Journal of Mathematical Research with Applications
Jan, 2022, Vol. 42, No. 1, pp. 57–72
DOI:10.3770/j.issn:2095-2651.2022.01.006
Http://jmre.dlut.edu.cn

A Differential-Difference Hierarchy Related to the Toda Lattice and Its Inverse Scattering Transformation

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Abstract By combined power evolution laws of the spectral parameter and the initial constants of integration, a new differential-difference hierarchy is presented from the Toda spectral problem. The hierarchy contains the classic Toda lattice equation, the nonisospectral Toda lattice equation and the mixed Toda lattice equation as reduced cases. The evolution of the scattering data in the inverse scattering transform is analyzed in detail and exact soliton solutions are computed through the corresponding inverse scattering transform.

Keywords Toda lattice; inverse scattering transformation; soliton solution

MR(2020) Subject Classification 37K40; 37K10

1. Introduction

Integrability is one of the most important characters of soliton equations. For decades, many methods have been developed to research the integrability of soliton equations. First of all, a soliton equation is called integrable, if it has a Lax pair [1]. Based on Lax pairs, many other integrable properties can be discussed such as infinitely many conservation laws [2–4], bi-Hamiltonian structures [5–7], nonlinearizations of Lax pairs [8, 9], Darboux and Bäcklund transformation [10], Lie algebras structures [11, 12], and so on. However, the Hirota bilinear method is the other powerful tool to study soliton equations which does not depend on Lax pairs [13, 14]. It can not only solve soliton equations, but also nonlinear systems of ordinary differential equations [15–17]. In general, a soliton equation is considered to be integrable if it can be transformed into bilinear derivative forms and owns N-soliton solutions ($N \geq 3$). It is interesting that many other solutions can also be gotten by the bilinear method such as lump and kink solutions of high dimensional soliton equations [18–20].

The inverse transformation (IST) [21] is a useful method to study soliton equations [22, 23]. Over the years, research has shown that all the methods mentioned above have relations with the IST. Sometimes a soliton equation is considered to be integrable if it can be solved through the IST. The IST can solve not only the classic soliton equations but also soliton equations with self-consistent sources [24], nonisospectral soliton equations [25–28] and nonisospectral soliton

Received November 13, 2020; Accepted January 28, 2021
Supported by the National Natural Science Foundation of China (Grant Nos. 11671177; 11771186) and Postgraduate Research & Practice Innovation Program of Jiangsu Province (Grant No. KYCX21_2566).
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equations with self-consistent sources [29]. More recently, there have been a lot of popular works on multi-component local and nonlocal soliton equations by the inverse scattering transformation [30–32]. One advantage of this powerful tool is that it can be applied to a whole hierarchy of evolution equations related to a certain spectral problem [27]. It is interesting that the exponential matrix solutions can also be obtained by this method [33–35].

In general, there are two sets of evolution equations, called the isospectral hierarchy and the nonisospectral hierarchy, respectively, generated from the same spectral problem [36–38]. Isospectral equations often describe the solitary waves in the lossless and uniform media, while the nonisospectral equations, the solitary waves in a certain type of nonuniform media [26–29,39]. Nonisospectral soliton equations presents τ-symmetries [12], and we often take $\lambda_t = \lambda^k, k \in \mathbb{N}$ as we construct continuous nonisospectral soliton equations. But this is not the case of the Toda lattice equation.

In 2011, Zhang et al. considered the following relation

$$\lambda_t = \begin{cases} \mu(\lambda^{k+1} - 2^{k+1}), & k \text{ is a positive odd number,} \\ \mu\lambda(\lambda^k - 2^k), & k \text{ is a postive even number,} \end{cases}$$

where λ_t is the derivative of λ with respect to t and μ is a constant. By these relations, two Toda hierarchies were obtained. They were proved to be integrable through the IST [27]. Is there any other evolution law of the spectral parameter which allows one to construct nonisospectral integrable lattice equations?

In this paper, we would like to consider a new relation between the time t and the spectral parameter

$$\lambda_t = \mu(\lambda^{k+1} - 4\lambda^{k-1}).$$

A new differential-difference hierarchy of mixed isospectral and nonisospectral equations is constructed from the Toda spectral problem with new integration conditions. The scattering data depending on t will be discussed in detail. The IST will be used to solve an initial-value problem for the differential-difference hierarchy in a systematic way.

The paper is organized as follows. In Section 2, we will construct a new differential-difference hierarchy from the Toda spectral problem. In Section 3, we will establish the IST theory for the new hierarchy and construct soliton solutions of the hierarchy through the IST. Some conclusions and remarks will be given in Section 4.

2. A new differential-difference hierarchy

Let $W_s = \{w(t,n) = (w_1, w_2, \ldots, w_s)^T\}$ be an s-dimensional vector field space, where $w_i = w_i(t,n)$, $1 \le i \le s$ are all real functions defined over $\mathbf{R} \times \mathbf{Z}$, and vanish rapidly as $|n| \to \infty$. E is the shift operator defined as $E^k f(n) = f(n+k)$, $k \in Z$ and n is a discrete variable.

Definition 2.1 *Let $f(t,n) = (f_1, f_2, \ldots, f_s)^T$, $g(t,n) = (g_1, g_2, \ldots, g_s)^T \in W_s$ be two vector* *functions. The inner product of them is defined as*

$$\langle f(t,n), g(t,n)\rangle = \sum_{n=-\infty}^{\infty}\sum_{j=1}^{s} f_j(t,n)g_j(t,n).$$

Definition 2.2 *Suppose that $\Psi : W_s \to W_s$ is an operator. Ψ^* is called the conjugate operator of Ψ, if*

$$\langle \Psi f(t,n), g(t,n)\rangle = \langle f(t,n), \Psi^* g(t,n)\rangle.$$

An operator Ψ is called skew-symmetric if

$$\langle \Psi f(t,n), g(t,n)\rangle = -\langle f(t,n), \Psi g(t,n)\rangle,$$

i.e., $\Psi^* = -\Psi$.

After the above preparation, let us introduce a new differential-difference hierarchy. Consider the spectral problem of the Toda lattice [40–42]

$$E\Phi = M\Phi, \quad M = \begin{pmatrix} 0 & 1 \\ -u(t,n) & \lambda - v(t,n) \end{pmatrix}, \quad \Phi = \begin{pmatrix} \phi_1(t,n) \\ \phi_2(t,n) \end{pmatrix}, \tag{2.1a}$$

and the time evolution

$$\Phi_t = N\Phi, \quad N = \begin{pmatrix} A(n) & B(n) \\ C(n) & D(n) \end{pmatrix}, \tag{2.1b}$$

where $u(t,n)$, $v(t,n)$, $A(n)$, $B(n)$, $C(n)$ and $D(n)$ defined over $\mathbf{R} \times \mathbf{Z}$ are smooth functions, λ is a spectral parameter and the subscript t denotes the derivative with respect to t. We assume that $(u(t,n), v(t,n))$ goes to $(1,0)$ rapidly as $|n| \to \infty$ for the physical background of the Toda lattice. The compatibility condition of (2.1) reads

$$M_t = (EN)M - MN, \tag{2.2}$$

which gives

$$u(t,n)B(n+1) + C(n) = 0, \tag{2.3a}$$

$$A(n+1) + (\lambda - v(t,n))B(n+1) - D(n) = 0, \tag{2.3b}$$

$$u(t,n)_t = u(t,n)D(n+1) - u(t,n)A(n) + (\lambda - v(t,n))C(n), \tag{2.3c}$$

$$v(t,n)_t = -C(n+1) - (\lambda - v(t,n))D(n+1) - u(t,n)B(n) + (\lambda - v(t,n))D(n) + \lambda_t. \tag{2.3d}$$

