The Laplacian matrix of $G$ is defined as $L(G) = D(G) - A(G)$. Clearly, $L(G)$ is a real symmetric, positive semidefinite matrix. It is not difficult to find that the row sum of $L(G)$ is 0, and so the smallest eigenvalue is equivalent to 0. For convenience, we always assume that the Laplacian eigenvalues of $G$ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. In [1], it is well known that $\mu_{n-1} > 0$ if and only if $G$ is connected and hence is called algebraic connectivity of $G$. The multiplicity of $\mu_i$ is denoted by $m_G(\mu)$, and the number of Laplacian eigenvalues in an interval $I$ is denoted by $m_G(I)$. The Laplacian spectrum of $G$ is a multiple set of Laplacian eigenvalues together with their multiplicities. We denote Spec$_L (G) = \{\mu_1^{k_1}, \mu_2^{k_2}, \ldots, \mu_r^{k_r}\}$ where $\mu_1, \mu_2, \ldots, \mu_{r-1}$ and $\mu_r$ are $r$ distinct Laplacian eigenvalues and $m_G(\mu_i) = k_i$ is the multiplicity of $\mu_i$ ($1 \leq i \leq r$) and $\sum_{i=1}^{r} k_i = n$.

For a graph $G$ of order $n$, a vertex of degree one is called a pendant vertex, and we write $p(G)$ for the number of pendant vertices of $G$. A vertex of $G$ is quasipendant vertex if it is adjacent to a pendant vertex, and we write $q(G)$ for the number of quasipendant vertices of $G$. Let $r(G)$ be the number of inner vertices of $G$. Let

$$V_P = \{v \in V(G) \mid v \text{ is a pendent vertex}\},$$
\[ V_Q = \{ v \in V(G) \mid v \text{ is a quasipendant vertex} \} \]

and

\[ V_R = V(G) \setminus (V_P \cup V_Q). \]

Then \( V_R \) is the set of the inner vertices of \( G \) which are not pendent vertices and quasipendant vertices. Obviously, \( |V_P| = p(G), \ |V_Q| = q(G) \) and \( |V_R| = r(G) = n - p(G) - q(G) \). Let \( L_R(G) \) be the principal submatrix of \( L(G) - E_n \) that corresponds to the inner vertices of \( G \), where \( E_n \) is an identity matrix of order \( n \). The nullity of \( L_R(G) \) denoted by \( \nu(L_R(G)) \), and \( C_n (n \geq 3) \) always represents the cycle. The diameter of \( G \), denoted by \( \text{diam}(G) \), is the maximum distance between any two vertices of \( G \). Meanwhile, the girth of \( G \), denoted by \( g \), is the length of the shortest cycle in \( G \). A unicyclic graph is a connected graph with the same number of edges and vertices. We denote by \( \mathcal{U}(n, g) \) the set of all connected unicyclic graphs with girth \( g (g \geq 3) \) on \( n \) vertices. For graph theoretic notations and terminologies not defined here, we refer the readers to [2].

In the past two decades, connected graphs with few distinct eigenvalues have been investigated for several graph matrices since such graphs always have pretty combinatorial properties. This problem was perhaps first raised by Doob [3]. Since then, a lot of publications (see [4–10]) have focused on graphs with fewer eigenvalues. In fact, a graph with fewer eigenvalues means that it has large multiplicity on some eigenvalues. Thus, characterizing graphs with largest multiplicity of eigenvalues is important for graphs with few distinct eigenvalues.

The multiplicity of Laplacian eigenvalue of graphs has attracted plenty of attention. Faria [11] proved that \( m_G(1) \) is bounded by \( p(G) - q(G) \), that is, \( m_G(1) \geq p(G) - q(G) \), which is also called Faria’s inequality. Then Andrade et al. [12] presented a unified approach on the Faria’s inequality for the Laplacian and signless Laplacian spectra. In [13], it was shown that for a tree \( T \) with order \( n \), if an integer \( \lambda > 1 \) is a Laplacian eigenvalue of \( T \), then \( m_T(\lambda) = 1 \) and \( \lambda \) divides \( n \). Additionally, Guo, Feng and Zhang [14] characterized all trees with \( n-6 \leq m_T(1) \leq n \). Also Barik, Lal and Pati [15] investigated the multiplicities of Laplacian eigenvalue 1 of a graph.

Base on above, we consider \( m_U(1) = n - 6 \) for a unicyclic graph \( U \) on \( n \geq 7 \) vertices in this paper, i.e., the unicyclic graph has five Laplacian eigenvalues different from 0 and 1 since 0 is a simple Laplacian eigenvalue of a connected graph, and obtain the following result:

**Theorem 1.1** Let \( U \) be a unicyclic graph on \( n \geq 7 \) vertices. Then \( m_U(1) = n - 6 \) if and only if \( U \) is isomorphic to one of \( H_1^1(a,b,c) \) \((a \geq 1, b \geq 1, c \geq 1)\), \( H_2^2(a,c) \) \((a \geq 0, c \geq 1)\), \( H_3^3(a,c) \) \((a \geq 1, c \geq 1)\), \( H_4^3(a,b) \) \((a \geq 0, b \geq 1)\), \( H_5^3(a,b) \) \((a \geq 1, b \geq 1)\), \( H_6^3(a,b) \) \((a \geq 1, b \geq 1)\), \( H_7^3(a,c) \) \((a \geq 0, c \geq 0)\), and \( H_8^3(a,b) \) \((a \geq 0, b \geq 0)\). All of these graphs are shown in Figure 1.

