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Compactness for Commutator of Fractional Integral on Non-homogeneous Morrey Spaces

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Abstract The aim of this paper is to establish the necessary and sufficient conditions for the compactness of fractional integral commutator $[b, I_\gamma]$ which is generated by fractional integral I_γ and function $b \in \mathrm{Lip}_\beta(\mu)$ on Morrey space over non-homogeneous metric measure space, which satisfies the geometrically doubling and upper doubling conditions in the sense of Hytönen. Under assumption that the dominating function λ satisfies weak reverse doubling condition, the author proves that the commutator $[b, I_\gamma]$ is compact from Morrey space $M_q^p(\mu)$ into Morrey space $M_t^s(\mu)$ if and only if $b \in \mathrm{Lip}_\beta(\mu)$.

Keywords non-homogeneous metric measure space; compactness; commutator of fractional integral; $\mathrm{Lip}_\beta(\mu)$; Morrey space

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1. Introduction

To unify both the spaces of homogeneous type in the sense of Coifman and Weiss [1, 2] and the non-doubling spaces whose measure satisfies the polynomial growth conditions [3–10], in 2010, Hytönen [11] first introduced a new class of metric measure spaces satisfying the so-called upper doubling and the geometrically doubling conditions (respectively, see Definitions 1.1 and 1.2 below). And the new-introduced space is now called non-homogeneous metric measure space. Since then, many classical results about the singular integral operators and function spaces on homogeneous space or non-homogeneous space have been proved still valid if the underlying spaces are replaced by the non-homogeneous metric measure spaces, for example, the readers can see [12–20] and their references therein.

In this paper, let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space in the sense of the Hytönen [11]. In the setting of this condition, we will mainly study the necessary and sufficient conditions of the compactness for the commutator $[b, I_\gamma]$ which is generated by fractional integral I_γ and function $b \in L^1_{\mathrm{loc}}(\mu)$ on Morrey space over $(\mathcal{X}, d, \mu)$.

Recall that a linear operator T from a Banach space X to a Banach space Y is compact if $\{Tx_j\}_{j=1}^\infty$ has a convergent subsequence in Y whenever $\{x_j\}_{j=1}^\infty$ is bounded sequence in X. In 2007, Betancor and Fariña [9] obtained the compactness of commutator $[b, I_\alpha]$ generated by

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fractional integral operators I_α and spaces RBMO(μ) under non-doubling measures. In 2008, Sawano and Shirai [10] proved that the multi-commutators generated by spaces RBMO(μ) and singular integrals or fractional integrals are compact on Morrey spaces if one of the RBMO functions can be approximated with compactly supported smooth functions. Nogayama and Sawano [21] obtained the necessary and sufficient conditions for the compact commutators, which are generated by Lipschitz functions and fractional integral operators I_α, on Morrey spaces $M^p_q(\mathbb{R}^n)$.

Before stating the main result of this paper, we need to recall some necessary notions.

Definition 1.1 ([11]) *A metric measure space $(\mathcal{X}, d, \mu)$ is said to be upper doubling if μ is a Borel measure on $\mathcal{X}$ and there exist a dominating function $\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$ and a positive constant C_λ such that, for each $x \in \mathcal{X}, r \to \lambda(x, r)$ is non-decreasing and, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,*

$$\mu(B(x, r)) \le \lambda(x, r) \le C_\lambda \lambda(x, r/2). \tag{1.1}$$

From [12], Hytönen et al. have showed that, there exists another dominating function $\tilde{\lambda}$ such that $\tilde{\lambda} \le \lambda$, $C_{\tilde{\lambda}} \le C_\lambda$ and

$$\tilde{\lambda}(x, r) \le C_{\tilde{\lambda}} \tilde{\lambda}(y, r), \tag{1.2}$$

where $d(x, y) \le r$ for all $x, y \in \mathcal{X}$. If there is no special explanation in this paper, we always assume λ as in (1.1) satisfies (1.2).