By simple calculation, we obtain

$$\begin{pmatrix} \ln u(t,n) \\ v(t,n) \end{pmatrix}_t = L_1 \begin{pmatrix} D(n) \\ B(n) \end{pmatrix} - \lambda L_2 \begin{pmatrix} D(n) \\ B(n) \end{pmatrix} + \lambda_t \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.4}$$

where

$$L_1 = \begin{pmatrix} E - E^{-1} & (E-1)v(t,n-1) \\ v(t,n)(E-1) & Eu(t,n)E - u(t,n) \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & E-1 \\ E-1 & 0 \end{pmatrix}.$$

To construct an integrable hierarchy, we expand $(D(n), B(n))^T$ as a series

$$\begin{pmatrix} D(n) \\ B(n) \end{pmatrix} = \sum_{j=0}^{k} \begin{pmatrix} d_{n,j} \\ b_{n,j} \end{pmatrix} \lambda^{k-j} \tag{2.5}$$

and let

$$\lambda_{t_k} = \mu(\lambda^{k+1} - 4\lambda^{k-1}), \tag{2.6}$$

where μ is a constant and k is an arbitrary non-negative integer by which we can generate a differential-difference hierarchy. In (2.6) and below, we add a subscript k for t to correspond to the time evolution relation (2.6). Comparing the coefficients of the same power of the λ in (2.4), we get

$$\begin{pmatrix} \ln u(t,n) \\ v(t,n) \end{pmatrix}_{t_k} = L_1 \begin{pmatrix} d_{n,k} \\ b_{n,k} \end{pmatrix},$$

$$L_2 \begin{pmatrix} d_{n,j+1} \\ b_{n,j+1} \end{pmatrix} = L_1 \begin{pmatrix} d_{n,j} \\ b_{n,j} \end{pmatrix}, \quad j \geq 2, j = 0,$$

$$L_2 \begin{pmatrix} d_{n,2} \\ b_{n,2} \end{pmatrix} = L_1 \begin{pmatrix} d_{n,1} \\ b_{n,1} \end{pmatrix} - 4 \begin{pmatrix} 0 \\ \mu \end{pmatrix},$$

$$L_2 \begin{pmatrix} d_{n,0} \\ b_{n,0} \end{pmatrix} = \begin{pmatrix} 0 \\ \mu \end{pmatrix}.$$

Under a new boundary condition

$$\begin{pmatrix} D(n) \\ B(n) \end{pmatrix}_{(u(n,t),v(n,t))=(1,0)} = \begin{pmatrix} n\mu\lambda^k + \nu\lambda^{k-1} \\ \nu\lambda^k + 2n\mu\lambda^{k-1} \end{pmatrix}, \tag{2.7}$$

we obtain a new differential-difference hierarchy

$$\begin{pmatrix} \ln u(n,t) \\ v(n,t) \end{pmatrix}_{t_k} = H_k = L^k \begin{pmatrix} 2\mu + \nu(v(n,t) - v(n-1,t)) \\ \mu v(n,t) + \nu(u(n+1,t) - u(n,t)) \end{pmatrix} - 4L^{k-1} \begin{pmatrix} 0 \\ \mu \end{pmatrix}, \tag{2.8}$$

where μ and ν are arbitrary constants satisfying $\mu^2 + \nu^2 \neq 0$ and $L = L_1 L_2^{-1}$. In fact, (2.8) can be looked as a new mixed Toda lattice hierarchy and μ presents the non-isospectral part while ν presents the isospectral part. For example,

- $k = 0$, $(\mu, \nu) = (0, 1)$.

In this case, (2.8) becomes

$$\begin{pmatrix} \ln u(n,t) \\ v(n,t) \end{pmatrix}_t = \begin{pmatrix} v(n,t) - v(n-1,t) \\ u(n+1,t) - u(n,t) \end{pmatrix},$$

which is the classic Toda lattice equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} x(n,t) = \exp\left[x(n-1,t) - x(n,t)\right] - \exp\left[x(n,t) - x(n+1,t)\right]$$

by taking $u(n,t) = \exp\left[x(n-1,t) - x(n,t)\right], v(n,t) = -x(n,t)$.

- $k = 1$, $(\mu, \nu) = (1, 0)$.

In this case, (2.8) becomes

$$\begin{pmatrix} \ln u(n,t) \\ v(n,t) \end{pmatrix}_t = \begin{pmatrix} v(n,t) + v(n-1,t) + 2(n+1)v(n,t) - 2nv(n-1,t) \\ v^2(n,t) + 2(n+2)u(n+1,t) - 2nu(n,t) - 4 \end{pmatrix}, \tag{2.9}$$

which is the first nonisospectral Toda lattice.

- $k=1$, $\mu\nu \neq 0$.

In this case, (2.8) gives

$$\begin{pmatrix} \ln u(n,t) \\ v(n,t) \end{pmatrix}_t = \begin{pmatrix} (E-1)v(n-1,t)a_{11} + (1+E^{-1})a_{12} \\ (Eu(n,t)E - u(n,t))a_{11} + v(n,t)a_{12} - 4\mu \end{pmatrix} \tag{2.10}$$

which is the first mixed Toda lattice, where

$$\begin{cases} a_{11} = 2n\mu + \nu v(n+1,t), \\ a_{12} = \mu v(n,t) + \nu[u(n+1,t) - u(n,t)]. \end{cases}$$

Here we note that the boundary condition (2.7) is necessary. It guarantees $H_k = 0$ when $(u(n,t), v(n,t)) = (1,0)$, which keeps the consistence of the both sides of (2.8) when $|n| \to \infty$.

Before moving to the next section, let us discuss some properties of the operators related to the differential-difference hierarchy.

Let

$$\Gamma = \begin{pmatrix} u(n,t)E^{-1} - Eu(n,t) & v(n,t)(1-E) \\ (E^{-1}-1)v(n,t) & E^{-1} - E \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1-E \\ E^{-1}-1 & 0 \end{pmatrix},$$

and

$$P = \begin{pmatrix} v(n,t) & (Eu(n,t)E - u(n,t))(E-1)^{-1} \\ 1 + E^{-1} & (E-1)v(n-1,t)(E-1)^{-1} \end{pmatrix}$$

be three operators $W_2 \to W_2$, then the following propositions hold.