Moreover, we also present some unicyclic graphs with \( m_U(1) = n - 7 \) (see Lemma 3.1).

2. Preliminaries

In this section, we introduce some lemmas which will be useful for the proof of main results.
Lemma 2.1 ([2]) If $e$ is an edge of the graph $G$ and $G' = G - e$, then
\[
\mu_1(G) \geq \mu_1(G') \geq \cdots \geq \mu_{n-1}(G) \geq \mu_{n-1}(G') \geq \mu_n(G) = \mu_n(G') = 0.
\]

Lemma 2.2 ([13]) Let $v$ be a pendant vertex of $\tilde{G}$ and let $G = \tilde{G} \setminus v$. Then the Laplacian eigenvalues of $G$ interlace the Laplacian eigenvalues of $\tilde{G}$.

Lemma 2.3 ([13]) Let $G$ be a connected graph with $p$ pendant vertices and $q$ quasipendant vertices. Then $m_G(1) = p - q + v(L_R(G))$.

Lemma 2.4 ([16]) Let $C_n$ be a cycle of order $n$. Then $\text{Spec}_L(C_n) = \{2 - 2\cos \frac{2\pi j}{n} | j = 0, 1, \ldots, n - 1\}$.

Lemma 2.5 ([17]) If $G$ is a connected graph with a cutpoint $v$, then $\mu_{n-1}(G) \leq 1$, where equality holds if and only if $v$ is adjacent to every vertex of $G$.

Let $G_{u:vH}$ be the graph obtained from $G$ and $H$ by joining a vertex $u$ of $G$ to a vertex $v$ of $H$. In particular, if $H = P_2(= uv)$, we denote by $G_{u:vw}$ for short.

Lemma 2.6 ([18]) Let $H$ be a graph, and $S_k$ a star on $k \geq 3$ vertices. Set $G = H_{u:vS_k}$.

1. If $v$ is a pendant vertex of $S_k$, then $m_G(1) = m_H(1) + k - 3$;
2. If $v$ is the center of $S_k$, then $m_G(1) = m_{H_{u:vw}}(1) + k - 2$.

Lemma 2.7 ([12]) Let $G$ be a connected graph on $n$ vertices. If $r(G) = 0$, that is, any internal vertex of $G$ is also a quasi- pendant vertex, then $m_G(1) = p(G) - q(G)$.

Figure 1 Graphs $H_1 \sim H_6$
Lemma 2.8 Let $D_n$ be the following determinant of order $n$.
\[
D_n = \begin{vmatrix}
1 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 1 & -1 & \cdots & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 \\
0 & 0 & -1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -1 \\
0 & 0 & 0 & \cdots & -1 & 1
\end{vmatrix}_{n \times n}
\]
Then
\[
\det D_n = \begin{cases}
(-1)^{\frac{n}{3}}, & \text{if } n \equiv 0 \pmod{3}; \\
(-1)^{\frac{n+1}{3}}, & \text{if } n \equiv 1 \pmod{3}; \\
0, & \text{otherwise}.
\end{cases}
\]
Proof Applying Laplacian Expansion Theorem in the first column of $D_n$ yields
\[
D_n = \begin{vmatrix}
1 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 1 & -1 & \cdots & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 \\
0 & 0 & -1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -1 \\
0 & 0 & 0 & \cdots & -1 & 1
\end{vmatrix}_{(n-1) \times (n-1)} + \begin{vmatrix}
-1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 1 & -1 & \cdots & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 \\
0 & 0 & -1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -1 \\
0 & 0 & 0 & \cdots & -1 & 1
\end{vmatrix}_{(n-1) \times (n-1)}
\]
\[
= D_{n-1} - D_{n-2} = -D_{n-3}
\]
From the recurrence relation above we get
\[
D_n = (-1)^{i}D_{n-3i}.
\tag{2.1}
\]
For a fixed $n$, we by induction on $i$ prove that the (2.1) holds. When $i = 1$, $D_n = (-1)^{1}D_{n-3}$. Clearly, (2.1) is true. Assume that the holds for $i < k$. If $i = k$, then
\[
D_n = (-1)^{k-1}D_{n-3(k-1)} = (-1)^{k-1}(-1)D_{n-3k} = (-1)^kD_{n-3k}.
\]
If $n \equiv 0 \pmod{3}$, then there exists a $k$, such that $n = 3k$, $D_n = D_{3k} = (-1)^k = (-1)^{\frac{n}{3}}$; Similarly, if $n \equiv 1 \pmod{3}$, it draws $D_n = (-1)^{\frac{n+1}{3}}$; otherwise, $D_n = 0$. The proof is completed. \qed