Definition 1.2 ([11]) *A metric space $(\mathcal{X}, d)$ is said to be geometrically doubling, if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.*

Remark 1.3 Let $(\mathcal{X}, d)$ be a metric space. Hytönen in [11] has showed that the geometrically doubling $(\mathcal{X}, d)$ is equivalent to the following statement: for any $\epsilon \in (0, 1)$ and any ball $B(x, r) \subset \mathcal{X}$, there is a finite ball covering $\{B(x_i, \epsilon r)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0 \epsilon^{-n}$, where $n := \log_2 N_0$.

Definition 1.4 ([20]) *Given $\beta \in (0, 1)$, the function $f : \mathcal{X} \to \mathbb{C}$ is said to satisfy the Lipschitz condition of β order provided that*

$$|f(x) - f(y)| \le C[\lambda(x, d(x, y))]^\beta, \quad \text{for any } x, y \in \mathcal{X}. \tag{1.3}$$

The smallest constant in (1.3) will be denoted by $\|f\|_{\mathrm{Lip}_\beta(\mu)}$.

Let $L^\infty_b(\mu)$ be the space of all $L^\infty(\mu)$ functions with bounded support. Then, for all $\gamma \in (0, 1)$ and $f \in L^\infty_b(\mu)$, the fractional integral I_γ on $(\mathcal{X}, d, \mu)$ is defined by

$$I_\gamma(f)(x) = \int_{\mathcal{X}} \frac{f(y)}{[\lambda(x, d(x, y))]^{1-\gamma}} \mathrm{d}\mu(y), \quad x \in \mathcal{X}. \tag{1.4}$$

Moreover, we refer to [15] for the bounded properties on I_γ. And, we have the following remark about I_γ.

Remark 1.5 (1) If we take

$$(\mathcal{X}, d, \mu) := (\mathbb{R}^n, |\cdot|, \mu), \quad \lambda(x, r) := Cr^d$$

with $d \in (0, n]$ and the measure μ to satisfy the non-doubling measure, then the I_γ defined in (1.4) is just the fractional integral operator under non-doubling measures [4].

(2) If we take

$$(\mathcal{X}, d, \mu) := (\mathbb{R}^n, |\cdot|, \mathrm{d}x),$$

then fractional integral operator I_γ defined as in (1.4) is just the classical fractional integral operator defined as follows:

$$\widetilde{I}_\gamma(f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}} \mathrm{d}y, \ x \in \mathbb{R}^n,$$

for more properties of $\widetilde{I}_\gamma(f)$, we can see [22–24].

In 1965, Calderón firstly introduced the commutator $[b, T](f) = bT(f) - T(bf)$ which was also called Calderón commutator, and obtained the boundedness of the commutator $[b, H]$ generated by the Hilbert transform and space $\mathrm{BMO}(\mathbb{R}^n)$ on $L^2(\mathbb{R}^n)$ (see [25]). In 1976, Coifman, Rochberg and Weiss [26] proved that the commutator $[b, T]$ which was generated by the classical Calderón-Zygmund operator and $b \in \mathrm{BMO}(\mathbb{R}^n)$ is bounded on the Lebesgue space $L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$. About the more development and the applications of the commutator, we can see [27–30].

Given a function $b \in \mathrm{Lip}_\beta(\mu)$. The commutator $[b, I_\gamma]$ closely related to the fractional integral operator I_γ is defined by

$$[b, I_\gamma](f)(x) = \int_{\mathcal{X}} \frac{b(x) - b(y)}{[\lambda(x, d(x, y))]^{1-\gamma}} f(y) \mathrm{d}\mu(y), \ x \in \mathcal{X}. \tag{1.5}$$

Definition 1.6 ([14]) *Let $k > 1$ and $1 < q \le p < \infty$. Then Morrey space $M_q^p(\mu)$ is defined by*

$$M_q^p(\mu) = \{f \in L^q_{\mathrm{loc}}(\mu) : \|f\|_{M_q^p(\mu)} < \infty\},$$

where

$$\|f\|_{M_q^p(\mu)} := \sup_B [\mu(kB)]^{\frac{1}{p} - \frac{1}{q}} \Big(\int_B |f(y)|^q \mathrm{d}\mu(y) \Big)^{\frac{1}{q}}. \tag{1.6}$$

Moreover, Cao and Zhou in [14] have showed that the norm of $\|f\|_{M_q^p(\mu)}$ is independent of the choices of k for $k > 1$.