Proposition 2.3 *The vector $(d_{n,j}, b_{n,j})^T$ in (2.5) satisfies another recurrence relation:*

$$(E-1)\begin{pmatrix} d_{n,j+1} \\ b_{n,j+1} \end{pmatrix} = P(E-1)\begin{pmatrix} d_{n,j} \\ b_{n,j} \end{pmatrix}, \quad j \geq 2.$$

Proof It is easy to verify that $(E-1)L_2^{-1}L_1 = P(E-1)$, which completes the proof. □

Proposition 2.4 The operators Γ and J are skew-symmetric and satisfy $P = \Gamma J^{-1}$.

Proof It is easy to verify $P = \Gamma J^{-1}$. Now let us prove T and J are both skew-symmetric. Let

$$f(n,t) = (f_1(n), f_2(n))^T, g(n,t) = (g_1(n), g_2(n))^T \in W_2.$$

Then we have

$$\begin{aligned} &\langle \Gamma f(n,t), g(n,t) \rangle \\ &= \sum_{n=-\infty}^{\infty} \{[u(n,t)f_1(n-1) - u(n+1,t)f_1(n+1) + v(n,t)f_2(n) - v(n,t)f_2(n+1)]g_1(n) + \\ &\quad [v(n-1,t)f_1(n-1) - v(n,t)f_1(n) + f_2(n-1) - f_2(n+1)]g_2(n)\} \\ &= \sum_{n=-\infty}^{\infty} \{f_1(n)[u(n+1,t)g_1(n+1) - u(n,t)g_1(n-1) + v(n,t)g_2(n+1) - v(n,t)g_2(n)] + \\ &\quad f_2(n)[v(n,t)g_1(n) - v(n-1,t)g_1(n-1) + g_2(n+1) - g_2(n-1)] \end{aligned}$$

$$= \langle f(n,t), -\Gamma g(n,t)\rangle\}.$$

Similarly, we can also obtain

$$\langle Jf(n,t), g(n,t)\rangle = \langle f(n,t), -Jg(n,t)\rangle. \quad \square$$

Finally, we remark that if we take $\phi_2(n,t) = \phi(n,z)$ and $\lambda = z + \frac{1}{z}$ in (2.1), the Lax pair of the Toda lattice can be rewritten as

$$\phi(n+1,z) + u(n,t)\phi(n-1,z) + v(n,t)\phi(n,z) = (z + \frac{1}{z})\phi(n,z), \tag{2.11a}$$

$$\phi(n,z)_t = C(n,t)\phi(n-1,z) + D(n,t)\phi(n,z). \tag{2.11b}$$

Eq. (2.11) will be our starting point of the IST procedure in the next section.

3. Solving the new differential-difference hierarchy

In this section, we will solve the whole differential-difference hierarchy by the IST. To make the paper self-contained, we will give a few results existing in some references.

3.1. The direct scattering problem

In this subsection, we will introduce the direct scattering theory of the Toda hierarchy in some references. Since all the results in the direct scattering part only have relations with the variable n, they also satisfy the whole differential-difference hierarchy. Moreover, we note that the bar does not denote the complex conjugation and instead we use $*$ to denote the complex conjugation.

Lemma 3.1 ([43]) *Suppose that the potentials $u(n,t)$ and $v(n,t)$ satisfy*

$$\sum_{n=-\infty}^{\infty} |n^j(u(n,t)-1)| < \infty \text{ and } \sum_{n=-\infty}^{\infty} |n^j v(n,t)| < \infty, \quad j = 0,1,2.$$

Then the spectral problem (2.11a) has two sets of Jost solutions $\phi_n(z), \bar{\phi}_n(z)$ and $\psi_n(z), \bar{\psi}_n(z)$ which are bounded for all values of n, and satisfy the following asymptotic behaviors:

- $n \to +\infty$

$$\begin{aligned} &\phi_n(z) \sim z^n, && \bar{\phi}_n(z) \sim z^{-n}, \\ &\tfrac{d}{dz}\phi_n(z) \sim nz^{n-1}, && \tfrac{d}{dz}\bar{\phi}_n(z) \sim -nz^{-n-1}; \end{aligned} \tag{3.1a}$$

- $n \to -\infty$

$$\begin{aligned} &\psi_n(z) \sim z^{-n}, && \bar{\psi}_n(z) \sim z^{n}, \\ &\tfrac{d}{dz}\psi_n(z) \sim -nz^{-n-1}, && \tfrac{d}{dz}\bar{\psi}_{n,z}(z) \sim nz^{n-1}. \end{aligned} \tag{3.1b}$$

Furthermore, $\phi_n(z)$ and $\psi_n(z)$ are analytic in the inside of unit circle, i.e., $|z| \le 1$ on complex plane of z, while $\bar{\phi}_n(z)$ and $\bar{\psi}_n(z)$ are analytic outside of the unit circle, i.e., $|z| > 1$. In addition, on unit circle, i.e., $|z| = 1$, $\phi_n(z)$ and $\psi_n(z)$ satisfy

$$\bar{\phi}_n(z) = \phi_n^*(z), \quad \bar{\psi}_n(z) = \psi_n^*(z).$$

Let

$$S_n = \prod_{j=n+1}^{\infty} u(j,t),$$

and

$$\theta_n(z) = \sqrt{S_n}\phi_n(z).$$

We define the discrete Wronskian as follows.

Definition 3.2 *The discrete Wronskian of spectral functions $\phi_n(z)$ and $\psi_n(z)$ is defined by*

$$W(\phi_n(z), \psi_n(z)) = S_{n-1}[\phi_n(z)\psi_{n-1}(z) - \phi_{n-1}(z)\psi_n(z)].$$

Noticing that $\phi_n(z), \bar{\phi}_n(z)$ and $\psi_n(z), \bar{\psi}_n(z)$ are two sets of linearly independent solutions to the second order difference equation (2.14a), we can suppose that they satisfy following linear relations:

$$\psi_n(z) = a(z)\bar{\phi}_n(z) + b(z)\phi_n(z), \tag{3.2a}$$

$$\bar{\psi}_n(z) = \bar{a}(z)\phi_n(z) + \bar{b}(z)\bar{\phi}_n(z). \tag{3.2b}$$

According to Eq. (3.2) and the asymptotic condition (3.1), we have

$$(z - \frac{1}{z})a(z) = W(\phi_n(z), \psi_n(z)), \quad (z - \frac{1}{z})b(z) = W(\psi_n(z), \bar{\phi}_n(z)),$$
$$(z - \frac{1}{z})\bar{a}(z) = -W(\bar{\phi}_n(z), \bar{\psi}_n(z)), \quad (z - \frac{1}{z})\bar{b}(z) = W(\phi_n(z), \bar{\psi}_n(z)),$$

and the zeros of $a(z)$ $(\bar{a}(z))$ obey the following lemma:

Lemma 3.3 ([43]) *The function $a(z)$ $(\bar{a}(z))$ has only a finite number of zeros at $z_1, z_2, \ldots, z_l$ in the unit circle.*

From the above lemma, $\phi_n(z_j)$ and $\psi_n(z_j)$ are linearly dependent, which means there exists a constant b_j such that

$$\psi_n(z_j) = b_j\phi_n(z_j), \quad j = 1, 2, \ldots, l.$$

Lemma 3.4 ([27]) *Assume that $a(z)$ and $\bar{a}(z)$ have only simple roots $z_1, z_2, \ldots, z_l$. Then we have*

$$\sum_{n=-\infty}^{\infty} S_n\phi_n^2(z_j) = -\frac{z_j a_z(z_j)}{b_j}, \quad j = 1, 2, \ldots, l,$$

where $a_z(z)$ denotes the derivative of $a(z)$ with respect to z.