Lemma 2.9 Let $M_n^m$ be the following determinant and $m \in R$.
\[
M_n^m = \begin{vmatrix}
1 & -1 & 0 & \cdots & 0 & -1 \\
-1 & 1 & -1 & \cdots & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 \\
0 & 0 & -1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -1 \\
-1 & 0 & 0 & \cdots & -1 & m
\end{vmatrix}_{n \times n}
\]
Then

\[ M'_n = \begin{cases} 
-2((-1)^{\frac{n-1}{3}} + 1) & \text{if } n \equiv 0 \Mod 3; \\
-2((-1)^{\frac{n-2}{3}} - 2) & \text{if } n \equiv 1 \Mod 3; \\
(m - 2) \cdot (-1)^{\frac{n-1}{3}} - 2 & \text{ otherwise.}
\end{cases} \]

**Proof** Applying Laplacian Expansion Theorem in the last column of \( M'_n \) yields

\[
M'_n = mD_{n-1} + (-1)^{n+1}.
\]

\[
= mD_{n-1} + (-1)^{n+1}.
\]

\[
= mD_{n-1} - 2D_{n-2} - 2
\]

If \( n \equiv 1 \Mod 3 \), then by Lemma 2.7 we get \( D_{n-1} = (-1)^{\frac{n-1}{3}} \) and \( D_{n-2} = 0 \). Thus, \( M'_n = (-1)^{\frac{n-1}{3}} m - 2 \). Similarly, if \( n \equiv 2 \Mod 3 \), then \( D_{n-1} = D_{n-2} = (-1)^{\frac{n-2}{3}} \), it therefore follows \( M'_n = (-1)^{\frac{n-2}{3}} (m - 2) - 2 \); if \( n \equiv 0 \Mod 3 \), then \( D_{n-1} = 0 \) and \( D_{n-2} = (-1)^{\frac{n-1}{3}} \). Hence \( M'_n = -2((-1)^{\frac{n-1}{3}} + 1) \). \( \square \)

**Lemma 2.10** Let \( U \) be a unicyclic graph on \( n \geq 4 \) vertices. If \( U \cong S_n^3 \) (see Figure 2), then

\[
m_U(1) = n - 3.
\]

![Figure 2 Graph \( S_n^3 \)](image)

**Proof** If \( U \cong S_n^3 \), one can get \( p(S_n^3) = n - 3 \) and \( q(S_n^3) = 1 \). Let \( L_R(S_n^3) \) be the principal
submatrix of $L(S_n^3) - E_n$ that corresponds to the inner vertices of $S_n^3$, then
\[
\det(L_R(S_n^3)) = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = D_2.
\]

It follows from Lemma 2.8 that $\det(L_R(S_n^3)) = 0$, which implies $\nu(L_R(S_n^3)) = 1$. Hence, from Lemma 2.3 we get $m_{S_n^3}(1) = n - 3 - 1 + 1 = n - 3$. □

**Lemma 2.11** Let $U$ be a unicyclic graph on $n \geq 7$ vertices. If $U$ is isomorphic to one of those graphs $C_1^3(a, b)$ $(a \geq 1, b \geq 1)$, $C_2^3(a, b)$ $(a \geq 0, b \geq 1)$ and $C_4(a, b)$ $(a \geq 0, b \geq 0)$, shown in Figure 3, then $m_U(1) = n - 5$.

![Graphs](image.png)

**Proof** If $U$ has the form $C_1^3(a, b)$ on $n = a + b + 3$ vertices, where $a \geq 1, b \geq 1$. Clearly, $C_1^3(a, b)$ has only one inner vertex with degree 2, which implies $\det(L_R(C_1^3(a, b))) = 1$, and so $\nu(L_R(C_1^3(a, b))) = 0$. Therefore, it follows from Lemma 2.3 that $m_{C_1^3(a,b)}(1) = n - 3 - 2 = n - 5$.

If $U$ has the form $C_2^3(a, b)$ on $n = a + b + 4$ vertices, where $a \geq 1, b \geq 1$, then $D_2 = 0$, it leads to $\nu(L_R(C_2^3(a, b))) = 1$. By Lemma 2.3, we have $m_{C_2^3(a,b)}(1) = n - 4 - 2 + 1 = n - 5$. Moreover, $a = 0$ implies $U \cong C_2^3(0, b)$ $(b \geq 1)$ with $n = b + 4$ vertices. And then, we obtain

\[
\det(L_R(C_2^3(0, b))) = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = M_2^3.
\]

Thus, it follows from Lemma 2.9 that $\det(L_R(C_2^3(0, b))) = -4$, which indicates $\nu(L_R(C_2^3(0, b))) = 0$. Therefore, by Lemma 2.3 we have $m_{C_2^3(0,b)}(1) = n - 4 - 1 = n - 5$. By the similar method as above, one can obtain $m_{C_4(a,b)}(1) = n - 5$ where $a \geq 0, b \geq 0$.

Sum up the above, we complete the proof. □

**3. Proof of the main result**

Before proving Theorem 1.1, we give some useful lemmas which needs to be used in the following.