Definition 1.7 ([11]) *Let $\tau > 1$ be some fixed constant. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the space* $\mathrm{RBMO}(\mu)$ *if there exists some constant $C > 0$ such that, for any ball B,*

$$\frac{1}{\mu(\tau B)} \int_B |f(x) - f_B| \mathrm{d}\mu(x) \le C, \tag{1.7}$$

and, for any two balls B and S such that $B \subset S$,

$$|f_B - f_S| \le C K_{B,S}, \tag{1.8}$$

here and in what follows,

$$f_B = \frac{1}{\mu(B)} \int_B f(x) \mathrm{d}\mu(x).$$

Then take

$$\|f\|_{\mathrm{RBMO}(\mu)} = \inf\{C : (1.7) \text{ and } (1.8) \text{ hold}\}.$$

Moreover, Hytönen [11] showed that the space $\mathrm{RBMO}(\mu)$ *is independent of the choice of* $\tau > 1$.

Definition 1.8 ([13]) *Let* $\theta \in (0,\infty)$. *A dominating function* λ *is said to satisfy* θ*-weak reverse doubling condition if, for all* $r \in (0, 2\mathrm{diam}(\mathcal{X}))$ *and* $a \in (1, 2\mathrm{diam}(\mathcal{X})/r)$, *there exists a number* $C(a) \in [1,\infty)$, *depending only on* a *and* $\mathcal{X}$, *such that, for all* $x \in \mathcal{X}$

$$\lambda(x, ar) \geq C(a)\lambda(x, r), \tag{1.9}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{[C(a^k)]^{\theta}} < \infty. \tag{1.10}$$

We are now in the position to state the main theorem of this paper as follows.

Theorem 1.9 *Let* $b \in L^1_{\mathrm{loc}}(\mu)$, $0 < \gamma < 1$, $0 < \beta < 1$, $1 < q \leq p < \infty$, $1 < t \leq s < \infty$, $\frac{q}{p} = \frac{t}{s}$ *and* $\frac{1}{s} = \frac{1}{p} - (\beta + \gamma)$. *Suppose that the dominating function* λ *satisfies* θ*-weak reverse doubling condition. Then the commutator* $[b, I_\gamma]$ *is a compact operator from the Morrey space* $M^p_q(\mu)$ *into the Morrey space* $M^s_t(\mu)$ *if and only if* $b \in \mathrm{Lip}_\beta(\mu)$.

Finally, we make some conventions on notion. Throughout the whole paper, C represents a positive constant being independent of the main parameters. For any subset $E \subset \mathcal{X}$, we use χ_E to denote its characteristic function. Given any $q \in (1,\infty)$, let $q' := q/(q-1)$ denote its conjugate index. Moreover, for any ball B, c_B and r_B represent the center and radius of ball B.

2. Preliminaries

To prove the main theorem of this paper, in this section, we will recall and establish some necessary lemmas. First, we need to establish the following lemma being sightly modified in [31].

Lemma 2.1 *Assume* $b \in L^1_{\mathrm{loc}}(\mu)$ *and* $\beta \in (0,1)$. *Then the following statements are mutually equivalent:*

(1) There exists a function $f \in \mathrm{Lip}_\beta(\mu)$ *such that* $b(x) = f(x)$ *for a.e.,* $x \in \mathcal{X}$.

(2) $\sup_B \frac{1}{\mu(B)[\lambda(c_B,r_B)]^\beta} \int_B |b(x) - m_B(b)|\mathrm{d}\mu(x) < \infty$, *where* $m_B(b)$ *represents the mean value of function* b *on ball* B, *that is,*

$$m_B(b) = \frac{1}{\mu(B)} \int_B b(y)\mathrm{d}\mu(y).$$

Proof (1)$\Rightarrow$(2). By the Definition 1.4 and $b(x) = f(x)$ with $f \in \mathrm{Lip}_\beta(\mu)$, we have