Now let us define the normalization constants and normalization eigenfunctions as follows.

Definition 3.5 *We call c_j the normalization constant for the eigenfunction $\theta_n(z_j)$ and $c_j\theta_n(z_j)$ the normalization eigenfunction, if*

$$\sum_{n=-\infty}^{\infty} c_j^2 S_n\phi_n^2(z_j) = \langle c_j\theta_n(z_j), c_j\theta_n(z_j)\rangle = 1,$$

where z_j is the simple root of $a(z_j)$ and it is also called discrete spectrum.

By Lemma 3.4, it is easy to find that

$$c_j^2 = -\frac{b_j}{z_j a_z(z_j)}, \quad j = 1, 2, \ldots l.$$

In addition to the discrete spectrum z_j of the square integrable eigenfunctions, there are continuous spectrum z corresponding to the eigenfunction which cannot be normalized. The continuous spectral is the complete circle $|z| = 1$.

Definition 3.6 *The set*

$$\{|z| = 1,\ R(z) = \frac{b(z)}{a(z)},\ z_j,\ c_j, \quad j = 1, 2, \ldots, l\} \tag{3.3}$$

is named the scattering data of the spectral problem (2.11a).

We have already gotten the expressions of the scattering data, and now let us consider the evolution of them. It is very important to establish the IST. Since the evolution of the spectral parameter λ is new, we will see that the evolution of the scattering data are also new, which leads to different solutions.

3.2. The time dependence of the scattering data

In this part we will determine the time dependence of the scattering data.

Lemma 3.7 ([27]) *Suppose that $\Phi(z)$ is a solution of (2.1a), M and N satisfy the zero curvature equation (2.2). Then*

$$P_n(z) = \Phi_t(z) - N\Phi(z)$$

solves (2.1a) as well.

Lemma 3.8 *The formula*

$$\phi_{n,t_k}(z) - C(n)\phi_{n-1}(z) - D(n)\phi_n(z) \tag{3.4}$$

is equivalent to

$$\theta_{n,t_k}(z) + \frac{1}{2}B(n+1)(\sqrt{u(n,t)}\theta_{n-1}(z) - \sqrt{u(n+1,t)}\theta_{n+1}(z)) + \frac{1}{2}\theta_n(z)(E-1)D(n). \tag{3.5}$$

Proof It is easy to obtain

$$S_{n,t_k}(t) = -[D(n+1) + D(n) - (\lambda - v(n,t))B(n+1)]S_n$$

by (2.3). Further, we get

$$\theta_{n,t_k}(z) = -\frac{1}{2}[D(n+1) + D(n) - (\lambda - v(n,t))B(n+1)]\theta_n(z) + \sqrt{S_n}\phi_{n,t_k}(z). \tag{3.6}$$

The spectral problem (2.11a) can be rewritten in term of $\theta_n(z)$ as

$$\sqrt{u(n+1,t)}\theta_{n+1}(z) + \sqrt{u(n,t)}\theta_{n-1}(z) + v(n,t)\theta_n(z) = \lambda\theta_n(z). \tag{3.7}$$

Now, multiplying Eq. (3.4) by $\sqrt{S_n}$ yields

$$\theta_{n,t_k}(z) + \frac{1}{2}\theta_n(z)(E-1)D(n) + [\sqrt{u(n,t)}\theta_{n-1}(z) - \frac{1}{2}(\lambda - v(n,t))\theta_n(z)]B(n+1),$$

where (3.6) has been used. Then, we complete the proof by using (3.7). □

Lemma 3.9 *The operator Γ owns an eigenvector $(\theta_n^2(z), \sqrt{u(n,t)}\theta_{n-1}(z)\theta_n(z))^T$, its corresponding eigenvalue is λ, i.e.,*

$$\Gamma\begin{pmatrix} \theta_n^2(z) \\ \sqrt{u(n,t)}\theta_{n-1}(z)\theta_n(z) \end{pmatrix} = \lambda\begin{pmatrix} \theta_n^2(z) \\ \sqrt{u(n,t)}\theta_{n-1}(z)\theta_n(z) \end{pmatrix}.$$

Proof On one hand, multiplying (3.7) by $\theta_n(z)$, we get

$$(E^{-1}-1)v(n,t)\theta_n^2(z) + (E^{-1}-E)\sqrt{u(n,t)}\theta_{n-1}(z)\theta_n(z) = \lambda(E^{-1}-1)\theta_n^2(z). \tag{3.8a}$$

On the other hand, multiplying (3.7) by $\sqrt{u(n,t)}\theta_{n-1}(z_j)$ and $\sqrt{u(n+1,t)}\theta_{n+1}(z_j)$ respectively, then the subtraction of them gives

$$\begin{aligned}&(u(n,t)E^{-1} - Eu(n,t))\theta_n^2(z) + v(n,t)(1-E)\sqrt{u(n,t)}\theta_{n-1}(z)\theta_n(z)\\ =&\lambda(1-E)\sqrt{u(n,t)}\theta_{n-1}(z)\theta_n(z).\end{aligned} \tag{3.8b}$$

We complete the proof by rewriting (3.8a) and (3.8b) in vector forms. □

Lemma 3.10 *Let z_j be the discrete spectrum of (2.11a) and $\tilde{\theta}(z_j) = c_j\theta(z_j)$ the normalization eigenfunction. Then the product of the eigenvector $(\tilde{\theta}_n^2(z_j), \sqrt{u(n,t)}\tilde{\theta}_{n-1}(z_j)\tilde{\theta}_n(z_j))^T$ and $(D_n, B_n)^T|_{\lambda=\lambda_j}$ has following relation:*

$$\left\langle (E-1)\begin{pmatrix} D(n) \\ B(n) \end{pmatrix}\Big|_{\lambda=\lambda_j}, \begin{pmatrix} \tilde{\theta}_n^2(z_j) \\ \sqrt{u(n,t)}\tilde{\theta}_{n-1}(z_j)\tilde{\theta}_n(z_j) \end{pmatrix}\right\rangle = (k+1)\mu\lambda_j^k - 4(k-1)\mu\lambda_j^{k-2},$$

where $\lambda_j = z_j + \frac{1}{z_j}$ and c_j, $j=1,2,\ldots l$ is the normalization constant.