**Lemma 3.1** Let $U$ be a unicyclic graph on $n \geq 7$ vertices, if $U$ is one of $U_2^3(a,b,c)$ $(a \geq 0, b \geq 1, c \geq 1)$, $U_3^3(a,b,c)$ $(a \geq 1, b \geq 1, c \geq 1)$, $U_3^3(a,b)$ $(a \geq 1, b \geq 1)$, $U_3^3(a,b,0)$ $(a \geq 1, b \geq 1, d \geq 1)$, $U_2^3(a,b,c,0)$ $(a \geq 1, b \geq 1, c \geq 1, d \geq 0)$, $U_3^3(a,b)$ $(a \geq 1, b \geq 1)$, $U_2^3(a,b,c)$ $(a \geq 1, b \geq 0, c \geq 1)$ and $U_1^3(a,b,c)$ $(a \geq 1, b \geq 1, c \geq 0)$, then $m_U(1) = n - 7$ (see
\begin{proof}
Let $U \in \mathcal{U}(n, g)$ be a unicyclic graph and let $L_R(U)$ be the principal submatrix of $L(U) - E_n$ corresponding to inner vertices of $U$.

If $U \cong U^3_2(a, b, c)$ ($a \geq 1, b \geq 1, c \geq 1$), then $n = a + b + c + 5$ and
\[
\det(L_R(U^3_2(a, b, c))) = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = D_2
\]
It follows from Lemma 2.8 that $\det(L_R(U^3_2(a, b, c))) = 0$, which implies $\nu(L_R(U^3_2(a, b, c))) = 1$.

From Lemma 2.3 we have $m_{U^3_2(a, b, c)}(1) = a + b + c - 3 + \nu(L_R(U^3_2(a, b, c))) = n - 7$. In addition, $a = 0$ implies $U \cong U^3_2(0, b, c)$ ($b \geq 1, c \geq 1$) with $n = b + c + 5$, then
\[
\det(L_R(U^3_2(0, b, c))) = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 2 \end{vmatrix} = M_3^2.
\]
It is deduced from Lemma 2.9 that $\det(L_R(U^3_2(0, b, c))) = -4$, which means $\nu(L_R(U^3_2(0, b, c))) = 0$. By Lemma 2.3 we have $m_{U^3_2(0, b, c)}(1) = b + c - 2 = n - 7$. Thus $m_{U^3_2(a, b, c)}(1) = n - 7$ where $a \geq 0, b \geq 1, c \geq 1$. With the same argument, we can obtain $m_{U^3_2(a, b)}(1) = n - 7$ where $a \geq 1, b \geq 1$.

If $U \cong U^3_2(a, b, c)$ ($a \geq 1, b \geq 1, c \geq 1$), then $n = a + b + c + 4$. It is easy to see that the inner vertex of $U^3_2(a, b, c)$ is the unique vertex with degree 2, and so, $\nu(L_R(U^3_2(a, b, c))) = 0$. Thus,

\begin{figure}
\centering
\includegraphics[width=\textwidth]{some_related_graphs}
\caption{Some related graphs}
\end{figure}
Proof. Assume that $G$ contains $P_8$ as its induced subgraph. Then by direct calculation we get

$$\text{Spec}_L(P_8) = \{0, 0.1522, 0.5858, 1.2346, 2, 2.7654, 3.4142, 3.8478\}. \quad (3.1)$$

It is not difficult to find that $G_0 = P_8 \cup (n-8)K_1$ is a spanning subgraph of $U$, and $U$ can be obtained from $G_0$ by adding $n-7$ edges: $e_1, e_2, \ldots, e_{n-7}$ (say). Let $U_i = G_0 + \{e_1, \ldots, e_i\}$ be a spanning subgraph of $U$ for $i = 1, 2, \ldots, n-7$. Obviously, $U \cong U_{n-7}$. Then we apply Lemma 2.1 repeatedly to get

$$\mu_j(U) \geq \mu_j(U_{n-8}) \geq \mu_j(U_{n-9}) \geq \cdots \geq \mu_j(U_1) \geq \mu_j(P_k), \quad (3.2)$$

for $j = 1, 2, \ldots, 8$.

By (3.1) we know that $m_{P_8}(1, n) = 5$, it therefore follows from (3.2) that $m_U(1, n) \geq 5$. In addition, by Lemma 2.5 we get $\mu_{n-1}(U) < 1$, and together with $m_U(0) = 1$ we have $m_U(1, n) \geq 2$. Thus, $m_U(1) = n - m_U[0, 1] - m_U(1, n) \leq n - 7$.
which contradicts \( m_U(1) = n - 6 \), and so we complete the proof. □

By Lemma 2.4 we can obtain that \( m_{C_n}(1) \leq 2 \), and if 1 is an eigenvalue of \( C_n \), it implies that \( 2 - 2 \cos \frac{2\pi j}{n} = 1 \) for some \( j \). Then we can deduce \( \cos \frac{2\pi j}{n} = \frac{1}{2} \), and further get \( j = \frac{n}{6} \) or \( j = \frac{5n}{6} \), i.e., \( n = 6t \) for \( t \geq 1 \). Hence, \( m_{C_n}(1) = 2 \) if and only if \( n = 6t \) for \( t \geq 1 \).