$$\begin{aligned}
&\frac{1}{\mu(B)[\lambda(c_B,r_B)]^\beta} \int_B |b(x) - m_B(b)|\mathrm{d}\mu(x) \\
&\quad \leq \frac{1}{\mu(B)} \frac{1}{\mu(B)} \frac{1}{[\lambda(c_B,r_B)]^\beta} \int_B \int_B |f(x) - f(y)|\mathrm{d}\mu(x)\mathrm{d}\mu(y) \\
&\quad \leq C\|f\|_{\mathrm{Lip}_\beta(\mu)} \frac{1}{\mu(B)} \frac{1}{\mu(B)} \frac{1}{[\lambda(c_B,r_B)]^\beta} \int_B \int_B [\lambda(x, d(x,y))]^\beta \mathrm{d}\mu(x)\mathrm{d}\mu(y)
\end{aligned}$$

$$\leq C\|f\|_{\mathrm{Lip}_\beta(\mu)} < \infty.$$

Taking the supremum over ball B, we get (2).

(2)⇒(1). Without loss of generality, we assume that there exists a function $f \in \mathrm{Lip}_\beta(\mu)$. For any $x,\ y \in B$, by Definition 1.4, we can deduce that

$$|f(x) - f(y)| \leq C[\lambda(x, d(x,y))]^\beta \leq C[\lambda(c_B, r_B)]^\beta.$$

Further, we get

$$\begin{aligned}
&\frac{1}{\mu(B)[\lambda(c_B,r_B)]^\beta}\int_B |f(x) - m_B(f)|\mathrm{d}\mu(x)\\
&\leq \frac{1}{\mu(B)[\lambda(c_B,r_B)]^\beta}\int_B |f(x)-f(y)|\mathrm{d}\mu(x) + \frac{1}{\mu(B)[\lambda(c_B,r_B)]^\beta}\int_B |f(y) - m_B(f)|\mathrm{d}\mu(x)\\
&\leq \frac{C}{\mu(B)[\lambda(c_B,r_B)]^\beta}\int_B |f(x)-f(y)|\mathrm{d}\mu(x) \leq C,
\end{aligned}$$

especially, we take $b = f$ with $f \in \mathrm{Lip}_\beta(\mu)$. Thus, we show that (1) holds. □

Now we recall the following lemma which is sightly modified from [10].

Lemma 2.2 *Let h be an integrable function on a bounded ball B_0. Suppose that there exists a non-decreasing function $\omega(\cdot)$ and a constant $\varepsilon \in (0,1]$ such that for every ball B and some constant f_B*

$$\frac{1}{\mu(B)}\int_B |h(x) - f_B|\mathrm{d}\mu(x) \leq \omega(r_B)[\lambda(c_B,r_B)]^\varepsilon.$$

Then there exists a function ν which almost everywhere equals to h such that

$$|\nu(x) - \nu(y)| \leq C\omega(d(x,y))[d(x,y)]^\varepsilon \tag{2.1}$$

holds for all $x, y \in B_0$, with the constant C only depending on ε.

Lemma 2.3 *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space, $\beta \in (0,1)$ and $b \in L^1_{\mathrm{loc}}(\mu)$. Then, the following statements are mutually equivalent:*

(i) $b \in \mathrm{Lip}_\beta(\mu)$.

(ii) The following two equalities hold:

$$\lim_{R\to\infty}\Big(\sup_{\substack{B:\ \text{balls},\\ r_B>R}} \frac{1}{[\lambda(c_B,r_B)]^{1+\beta}}\int_B |b(x) - m_B(b)|\mathrm{d}\mu(x)\Big) = 0 \tag{2.2}$$

and

$$\lim_{t\downarrow 0}\Big(\sup_{\substack{B:\ \text{balls},\\ r_B<t}} \frac{1}{[\lambda(c_B,r_B)]^{1+\beta}}\int_B |b(x) - m_B(b)|\mathrm{d}\mu(x)\Big) = 0. \tag{2.3}$$

Proof (i)⇒(ii) is obvious, the details are omitted here.