Proof For the $\tilde{\theta}(z_j)$ is a normalization eigenfunction, we have

$$\begin{aligned}&\left\langle (E-1)\begin{pmatrix} D(n) \\ B(n) \end{pmatrix}\Big|_{\lambda=\lambda_j}, \begin{pmatrix} \tilde{\theta}_n^2(z_j) \\ \sqrt{u(n,t)}\tilde{\theta}_{n-1}(z_j)\tilde{\theta}_n(z_j) \end{pmatrix}\right\rangle\\
=&\left\langle (E-1)\sum_{j=0}^{k}\begin{pmatrix} d_{n,j} \\ b_{n,j} \end{pmatrix}\lambda_j^{k-j}, \begin{pmatrix} \tilde{\theta}_n^2(z_j) \\ \sqrt{u(n,t)}\tilde{\theta}_{n-1}(z_j)\tilde{\theta}_n(z_j) \end{pmatrix}\right\rangle\\
=&\left\langle (E-1)\sum_{j=0}^{k}\begin{pmatrix} d_{n,0} \\ b_{n,0} \end{pmatrix}\lambda_j^{k}, \begin{pmatrix} \tilde{\theta}_n^2(z_j) \\ \sqrt{u(n,t)}\tilde{\theta}_{n-1}(z_j)\tilde{\theta}_n(z_j) \end{pmatrix}\right\rangle\\
=&\left\langle (E-1)\Big[\sum_{j=0}^{k}\begin{pmatrix} \mu n \\ \nu \end{pmatrix}\lambda_j^{k} - 4\sum_{j=2}^{k}\begin{pmatrix} \mu n \\ \nu \end{pmatrix}\lambda_j^{k-2}\Big], \begin{pmatrix} \tilde{\theta}_n^2(z_j) \\ \sqrt{u(n,t)}\tilde{\theta}_{n-1}(z_j)\tilde{\theta}_n(z_j) \end{pmatrix}\right\rangle\\
=&\mu(k+1)\lambda_j^k - 4\mu(k-1)\lambda_j^{k-2},\end{aligned}$$

where we have used Propositions 2.3, 2.4 and Lemma 3.9. □

Theorem 3.11 *The scattering data (3.3) of the spectral problem (2.11a) possesses the following time evolutions relations:*

$$R(t,z) = R(0,z)\exp\Big\{\int_0^t [\nu(z^2(\xi) - z^{-2}(\xi)) + 2\mu(z(\xi) - z^{-1}(\xi))]\lambda^{k-1}(\xi)\mathrm{d}\xi\Big\}, \tag{3.9a}$$

$$c_j^2(t)=c_j^2(0)\exp\Big\{\int_0^t \nu\lambda_j^k(\xi)(z_j(\xi)-z_j^{-1}(\xi))+\mu(k+2)\lambda_j^k(\xi)-4\mu(k+z_j^{-2}(\xi))\lambda_j^{k-2}(\xi)\mathrm{d}\xi\Big\}. \tag{3.9b}$$

Proof If Φ is a solution of (2.1a), then

$$P_n(z)=\Phi_t(z)-N\Phi(z)$$

is still a solution of (2.1a) by Lemma 3.7. Hence there exist two constants α and β such that

$$\psi_{n,t}(z)-C(n)\psi_{n-1}(z)-D(n)\psi_n(z)=\alpha\bar{\psi}_n(z)+\beta\psi_n(z) \tag{3.10}$$

where $\psi_n(z)$ is independent of $\bar{\psi}_n(z)$. Noticing that $\psi_n(z)$ goes to z^{-n}, while $\bar{\psi}_n(z)$ goes to z^n when $n\to-\infty$, we immediately get $\alpha=0$. Thus, Eq. (3.10) is reduced to

$$\psi_{n,t}(z)-C(n)\psi_{n-1}(z)-D(n)\psi_n(z)=\beta\psi_n(z). \tag{3.11}$$

Letting $n\to-\infty$ and comparing the coefficient of n in (3.11), we can obtain

$$\beta=(2\mu+\nu z)z\lambda^{k-1},$$

by taking advantage of the asymptotic condition (2.7). So Eq. (3.11) is changed into

$$\psi_{n,t_k}(z)+[\nu\lambda+2\mu(n+1)]\lambda^{k-1}\psi_{n-1}(z)-(n\mu\lambda+\nu+2\mu z+\nu z^2)\lambda^{k-1}\psi_n(z)=0.$$

Replacing $\psi_n(z)$ by $a(z)\bar{\phi}_n(z)+b(z)\phi_n(z)$ and letting $n\to+\infty$, we find that

$$\begin{aligned}&a(z)_{t_k}z^{-n}+b(z)_{t_k}z^n+nb(z)z^{n-1}z_{t_k}+[2(n+1)\mu+\nu\lambda-\mu n\lambda z-\nu z]\lambda^{k-1}b(z)z^{n-1}\\&\quad=(2\mu+\nu z)z\lambda^{k-1}b(z)z^n,\end{aligned}$$

where (3.1) has been used. Comparing the coefficient of n, z^{-n} and z^n in the above equation yields

$$\begin{cases}a(t,z)=a(0,z),\\ b(t,z)=b(0,z)\exp\{\int_0^t[\nu(z(\xi)+z^{-1}(\xi))+2\mu]\lambda^{k-1}(\xi)(z(\xi)-z^{-1}(\xi))\mathrm{d}\xi\},\end{cases}$$

which means

$$R(t,z)=R(0,z)\exp\Big\{\int_0^t[\nu(z(\xi)+z^{-1}(\xi))+2\mu]\lambda^{k-1}(\xi)(z(\xi)-z^{-1}(\xi))\mathrm{d}\xi\Big\}.$$

Next, we will consider time evolutions of the discrete scattering data. Taking $z=z_j$ in (3.10), we have the linear relationship

$$\phi_{n,t_k}(z_j)-C(n)|_{\lambda=\lambda_j}\phi_{n-1}(z_j)-D(n)|_{\lambda=\lambda_j}\phi_n(z_j)=\alpha\phi_n(z_j)+\beta\bar{\phi}_n(z_j), \tag{3.12}$$

where $\phi_n(z_j)$ and $\bar{\phi}_n(z_j)$ follow the asymptotic condition (3.1a). We can find that $\beta=0$ because $\phi_n(z_j)$ tends to z_j^n, while $\bar{\phi}_n(z_j)$ tends to z_j^{-n} when $n\to+\infty$. Thus Eq. (3.12) reads

$$\phi_{n,t_k}(z_j)-C(n)\phi_{n-1}(z_j)-D(n)\phi_n(z_j)=\alpha\phi_n(z_j).$$

By Lemma 3.8, we have

$$\begin{aligned}&\frac{1}{2}(B(n+1)-B(n))\sqrt{u(n,t)}\theta_{n-1}(z_j)+\frac{1}{2}(D(n+1)-D(n))\theta_n(z_j)\\&\quad=\alpha\theta_n(z_j)-\theta_{n,t_k}(z_j)-\frac{1}{2}(B(n)\sqrt{u(n,t)}\theta_{n-1}(z_j)-B(n+1)\sqrt{u(n+1,t)}\theta_{n+1}(z_j)).\end{aligned} \tag{3.13}$$

Multiplying the above equation by $2\theta_n(z_j)$ and summing it, we obtain

$$(2\alpha - \frac{d}{dt_k})\langle\theta_n(z_j), \theta_n(z_j)\rangle = \Big\langle (E-1)\begin{pmatrix} D(n) \\ B(n) \end{pmatrix}, \begin{pmatrix} \theta_n^2(z_j) \\ \sqrt{u(n,t)}\theta_{n-1}(z_j)\theta_n(z_j) \end{pmatrix} \Big\rangle.$$

Let $\theta_n(z_j)$ be a normalization eigenfunction. We have

$$2\alpha = \Big\langle (E-1)\begin{pmatrix} D(n) \\ B(n) \end{pmatrix}, \begin{pmatrix} \theta_n^2(z_j) \\ \sqrt{u(n,t)}\theta_{n-1}(z_j)\theta_n(z_j) \end{pmatrix} \Big\rangle.$$

Finally, we obtain

$$\alpha = \frac{k+1}{2}\mu\lambda_j^k - 2(k-1)\mu\lambda_j^{k-2},$$

by Lemma 3.10.