It is worth mentioning that if \( U \) is a cycle, then \( m_U(1) = n - 6 \), then \( n - 6 = 2 \), i.e., \( n = 8 \), but 6 \( \nmid 8 \), a contradiction. Hence, \( m_{C_n}(1) = 2 \) if and only if \( n = 6t \) for \( t \geq 1 \).

Let \( U \in \mathcal{U}(n, g) \) be a unicyclic graph with \( m_U(1) = n - 6 \). If either \( \text{diam}(U) \geq 6 \) or \( g \geq 7 \), then \( U \) must contain \( P_8 \) as its induced subgraph. Thus, from Lemma 3.2 we have the following corollary.

**Corollary 3.3** Let \( U \in \mathcal{U}(n, g) \) be a unicyclic graph on \( n \geq 7 \) vertices. If \( m_U(1) = n - 6 \), then \( \text{diam}(U) \leq 5 \) and \( g \leq 6 \).

**Lemma 3.4** Let \( U \in \mathcal{U}(n, 3) \) be a unicyclic graph on \( n \geq 7 \) vertices. Then \( m_U(1) = n - 6 \) if and only if \( U \) is one of \( H_2^2(a, b, c) \) \( (a \geq 1, b \geq 1, c \geq 1) \), \( H_3^2(a, c) \) \( (a \geq 0, c \geq 1) \), \( H_3^3(a, e) \) \( (a \geq 1, c \geq 1) \), \( H_3^3(a, b) \) \( (a \geq 0, b \geq 1) \) and \( H_3^3(b, a) \) \( (a \geq 1, b \geq 1) \) (see Figure 1).

**Proof** Let \( U \in \mathcal{U}(n, 3) \). Then \( \text{diam}(U) \leq 5 \) by Corollary 3.3. For the sake of clarity, we here discuss it by the diameter of \( U \) below.

Case 1. \( \text{diam}(U) \leq 3 \).

When \( \text{diam}(U) = 2 \), one can find that \( U \cong S_3^n \), it is clearly impossible since \( m_{S_3}(1) = n - 3 \) by Lemma 2.10; when \( \text{diam}(U) = 3 \), \( U \) has one of forms \( C_3^2(a, b) \) \( (a \geq 0, b \geq 1) \) and \( H_3^3(a, b, c) \) with \( n = a + b + c + 3 \). Then by Lemma 2.11, we can see that \( m_{C_3^2(a, b)}(1) = n - 5 \), which is a contradiction. Hence, \( U \) has just the form of \( H_2^2(a, b, c) \) (see Figure 1). Clearly, \( a \geq 1, b \geq 1 \) and \( c \geq 1 \) since if one or two of \( a, b, c \) equal(s) zero, then \( U \) has one of the forms \( S_3^n \) and \( C_3^2(a, b) \). From Lemmas 2.10 and 2.11, it is also impossible. We notice that \( H_2^2(a, b, c) \) has no inner vertex. So, it follows from Lemma 2.7 that

\[
m_{H_2^2(a, b, c)}(1) = p(H_2^2(a, b, c)) - q(H_2^2(a, b, c)) = a + b + c - 3 = n - 6.
\]

Case 2. \( \text{diam}(U) = 4 \).

When \( \text{diam}(U) = 4 \), assume that \( U \) contains \( I_1 \) as its induced subgraph (see Figure 5). Then by simple computation we get

\[
\text{Spec}_L(I_1) = \{0, 0.3065, 0.3820, 1.6703, 2.6180, 3, 3.3297, 4.6935\}.
\]

By the same reasoning as \( P_8 \), we get \( m_U[0, 1] \geq 2 \) and \( m_U(1, n] \geq 5 \). It follows

\[
m_U(1) = n - m_U[0, 1] - m_U(1, n] \leq n - 7
\]

it is a contradiction. For the same reasoning as \( I_1 \), one can prove that \( U \) does not contain \( I_2 \) as its induced subgraph from Table 1. Thus, \( U \) has one of forms \( U_2^3(a, b, c) \) \( (a \geq 0, b \geq 0, c \geq 1) \), \( U_3^3(a, b, c) \) \( (a \geq 1, b \geq 0, c \geq 1) \) and \( U_3^3(a, b, c, d) \) \( (a \geq 1, b \geq 1, c \geq 0, d \geq 1) \). According to Lemma 3.1, one can easily get \( m_{U_2^3(a, b, c)}(1) = n - 7 \) with \( a \geq 0, b \geq 1 \) and \( c \geq 1 \), \( m_{U_3^3(a, b, c)}(1) = n - 7 \) with \( a \geq 1, b \geq 1 \) and \( c \geq 1 \), and \( m_{U_3^3(a, b, c, d)}(1) = n - 7 \) with \( a \geq 1, b \geq 1 \) and \( d \geq 1 \), which are
all impossible since $m_U(1) = n - 6$. Furthermore, if $U \cong U^5_3(a, b, c, d)$ with $a \geq 1, b \geq 1, c \geq 1$ and $d \geq 1$, then $m_{U^5_3(a, b, c, d)}(1) = n - 8$ by Lemma 2.3, it is also impossible. Consequently, $U$ has one of the forms $U^7_3(a, 0, c)$ with $a \geq 0, c \geq 1$ and $U^3_3(a, 0, c)$ with $a \geq 1, c \geq 1$.