(ii)⇒(i) (2.3) implies that

$$b = \lim_{j\to\infty} m_{B(\cdot,2^{-j})}(b)$$

belongs to the space $\mathrm{Lip}_\beta(\mu)$, further, (2.2) implies

$$b = \lim_{j\to\infty}(m_{B(\cdot,2^{-j})}(b) - m_{B(\cdot,2^{j})}(b))$$

in $\mathrm{Lip}_\beta(\mu)$. Thus, we have $b \in \mathrm{Lip}_\beta(\mu)$. □

Also, we need to establish the following lemma.

Lemma 2.4 *Let $1 < q \le p < \infty$ and $1 < t \le s < \infty$. Assume that $K : (\mathcal{X} \times \mathcal{X}) \setminus \{(x, y) : x = y\} \to \mathbb{C}$ is a compactly supported L^∞-function and satisfies the following conditions:*

$$|K(x,y)| \le \frac{C}{\lambda(x, d(x,y))}, \tag{2.4}$$

and, for all x, x', $y \in \mathcal{X}$ with $d(x,y) \ge 2d(x,x')$ and $\delta \in (0,1]$,

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \le C\frac{[d(x,x')]^\delta}{[d(x,y)]^\delta \lambda(x, d(x,y))}. \tag{2.5}$$

Then

$$Tf(x) = \int_{\mathcal{X}} K(x,y) f(y) \mathrm{d}\mu(y) \tag{2.6}$$

is a compact operator from Morrey space $M^p_q(\mu)$ into Morrey space $M^s_t(\mu)$.

To prove the Lemma 2.4, we also need the following lemma whose proof is similar to [10, Proposition 2.1].

Lemma 2.5 *Let $1 < q$, $p < \infty$. Suppose that the kernel function $K : (\mathcal{X} \times \mathcal{X}) \setminus \{(x, y) : x = y\} \to \mathbb{C}$ satisfies*

$$\|K\|_{L^q(\mathcal{X}_x; L^{p'}(\mathcal{X}_y))} := \Big\{ \int_{\mathcal{X}} \Big(\int_{\mathcal{X}} |K(x,y)|^{p'} \mathrm{d}\mu(y) \Big)^{\frac{q}{p'}} \mathrm{d}\mu(x) \Big\}^{\frac{1}{q}} < \infty.$$

Then the operator T defined as in (2.6) is compact operator from $L^p(\mu)$ into $L^q(\mu)$.

Proof of Lemma 2.4 Let B be a large doubling ball such that $\mathrm{supp}(k) \subset B \times B$. Define the following three linear operators T_1, T_2 and T_3:

$$T_1 : \ f \in \mathcal{M}^p_q(\mu) \mapsto \chi_B \cdot f \in L^q(\mu),$$

$$T_2 : \ f \in L^q(\mu) \mapsto Tf \in L^s(\mu)$$

and

$$T_3 : \ f \in L^s(\mu) \mapsto \chi_B \cdot f \in M^s_t(\mu).$$

By applying Lemma 2.5 and the fact that $T = T_3 T_2 T_1$, the proof of Lemma 2.4 is completed. □

3. Necessity of Theorem 1.9

To prove the necessity of Theorem 1.9, we first recall some notion and signs.

Assume that $b \in \mathrm{RBMO}(\mu)$ with $\|b\|_{\mathrm{RBMO}(\mu)} = 1$. And define

$$\mathrm{osc}_\beta(b, B) = \frac{1}{[\mu(B)]^{1+\beta}} \int_B |b(x) - m_B(b)| \mathrm{d}\mu(x),$$

where $\beta \in (0,1)$ and B is a doubling ball. If there is no special instruction, we always assume that $\mathrm{osc}_\beta(b, B) > 0$ in the latter of the paper.