Noting that $\theta_n(z_j) \to c_j(t)z_j^n$ as $n \to +\infty$, from (3.13) we have

$$\frac{c_{jt_k}}{c_j}z_j + nz_{jt_k} + \frac{1}{2}[\nu\lambda_j + 2\mu(n+1)]\lambda_j^{k-1}(1-z_j^2) = \frac{1}{2}\mu[k\lambda_j^2 - 4(k-1)]z_j\lambda_j^{k-2},$$

which gives

$$\frac{c_{jt_k}}{c_j} = \frac{1}{2}\nu\lambda_j^k(z_j - \frac{1}{z_j}) + \mu(\frac{1}{2}k+1)\lambda_j^k - 2(\frac{1}{z_j^2} + k)\mu\lambda_j^{k-2}. \quad \square$$

Since we have already obtained the evolution of the scattering data, next we will recover the potential $u(n,t)$ by the IST.

3.3. Exact solutions

In this part, we will derive the exact expression of the reflectionless potentials $u(n,t)$. Letting

$$\phi_n(z) = \sum_{j=n}^{\infty} K_{n,j}z^j, \quad \bar{\phi}_n(z) = \sum_{j=n}^{\infty} K_{n,j}z^{-j},$$

we can get

$$u(n,t)K_{n-1,n-1} = K_{n,n},$$

$$u(n,t)K_{n-1,n} + v(n,t)K_{n,n} = K_{n,n+1},$$

$$K_{n+1,n+1} + u(n,t)K_{n-1,n+1} + v(n,t)K_{n,n+1} = K_{n,n} + K_{n,n+2},$$

$$\cdots$$

by (2.11a). Through the above recurrence, we can easily obtain $K_{n,j}$ and recover $u(n,t)$ and $v(n,t)$ as

$$u(n,t) = \frac{K_{n,n}}{K_{n-1,n-1}}, \quad v(n,t) = \frac{K_{n,n+1}}{K_{n,n}} - \frac{K_{n-1,n}}{K_{n-1,n-1}}. \tag{3.14}$$

To formulate the exact expression of the potential $u(n,t)$, let us consider the following proposition.

Proposition 3.12 ([27]) *Let S and Q be two $l \times l$ matrices and* $\mathrm{R}(Q) = 1$. *Then*

$$|S+Q| = |S| + \mathrm{tr}(S^*Q),$$

where $\mathrm{R}(Q)$ *means the rank of matrix* Q*, and* S^* *is the adjoint matrix of matrix* S*. Especially, when* $|S| \neq 0$*, we have*

$$|S+Q| = |S|[1+\mathrm{tr}(S^{-1})Q].$$

Similar to the isospectral case [43], $K_{n,m}$ can be formulated by the discrete Gel'fand-Levitan-Marchenko equation. We skip the details and only give the following theorem.

Theorem 3.13 ([43]) *Given the scattering data (3.3) of the spectral problem (2.11a) and*

$$F_m = \sum_{j=1}^{l} c_j^2 z_j^m + \frac{1}{2\pi i}\oint_{|z|=1} R(z) z^{m-1}\mathrm{d}z,$$

then $\hat{K}_{n,m} = \frac{K_{n,m}}{K_{n,n}}$ *solves the discrete Gel'fand-Levitan-Marchenko equation*

$$\hat{K}_{n,m} + F_{n,m} + \sum_{s=n+1}^{\infty} \hat{K}_{n,s} F_{s+m} = 0, \quad m > n, \tag{3.15a}$$

and

$$K_{n,n}^{-1} = 1 + F_{2n} + \sum_{j=n+1}^{\infty} K_{n,j} F_{j+n}, \quad m = n. \tag{3.15b}$$

When the reflection coefficient $R(t, z(t)) = 0$*, Eq. (3.15a) is*

$$g_{n,j}(t) + c_j(t) z_j^n + \sum_{k=1}^{l} c_j(t) c_k(t) \frac{z_j^{n+1} z_k^{n+1}}{1 - z_j z_k} g_{n,k}(t) = 0, \quad j = 1, 2, \ldots l, \tag{3.16}$$

by letting

$$\hat{K}_{n,m}(t) = \sum_{j=1}^{l} c_j(t) z_j^m g_{n,j}(t). \tag{3.17}$$

By Theorem 3.13, we can get the expression of $\hat{K}_{n,m}$. Furthermore, when the reflection coefficient is zero, we can get $\hat{K}_{n,m}$ by solving a system of linear equations.

Let $D_n = [(D_n(t))_{j,k}]$ be an $l \times l$ matrix with entries as follows

$$(D_n(t))_{j,k} = c_j(t) c_k(t) \frac{z_j^{n+1} z_k^{n+1}}{1 - z_j z_k}, \quad 1 \leq j, \ k \leq l.$$

Then we have following theorem.

Theorem 3.14 $\hat{K}_{n,m}$ *and* $K_{n,n}$ *can be expressed as*

$$\hat{K}_{n,m}(t) = -\mathrm{tr}[(I + D_n(t))^{-1} h_n(t) h_m(t)^T], \quad m > n,$$
$$K_{n,n}(t) = \frac{\det(I + D_n(t))}{\det(I + D_{n-1}(t))}.$$

Proof Eq. (3.16) can be rewritten as

$$g_n(t) = -(I + D_n(t))^{-1} h_n(t), \tag{3.18}$$

where $g_n(t) = (g_{n,1}(t), g_{n,2}(t), \ldots, g_{n,l}(t))^T$ and $h_n(t) = (c_1(t)z_1^n, c_2(t)z_2^n, \ldots, c_l(t)z_l^n)^T$. By (3.17), we obtain

$$\hat{K}_{n,m}(t) = -\mathrm{tr}[(I + D_n(t))^{-1}h_n(t)h_m(t)^T], \quad m > n,$$

where tr means the trace of a matrix.