For convenience, we denote by $U^2_3(a, 0, c) = H^2_3(a, c)$ and $U^3_3(a, 0, c) = H^3_3(a, c)$ (see Figure 1) now. For the graph $H^3_3(a, c)$ ($a \geq 1, c \geq 1$), it can be obtained from $S^3_{a+3}$ and $S_{c+2}$ by joining the center of $S^3_{a+3}$ to a pendant vertex of $S_{c+2}$. Thus, it follows from Lemma 2.6 (1) that $m_{H^3_3(a, c)}(1) = m_{S^3_{a+3}}(1) + c + 2 - 3 = n - 6$. Moreover, $a = 0$ implies $U \cong H^2_3(0, c)$, it can be obtained from $C_3$ and $S_{c+2}$ by joining an arbitrary vertex of $C_3$ to a pendant vertex of $S_{c+2}$. It therefore follows from Lemma 2.6 (1) that $m_{H^2_3(0, c)}(1) = c - 1 = n - 6$. For the graph $H^3_3(a, c)$, it is deduced from Lemma 2.3 that $m_{H^3_3(a, c)}(1) = n - 6$.

Case 3. $\text{diam}(U) = 5$.

If $\text{diam}(U) = 5$, by similar reasoning as $I_1$, one can prove that $U$ does not contain $I_3, I_4, I_5, I_6, I_7, I_8$ as its induced subgraphs in terms of Table 1. Therefore, $U$ has one of forms $H^4_3(a, b)$ with $a \geq 0, b \geq 1$, $H^3_3(a, b)$ with $a \geq 1, b \geq 1$ and $U^4_3(a, b)$ with $a \geq 1, b \geq 1$.

If $U \cong U^4_3(a, b)$ ($a \geq 1, b \geq 1$), it follows from Lemma 3.1 that $m_{U^3_3}(1) = n - 7$, which contradicts $m_U(1) = n - 6$. Further, if $U \cong H^3_3(a, b)$ ($a \geq 1, b \geq 1$), then $n = a + b + 6$ and

$$L_R(H^3_3(a, b)) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

By direct calculation, we can get $\nu(L_R(H^3_3(a, b))) = 2$. It follows from Lemma 2.3 that

$m_{H^3_3(a, b)}(1) = a + b - 2 + \nu(L_R(H^3_3(a, b))) = n - 6$.

If $U \cong H^3_3(0, b)$ ($b \geq 1$) with $n = b + 6$, then

$$L_R(H^3_3(0, b)) = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

It follows $\nu(L_R(H^3_3(0, b))) = 1$. By Lemma 2.3 again, we have $m_{H^3_3(0, b)}(1) = n - 6$.

If $U \cong H^3_3(a, b)$ ($a \geq 1, b \geq 1$) with $n = a + b + 5$, then

$$L_R(H^3_3(a, b)) = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Similarly, we can get $\nu(L_R(H^3_3(a, b))) = 1$, which leads to $m_{H^3_3(a, b)}(1) = a + b - 1 = n - 6$ by Lemma 2.3.

Conversely, from Lemma 2.3 the conclusion holds. The proof is completed. □
Unicyclic graphs with five Laplacian eigenvalues different from 0 and 1

Figure 5 Graphs $I_i$ ($1 \leq i \leq 9$)

<table>
<thead>
<tr>
<th>$I_i$</th>
<th>$m$</th>
<th>$\xi$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
<th>$\lambda_4$</th>
<th>$\lambda_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_1$</td>
<td>0</td>
<td>0.3065</td>
<td>0.3820</td>
<td>1.6703</td>
<td>2.6180</td>
<td>3.3297</td>
</tr>
<tr>
<td>$I_2$</td>
<td>0</td>
<td>0.3820</td>
<td>0.4280</td>
<td>1.2285</td>
<td>2.2799</td>
<td>3.8123</td>
</tr>
<tr>
<td>$I_3$</td>
<td>0</td>
<td>0.2243</td>
<td>0.5858</td>
<td>1.4108</td>
<td>2.7237</td>
<td>3.0000</td>
</tr>
<tr>
<td>$I_4$</td>
<td>0</td>
<td>0.2137</td>
<td>0.6177</td>
<td>1.4977</td>
<td>2.3537</td>
<td>3.0000</td>
</tr>
<tr>
<td>$I_5$</td>
<td>0</td>
<td>0.2593</td>
<td>0.7150</td>
<td>1.3232</td>
<td>1.5891</td>
<td>3.1143</td>
</tr>
<tr>
<td>$I_6$</td>
<td>0</td>
<td>0.2434</td>
<td>0.6972</td>
<td>1.1798</td>
<td>2.0000</td>
<td>3.1386</td>
</tr>
<tr>
<td>$I_7$</td>
<td>0</td>
<td>0.2588</td>
<td>0.6436</td>
<td>1.1385</td>
<td>2.1603</td>
<td>3.1943</td>
</tr>
<tr>
<td>$I_8$</td>
<td>0</td>
<td>0.3004</td>
<td>0.4915</td>
<td>1.3204</td>
<td>2.2391</td>
<td>2.8258</td>
</tr>
<tr>
<td>$I_9$</td>
<td>0</td>
<td>0.2955</td>
<td>0.5979</td>
<td>1.1449</td>
<td>2.3295</td>
<td>2.4734</td>
</tr>
</tbody>
</table>