Let

$$c_0 = m_B(\mathrm{sgn}(b - m_B(b))) \in [-1, 1]. \tag{3.1}$$

Define

$$f = [\mu(B)]^{-\frac{1}{p}}(\mathrm{sgn}(b - m_B(b)) - c_0)\chi_B. \tag{3.2}$$

Then we can get the following properties:

$$f \cdot (b - m_B(b)) \geq 0, \quad \int_{\mathcal{X}} f(y)\mathrm{d}\mu(y) = 0, \quad |f| \leq 2[\mu(B)]^{-\frac{1}{p}}\chi_B,$$

and

$$\begin{aligned}
&\int_{\mathcal{X}} (b(y) - m_B(b))f(y)\mathrm{d}\mu(y) = \int_{\mathcal{X}} |b(y) - m_B(b)||f(y)|\mathrm{d}\mu(y) \\
&\leq 2[\mu(B)]^{1-\frac{1}{p}} m_B(|b - m_B(b)|) \leq C[\mu(B)]^{1-\frac{1}{p}}.
\end{aligned} \tag{3.3}$$

Here we have used the properties of the RBMO norm for (3.3).

Now we should establish the following lemmas being used in the proof of Theorem 1.9.

Lemma 3.1 *For all $x \in \mathcal{X} \setminus 6B$, then the following inequality*

$$I_\gamma((b - m_B(b))f)(x) \leq C\frac{[\mu(B)]^{1-\frac{1}{p}}}{[\lambda(x, d(x, c_B))]^{1-\gamma}}$$

holds.

Proof For any $x \in \mathcal{X} \setminus 6B$. By applying the definition of the fractional integral, Definition 1.1 and (3.3), we have

$$\begin{aligned}
I_\gamma((b - m_B(b))f)(x) &= \int_B \frac{b(y) - m_B(b)}{[\lambda(x, d(x, y))]^{1-\gamma}} f(y)\mathrm{d}\mu(y) + \int_{\mathcal{X}\setminus B} \frac{b(y) - m_B(b)}{[\lambda(x, d(x, y))]^{1-\gamma}} f(y)\mathrm{d}\mu(y) \\
&= \int_B \frac{b(y) - m_B(b)}{[\lambda(x, d(x, y))]^{1-\gamma}} f(y)\mathrm{d}\mu(y) \\
&\leq \frac{C}{[\lambda(x, d(x, c_B))]^{1-\gamma}} \int_B (b(y) - m_B(b))f(y)\mathrm{d}\mu(y) \\
&\leq C\frac{[\mu(B)]^{1-\frac{1}{p}}}{[\lambda(x, d(x, c_B))]^{1-\gamma}}.
\end{aligned}$$

Thus, the proof of the Lemma 3.1 is completed. □

Lemma 3.2 *For any $x \in \mathcal{X} \setminus 6B$, and $\epsilon \in (0, 1)$, then the following equality*

$$|I_\gamma f(x)| \leq C\frac{[\mu(B)]^{1-\frac{1}{p}}}{[\lambda(x, d(x, c_B))]^{1-\gamma}} \frac{r_B^\epsilon}{[d(x, c_B)]^\epsilon}$$

holds.

Proof By using the fact $\int_{\mathcal{X}} f(y)\mathrm{d}\mu(y) = 0$, we can get

$$I_\gamma(f)(x) = \int_{\mathcal{X}} \frac{f(y)}{[\lambda(x, d(x, y))]^{1-\gamma}}\mathrm{d}\mu(y) - \int_{\mathcal{X}} \frac{f(y)}{[\lambda(x, d(x, c_B))]^{1-\gamma}}\mathrm{d}\mu(y)$$

$$=\int_{\mathcal{X}}\Big[\frac{f(y)}{[\lambda(x,d(x,y))]^{1-\gamma}}-\frac{f(y)}{[\lambda(x,d(x,c_B))]^{1-\gamma}}\Big]\mathrm{d}\mu(y).$$

Via (1.1) and (1.2), for $x\in\mathcal{X}\setminus 6B$ and $y\in B$, we can deduce that

$$\lambda(x,d(x,y))\le\lambda(x,d(x,c_B)+d(y,c_B))\le C_\lambda\lambda(x,d(x,c_B))$$

and

$$\lambda(x,d(x,c_B))\le\lambda(x,d(y,c_B)+d(x,y))\le C_\lambda\lambda(x,d(x,y)),$$