When $m = n$, we obtain

$$K_{n,n}^{-1}(t) = 1 + \sum_{j=1}^{l} c_j^2(t)z_j^{2n} + \sum_{j,k=1}^{l} c_j^2(t)c_k(t)\frac{z_j^{n+1}z_k^{n+1}}{1 - z_j z_k} z_j^n g_{n,k}(t) \tag{3.19}$$

from (3.15b). Multiplying the j-th equation of (3.16) with $c_j(t)z_j^n$ and summing them with respect to j, then we see that the subtraction of the resulting expression and (3.19) gives

$$K_{n,n}^{-1}(t) = 1 - \sum_{j=1}^{l} c_j(t)z_j^n g_{n,j}(t).$$

Taking the place of $g_{n,j}(t)$ with its expression (3.18), the above formula can be rewritten as

$$K_{n,n}^{-1}(t) = 1 + h_n^T(t)(I + D_n(t))^{-1}h_n(t). \tag{3.20}$$

The trace of both sides of Eq. (3.20) gives

$$K_{n,n}^{-1} = 1 + \mathrm{tr}[(I + D_n(t))^{-1}h_n(t)h_n^T(t)].$$

Noticing that

$$\det(I + D_{n-1}(t)) = \det\left[(I + D_n(t)) + D_{n-1}(t) - D_n(t)\right]$$

and $D_{n-1}(t) - D_n(t) = h_n(t)h_n^T(t)$, we can easily get

$$\det(I + D_{n-1}(t)) = \det(I + D_n(t))\{1 + \mathrm{tr}[(1 + D_n(t))^{-1}h_n(t)h_n^T(t)]\}$$

by using Proposition 3.12. Thus, we complete the proof. □

Therefore, the solution of the differential-difference hierarchy can be given as follows

$$u(n,t) = \frac{\det(I + D_n(t))\det(I + D_{n-2}(t))}{\det(I + D_{n-1}(t))^2}$$

by (3.14).

Especially, when $l = 1$, it yields

$$u(n,t) = \frac{(1 - z_1^2 + c_1^2(t)z_1^{2n+2})(1 - z_1^2 + c_1^2(t)z_1^{2n-2})}{(1 - z_1^2 + c_1^2(t)z_1^{2n})^2}.$$

Substituting the relation of c_j depending on the time t into (3.9b), we can obtain explicit solutions for the hierarchy. For example,

- $k = 0$, $(\mu, \nu) = (0, 1)$.

In this case, we suppose that $z_1 = e^{\frac{\kappa_j}{2}}$ is a constant for $\lambda_t = 0$. Further, it is easy to obtain

$$c_j^2 = c_j^2(0)\exp\left[(e^{\kappa_j} - e^{-\kappa_j})t\right].$$

Letting $c_j^2(0) = 1$, we obtain the potential of isospectral Toda lattice

$$u_n = \frac{[1 - e^{\kappa_1} + e^{(e^{\kappa_1} - e^{-\kappa_1})t + (n-1)\kappa_1}][1 - e^{\kappa_1} + e^{(e^{\kappa_1} - e^{-\kappa_1})t + (n+1)\kappa_1}]}{[1 - e^{\kappa_1} + e^{(e^{\kappa_1} - e^{-\kappa_1})t + n\kappa_1}]^2}.$$

- $k=1$, $(\mu,\nu)=(1,0)$.

In this case, we can easily find $z_1=\frac{1+ce^{2t}}{1-ce^{2t}}$ and

$$c_1^2=c_1^2(0)e^{2t}\frac{(1+ce^{2t})}{(1-ce^{2t})^3},$$

where c is a constant. Let $c=2$, $c_1^2(0)=1$ and

$$r(t,n)=(1-2e^{2t})^{2n+3}-(1+2e^{2t})^2(1-2e^{2t})^{2n+1}+e^{2t}(1+2e^{2t})^{2n+1}.$$

Then the potential $u(n,t)$ of Eq. (2.9) can be recovered as

$$u(n,t)=\frac{r(t,n-1)r(t,n+1)}{r^2(t,n)}.$$

- $k=1$, $\mu\nu\neq 0$.

In this case, $z_1=\frac{1+ce^{2t}}{1-ce^{2t}}$ and

$$c_1^2(t)=\frac{(1+ce^{2t})^{\mu}}{(1-ce^{2t})^{3\mu}}\exp(\frac{4c\nu e^{2t}}{1-c^2e^{4t}}+2\mu t),$$

the potential $u(n,t)$ of the Eq. (2.10) is recovered as

$$u(n,t)=\frac{s(n-1)s(n+1)}{s^2(n)},$$

where

$$s(n)=1-\frac{(1+ec^{2t})^2}{(1-ce^{2t})^2}+\frac{(1+ce^{2t})^{2n+\mu}}{(1-ce^{2t})^{2n+3\mu}}\exp(\frac{4c\nu e^{2t}}{1-c^2e^{4t}}+2\mu t).$$

4. Conclusions and remarks

When we consider continuous soliton equations, we always assume that potentials go to zero as x go to infinity. But this is not the case for some discrete soliton equations because of physical background. As we all know, the potential of the Toda lattice $(u(n,t),v(n,t))$ goes to $(1,0)$ as n goes to infinity. This results in the difficulty of choosing suitable evolution relations of the spectral parameter to construct the nonisospectral integrable lattice.

In this paper, by the new evolution relations of the parameter λ and the boundary conditions, we have derived a novel differential-difference hierarchy from the Toda spectral problem through the discrete zero curvature equation. The obtained differential-difference hierarchy is Liouville integrable and its exact soliton solutions have been presented through the IST.

Acknowledgements We thank the referees for their time and comments.

References

[1] P. D. LAX. *Integrals of nonlinear equations of evolution and solitary waves.* Comm. Pure Appl. Math., 1968, **21**: 467–490.

[2] R. M. MIURA, C. S. GARDENER, M. D. KRUSKAL. *Korteweg-de Vries equations and generalization. II. Existence of conservation laws and constants of motion.* J. Mathematical Phys., 1968, **9**: 1204–1209.

[3] Jianbing ZHANG, Jie JI, Dengyuan CHEN. *Conservation laws and symmetries of the Levi equation.* Commun. Theor. Phys. (Beijing), 2008, **50**(6): 1261–1264.

[4] Jianbing ZHANG, Yingyin GONGYE, Wenxiu MA. *Conservation laws and τ-symmetry algebra of the Gerdjikov-Ivanov soliton hierarchy*. Bull. Malays. Math. Sci. Soc., 2020, **43**(1): 111–123.

[5] P. M. SANTINI, A. S. FOKAS. *Recursion operator and bi-Hamiltonian structure in multidimensions (I)*. Comm. Math. Phys., 1988, **115**: 375–419.

[6] A. S. FOKAS, P. M. SANTINI. *Recursion operator and bi-Hamiltonian structure in multidimensions (II)*. Comm. Math. Phys., 1988, **116**: 449–474.

[7] Jianbing ZHANG, Jie JI, Yuqin YAO. *From the conservation laws to the Hamiltonian structures of discrete soliton systems*. Phys. Scr., 2011, **84**: 015001.

[8] Cewen CAO. *A cubic system which generates Bargmann potential and N-gap potential*. Chinese Quart. J. Math., 1988, **3**(1): 90–96.