Table 1 The Laplacian spectra of $I_i$ ($1 \leq i \leq 9$)

Lemma 3.5 Let $U \in \mathcal{U}(n, 4)$ be a unicyclic graph on $n \geq 7$ vertices. Then $m_U(1) = n - 6$ if and only if $U \cong H_1^4(a, b)$ ($a \geq 1, b \geq 1$), or $U \cong H_2^4(a, b)$ ($a \geq 0, b \geq 1$) (see Figure 1).

Proof Let $U$ be a unicyclic graph of $\mathcal{U}(n, 4)$ with order $n \geq 7$. Then diam($U$) $\leq 5$ by Corollary 3.3. According to Table 2, we can deduce that $U$ does not contain $I_{10}$, $I_{11}$, $I_{12}$ and $I_{13}$ (see Figure 6) as its induced subgraphs by the same argument as $I_1$. Then $U$ must have one of forms $U_1^4(a, b, c, d)$ ($a \geq 0, b \geq 0, c \geq 0, d \geq 0$), $U_2^4(a, b)$ ($a \geq 1, b \geq 1$), $U_3^4(a, b, c)$ ($a \geq 1, b \geq 0, c \geq 1$) and $H_2^4(a, b)$ ($a \geq 0, b \geq 1$) (see Figures 1 and 4).

When $U$ has the form of $U_3^4(a, b, c, d)$, if $a \geq 1, b \geq 1, c \geq 1$ and $d \geq 1$, then $r(U_3^4(a, b, c, d)) = 0$. It follows from Lemma 2.7 that

$$m_{U_3^4(a, b, c, d)}(1) = p(U_3^4(a, b, c, d)) - q(U_3^4(a, b, c, d)) = a + b + c + d - 4 = n - 8,$$

which contradicts $m_U(1) = n - 6$; if one of $a, b, c, d$ is equal to zero, without loss of generality, we may assume $d = 0$, then $U \cong U_3^4(a, b, c, 0)$ ($a \geq 1, b \geq 1, c \geq 1$). By Lemma 3.1, one can get
m_{U_1(a,b,c,0)}(1) = n - 7, it is also a contradiction. In addition, we see that $U \not\cong C_4(a,b)$ ($a \geq 0, b \geq 0$) due to $m_{C_4(a,b)}(1) = n - 5$ in terms of Lemma 2.11. Thus, $U$ has just the form $H^1_4(a,b)$ ($a \geq 1, b \geq 1$). It follows from Lemma 2.3 that $m_{H^1_4(a,b)} = n - 6$.

When $U$ has the form of $U_2^4(a,b)$ ($a \geq 1, b \geq 1$) or $U_3^4(a,b,c)$ ($a \geq 1, b \geq 0, c \geq 1$), then by Lemma 3.1 we find that it is impossible as $m_{U_2^4(a,b)}(1) = m_{U_3^4(a,b,c)}(1) = n - 7$.

When $U$ has the form of $H^2_4(a,b)$, if $a \geq 1, b \geq 1$, then

$$L_R(H^2_4(a,b)) = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

By direct calculation, one can deduce $\nu(L_R(H^2_4(a,b))) = 1$, and so, it follows from Lemma 2.3 that $m_{H^2_4(a,b)}(1) = a + b - 2 + \nu(L_R(H^2_4(a,b))) = n - 6$; if $a = 0$, then $U \cong H^2_4(0,b)$. Clearly, $H^2_4(0,b)$ can be obtained from $C_4$ and $S_{b+1}$ by joining an arbitrary vertex of $C_4$ to the center of $S_{b+1}$. It therefore follows from Lemma 2.6 (2) that $m_{H^2_4(b)}(1) = m_{C_4 \cup v = v}(1) + b + 1 - 2 = n - 6$.

| $I_{10}$ | 0 | 0.2765 | 1.3323 | 2.0000 | 2.5219 | 3.2920 | 4.5772 |
| $I_{11}$ | 0 | 0.3581 | 0.6918 | 1.2843 | 2.0000 | 2.4091 | 3.8877 | 5.3689 |
| $I_{12}$ | 0 | 0.3636 | 0.5858 | 1.3478 | 2.0000 | 3.2222 | 3.4142 | 5.0664 |
| $I_{13}$ | 0 | 0.3432 | 0.6639 | 1.1805 | 2.2491 | 2.9045 | 3.5994 | 5.0594 |
| $I_{14}$ | 0 | 0.3820 | 1.3820 | 1.5858 | 2.6180 | 3.6180 | 4.4142 |
| $I_{15}$ | 0 | 0.4215 | 0.6228 | 1.3204 | 1.7261 | 2.8258 | 4.3623 | 4.6511 |
| $I_{16}$ | 0 | 0.4679 | 0.7369 | 1.4843 | 1.6527 | 3.1826 | 3.8794 | 4.5962 |
Conversely, it follows by the discussion above. The proof is completed. □

Lemma 3.6 Let \( U \in \mathcal{U}(n, 5) \) be a unicyclic graph on \( n \geq 7 \) vertices. Then \( m_U(1) = n - 6 \) if and only if \( U \cong H^5_1(a, c) \ (a \geq 0, c \geq 0) \) (see Figure 1).