that is, $\lambda(x,d(x,y))\sim\lambda(x,d(x,c_B))$. Hence, we can get

$$\begin{aligned}&\Big|\frac{1}{[\lambda(x,d(x,y))]^{1-\gamma}}-\frac{1}{[\lambda(x,d(x,c_B))]^{1-\gamma}}\Big|\\&=\Big|\frac{1}{-1+\gamma}\int_{\lambda(x,d(x,y))}^{\lambda(x,d(x,c_B))}\frac{\mathrm{d}t}{t^{2-\gamma}}\Big|\\&\le\frac{C}{[\lambda(x,d(x,c_B))]^{2-\gamma}}|\lambda(x,d(x,y)+d(c_B,y))-\lambda(x,d(x,y))|\\&\le\frac{C}{[\lambda(x,d(x,c_B))]^{2-\gamma}}\Big[\frac{d(c_B,y)}{d(x,y)}\Big]^{\epsilon}\lambda(x,d(x,y))\\&\le\frac{C}{[\lambda(x,d(x,c_B))]^{1-\gamma}}\Big[\frac{r_B}{d(c_B,x)}\Big]^{\epsilon},\end{aligned}$$

where we have used the following fact [13, Remark 1.4 (iii)]

$$|\lambda(y,r+t)-\lambda(x,r)|\le C_\lambda\Big[\frac{d(x,y)+t}{r}\Big]^{\epsilon}\lambda(x,r),\quad\text{for } r\in(0,\infty),\ t\in[0,r]\ \text{ and } d(x,y)\in[0,r].$$

Further, we have

$$\begin{aligned}|I_\gamma(f)(x)|\le&\frac{C}{[\lambda(x,d(x,c_B))]^{1-\gamma}}\Big[\frac{r_B}{d(c_B,x)}\Big]^{\epsilon}\int_{\mathcal{X}}|f(y)|\mathrm{d}\mu(y)\\\le&C\frac{[\mu(B)]^{-\frac1p}}{[\lambda(x,d(x,c_B))]^{1-\gamma}}\Big[\frac{r_B}{d(c_B,x)}\Big]^{\epsilon}\int_B|b(y)-m_B(b)|\mathrm{d}\mu(y)\\\le&C\frac{[\mu(B)]^{1-\frac1p}}{[\lambda(x,d(x,c_B))]^{1-\gamma}}\Big[\frac{r_B}{d(c_B,x)}\Big]^{\epsilon}.\end{aligned}$$

The proof of Lemma 3.2 is completed. □

Lemma 3.3 *For any $x\in\mathcal{X}\setminus 6B$,*

$$I_\gamma((b-m_B(b))f)(x)\gtrsim\frac{[\mu(B)]^{1+\beta-\frac1p}}{[\lambda(x,d(x,c_B))]^{1-\gamma}}\mathrm{osc}_\beta(b,B)(1-|c_0|).$$

Proof Since $(b(y)-m_B(b))f(y)\ge0$, we can get

$$\begin{aligned}(b(y)-m_B(b))f(y)=&[\mu(B)]^{-\frac1p}(b(y)-m_B(b))[(\mathrm{sgn}(b-m_B(b))-c_0)\chi_B]\\=&[\mu(B)]^{-\frac1p}|b(y)-m_B(b)|-[\mu(B)]^{-\frac1p}c_0(b(y)-m_B(b))\\\ge&[\mu(B)]^{-\frac1p}|b(y)-m_B(b)|-[\mu(B)]^{-\frac1p}|c_0||b(y)-m_B(b)|\\=&[\mu(B)]^{-\frac1p}(1-|c_0|)|b(y)-m_B(b)|.\end{aligned}$$