[9] Cewen CAO. *Nonlinearization of Lax system for the AKNS hierarchy*. Sci. China Ser. A, 1990, **33**(5): 528–542.

[10] Chaohao GU, Hesheng HU, Zixiang ZHOU. *Darboux Transformation in Soliton Theory and Its Geometric Applications*. Shanghai Scientific and Technical Press, Shanghai, 1999. (in Chinese)

[11] J. M. ALBERTY, T. KOIKAWA, R. SASAKI. *Canonical structure of soliton equations (I)*. Phys. D, 1982, **5**(1): 43–65.

[12] Jianbing ZHANG, Yingyin GONGYE, Wenxiu MA. *A τ-symmetry algebra of the generalized derivative nonlinear schrödinger soliton hierarchy with an arbitrary parameter*. Symmetry, 2018, **10**: 535.

[13] R. HIROTA. *Exact solutions of the Korteweg-de Vries equation for multiple collisions of solitons*. Phys. Rev. Lett., 1971, **27**: 1192–1194.

[14] Jianbing ZHANG, Yingyin GONGYE, Shouting CHEN. *Soliton solutions to the coupled Gerdjikov-Ivanov equation with rogue-wave-like phenomena*. Chin. Phys. Lett., 2017, **34**: 090201.

[15] Jianbing ZHANG, Shouting CHEN, Qi LI. *Bilinear approaches for a finite-dimensional Hamiltonian system*. Phys. Scr., 2013, **88**: 065006.

[16] Jianbing ZHANG, Dajun ZHANG, Qing SHEN. *Bilinear approach for a symmetry constraint of the modified KdV equation*. Appl. Math. Comput., 2011, **218**(8): 4494–4500.

[17] Jianbing ZHANG, Dajun ZHANG, Dengyuan CHEN. *Solving the KdV equation under Bargmann constraint via bilinear approach*. Commun. Theor. Phys., 2010, **53**: 211–217.

[18] Jianbing ZHANG, Wenxiu MA. *Mixed lump-kink solutions to the BKP equation*. Comput. Math. Appl., 2017, **74**(3): 591–596.

[19] Wenxiu MA, Yi ZHANG, Yaning TANG. *Symbolic Computation of Lump Solutions to a combined equation involving three types of nonlinear terms*. East Asian J. Appl. Math., 2020, **10**(4): 732–745.

[20] Jinyun YANG, Wenxiu MA. *Determining lump solutions for a combined soliton equation in* $(2+1)$*-dimensions*. Eur. Phys. J. Plus, 2020, **135**: 494.

[21] C. S. GARDENER, J. M. GREENE, M. D. KRUSKAL, et al. *Method for solving the KdV equation*. Phys. Rev. Lett., 1967, **19**: 1095–1097.

[22] M. J. ABLOWITZ, H. SEGUR. *Soliton and the Inverse Scattering Transform*. SIAM, Philadephia, 1981.

[23] M. J. ABLOWITZ, P. A. CLARKSON. *Soliton, Nonlinear Evolution Equation and Inverse Scattering*. Cambridge University Press, Cambridge, 1991.

[24] Yunbo ZENG, Wenxiu MA, Runliang LIN. *Integration of the soliton hierarchy with self-consistent sources*. J. Math. Phys., 2000, **41**(8): 5453–5489.

[25] Tongke NING, Dengyuan CHEN, Dajun ZHANG. *The exact solutions for the nonisospectral AKNS hierarchy through the inverse scattering transform*. Phys. A, 2004, **339**(3-4): 248–266.

[26] H. H. CHEN, Chuansheng LIU. *Solitons in nonuniform media*. Phys. Rev. Lett., 1976, **37**(11): 693–697.

[27] Jianbing ZHANG, Dajun ZHANG, Dengyuan CHEN. *Exact solutions to a mixed Toda lattice hierarchy through the inverse scattering transformation*. J. Phys. A, 2011, **44**(11): 115201, 14 pp.

[28] F. CALOGERO, A. DEGASPERIS. *Extension of the spectral transform method for solving nonlinear evolution equations*. Lett. Nuovo Cimento (2), 1978, **22**(4): 131–137.

[29] Qi LI, Dajun ZHANG, Dengyuan CHEN. *Solving the hierarchy of the nonisospectral KdV equation with self-consistent sources via the inverse scattering transform*. J. Phys. A, 2008, **41**(35): 355209, 14 pp.

[30] Jianbing ZHANG, Zexuan ZHANG. *A Riemann-Hilbert approach to the Multi-component Kaup-Newell equation*. Adv. Math. Phys., 2020, **2020**: 8844078.

[31] Wenxiu MA, Yehui HUANG, Fudong WANG. *Inverse scattering transforms and soliton solutions of nonlocal reverse-space nonlinear Schrödinger hierarchies*. Stud. Appl. Math., 2020, **145**(3): 563–585.

[32] Wenxiu MA. *Inverse scattering and soliton solutions of nonlocal reverse-spacetime nonlinear Schrödinger hierarchies*. Proc. Amer. Math. Soc., 2021, **149**(1): 251–263.

[33] T. AKTOSUN, C. van der. MEE. *Explicit solutions to the Korteweg-de Vries equation on the half-line*. Inverse Problems, 2006, **22**(6): 2165–2174.

[34] Jianbing ZHANG, Canyuan GU, Wenxiu MA. *Generalized matrix exponential solutions to the AKNS hierarchy*. Adv. Math. Phys., 2018, Art. ID 1375653, 9 pp.

[35] Jianbing ZHANG, Wenxiu MA. *General matrix exponent solutions to the coupled derivative nonlinear Schrödinger equation on half-line*. Modern Phys. Lett. B, 2019, **33**(5): 1950055, 10 pp.

[36] Yishen LI. *A class of evolution equations and the spectral deformation*. Sci. Sinica Ser. A, 1982, **25**(9): 911–917.

[37] Wenxiu MA. *An approach for constructing nonisospectral hierarchies of evolution equations*. J. Phys. A, 1992, **25**(12): L719–L726.

[38] M. R. GUPTA. *Exact inverse scattering solution of a nonlinear evolution equation in a nonuniform medium*. Phys. Lett. A, 1979, **72**(6): 420–422.

[39] W. L. CHAN, K. S. LI. *Nonpropagating Solitons of the variable coefficients and nonisospectral KdV equation*. J. Math. Phys., 1989, **30**(11): 2521–2526.

[40] M. WADATI. *Transformation theories for nonlinear discrete systems*. Suppl. Prog. Theor. Phys., 1976, **59**: 36–63.

[41] M. BLASZAK, K. MARCINIAK. *R-matrix approach to lattice integrable systems*. J. Math. Phys., 1994, **35**: 4661–4682.

[42] Wenxiu MA, B. FUCHSSTEINER. *Algebraic structure ofdiscrete zero curvature equations and master symmetries of discreteevolution equations*. J. Math. Phys., 1999, **40**: 2400–2418.

[43] Dengyuan CHEN. *Introduction to Soliton*. Science Press, Beijing, 2006. (in Chinese)