Proof Let \( U \) be a graph of \( \mathcal{U}(n, 5) \) with order \( n \geq 7 \). Then by Corollary 3.3 \( \text{diam}(U) \leq 5 \). Using the same argument as \( I_1 \), we can obtain that \( U \) does not contain \( I_{14} \) and \( I_{15} \) as its induced subgraphs from Table 2. Therefore, \( U \) has the form of \( U^5_5(a, b, c) \). By Lemma 3.1, one can obtain that \( m_{U^5_5(a,b,c)}(1) = n - 7 \) if \( a \geq 1, b \geq 1, c \geq 0 \). Thus, \( U \) should only take the form of \( H^5_1(a, c) \ (a \geq 0, c \geq 0) \). According to the symmetry, we may assume that \( a = 0 \) and \( c \geq 1 \), then \( U \cong H^5_1(0, c) \) with \( n = c + 5 \), and so

\[
\det(L_R(H^5_1(0, c))) = \begin{vmatrix}
1 & -1 & 0 & 0 \\
-1 & 1 & -1 & 0 \\
0 & -1 & 1 & -1 \\
0 & 0 & -1 & 1 \\
\end{vmatrix} = D_4
\]

In light of Lemma 2.8, we have \( \det(L_R(H^5_1(0, c))) = -1 \), which implies \( \nu(L_R(H^5_1(0, c))) = 0 \). Therefore, \( m_{H^5_1(0, c)}(1) = c - 1 + \nu(L_R(H^5_1(0, c))) = n - 6 \). If \( a \geq 1 \) and \( c \geq 1 \), adopting the same way as above, we obtain \( m_{H^5_1(a,c)}(1) = a + c - 2 + \nu(L_R(H^5_1(a,c))) = n - 6 \).

Conversely, it follows by Lemma 2.3. The proof is completed. □

Lemma 3.7 Let \( U \in \mathcal{U}(n, 6) \) be a unicyclic graph on \( n \geq 7 \) vertices. Then \( m_U(1) = n - 6 \) if and only if \( U \cong H^6_1(a, b) \ (a \geq 0, b \geq 0) \) (see Figure 1).

Proof Let \( U \) be a unicyclic graph of \( \mathcal{U}(n, 6) \) with order \( n \geq 7 \). Then \( \text{diam}(U) \leq 5 \) by Corollary 3.3. It can be shown in the same way as \( I_1 \), \( U \) does not contain \( I_{16} \) as its induced subgraph by Table 2. Besides, from Lemma 3.2, \( U \) cannot include \( I_{17} \) as its induced subgraph either because \( P_5 \) is a subgraph of \( I_{17} \). Therefore, \( U \) can only take the form of \( H^6_1(a, b) \). If \( a = 0 \) or \( b = 0 \), without loss of generality, we assume \( a = 0 \), then \( U \cong H^6_1(0, b) \) with \( n = b + 6 \), which can be obtained from \( C_6 \) by attaching pendant vertices at any vertex of \( C_6 \), and

\[
\det(L_R(H^6_1(0, b))) = \begin{vmatrix}
1 & -1 & 0 & 0 & 0 \\
-1 & 1 & -1 & 0 & 0 \\
0 & -1 & 1 & -1 & 0 \\
0 & 0 & -1 & 1 & -1 \\
0 & 0 & 0 & -1 & 1 \\
\end{vmatrix} = D_5
\]

It follows from Lemma 2.8 that \( \det(L_R(H^6_1(0, b))) = 0 \), which implies \( \nu(L_R(H^6_1(0, b))) = 1 \), and so \( m_{H^6_1(0, b)}(1) = n - 6 \); if \( a \geq 1 \) and \( b \geq 1 \), then \( U \cong H^6_1(a, b) \) with \( n = a + b + 6 \). It also follows from Lemma 2.3 that

\[
m_{H^6_1(a,b)}(1) = a + b - 2 + \nu(L_R(U^5_1(a,b))) = n - 6.
\]

Conversely, it is obvious by the discussion above. The proof is completed. □
Proof of Theorem 1.1 Let $U \in \mathcal{U}(n, g)$ be a unicyclic graph on $n \geq 7$ vertices. If $U$ satisfies $g \geq 7$, then $U$ must contain $P_8$ as its subgraph due to $U \not\cong C_n$, which obviously contradicts Lemma 3.2. So we have $3 \leq g \leq 6$. Furthermore, together with Lemmas 3.4–3.7, the proof therefore follows. □

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