Further, by applying the definition of fractional integral as in (1.4), we can get

$$\begin{aligned}
I_\gamma((b-m_B(b))f)(x) \geq & \frac{1-|c_0|}{[\mu(B)]^{\frac{1}{p}}} \int_B \frac{|b(y)-m_B(b)|}{[\lambda(x,d(x,y))]^{1-\gamma}} \mathrm{d}\mu(y) \\
\gtrsim & \frac{1-|c_0|}{[\mu(B)]^{\frac{1}{p}}} \frac{1}{[\lambda(x,d(x,c_B))]^{1-\gamma}} \int_B |b(y)-m_B(b)| \mathrm{d}\mu(y) \\
\gtrsim & \frac{1-|c_0|}{[\mu(B)]^{\frac{1}{p}}} \frac{[\mu(B)]^{1+\beta}}{[\lambda(x,d(x,c_B))]^{1-\gamma}} \frac{1}{[\mu(B)]^{1+\beta}} \int_B |b(y)-m_B(b)| \mathrm{d}\mu(y) \\
\gtrsim & \frac{[\mu(B)]^{1+\beta-\frac{1}{p}}}{[\lambda(x,d(x,c_B))]^{1-\gamma}} \mathrm{osc}_\beta(b,B)(1-|c_0|).
\end{aligned}$$

Thus, the proof of the Lemma 3.3 is completed. □

Lemma 3.4 *Let $\varsigma \gg \varsigma' \gg 1$ be sufficiently large and $\iota \in (0,1)$. Assume that c_0 as in (3.1) satisfies $|c_0| < \iota$. Then*

$$\Big(\int_{\varsigma' r_B \leq d(y,c_B) \leq \varsigma r_B} |[b,I_\gamma]f(y)|^t \mathrm{d}\mu(y)\Big)^{\frac{1}{t}} \gtrsim \frac{[\mu(B)]^{1+\beta-\frac{1}{p}+\frac{1}{t}}}{[\lambda(c_B,\varsigma r_B)]^{1-\gamma}} \mathrm{osc}_\beta(b,B)(1-\iota), \tag{3.4}$$

where implicit constant is independent of B, b, ς and ς'.

Proof By Minkowski inequality, write

$$\begin{aligned}
& \Big(\int_{\varsigma' r_B \leq d(y,c_B) \leq \varsigma r_B} |[b,I_\gamma]f(y)|^t \mathrm{d}\mu(y)\Big)^{\frac{1}{t}} \\
& = \Big(\int_{\varsigma' r_B \leq d(y,c_B) \leq \varsigma r_B} |I_\gamma(b-m_B(b))f(y) - (b(y)-m_B(b))I_\gamma(f)(y)|^t \mathrm{d}\mu(y)\Big)^{\frac{1}{t}} \\
& \geq \Big(\int_{\varsigma' r_B \leq d(y,c_B) \leq \varsigma r_B} |I_\gamma(b-m_B(b))f(y)|^t \mathrm{d}\mu(y)\Big)^{\frac{1}{t}} - \\
& \quad \Big(\int_{\varsigma' r_B \leq d(y,c_B) \leq \varsigma r_B} |(b(y)-m_B(b))I_\gamma(f)(y)|^s \mathrm{d}\mu(y)\Big)^{\frac{1}{t}} \\
& := \mathrm{D}_1 + \mathrm{D}_2.
\end{aligned}$$

By applying the Lemma 3.3, we have

$$\begin{aligned}
\mathrm{D}_1 \gtrsim & [\mu(B)]^{1+\beta-\frac{1}{p}} \Big(\int_{\varsigma' r_B \leq d(y,c_B) \leq \varsigma r_B} \frac{\mathrm{d}\mu(y)}{[\lambda(y,d(y,c_B))]^{(1-\gamma)s}}\Big)^{\frac{1}{s}} \mathrm{osc}_\beta(b,B)(1-|c_0|) \\
\gtrsim & [\mu(B)]^{1+\beta-\frac{1}{p}} \Big(\int_{\varsigma' r_B \leq d(y,c_B) \leq \varsigma r_B} \frac{\mathrm{d}\mu(y)}{[\lambda(y,d(y,c_B))]^{(1-\gamma)s}}\Big)^{\frac{1}{s}} \mathrm{osc}_\beta(b,B)(1-|c_0|) \\
\gtrsim & [\mu(B)]^{1+\beta-\frac{1}{p}} \Big(\int_{\varsigma' r_B \leq d(y,c_B) \leq \varsigma r_B} \frac{\mathrm{d}\mu(y)}{[\lambda(c_B,d(y,c_B))]^{(1-\gamma)s}}\Big)^{\frac{1}{s}} \mathrm{osc}_\beta(b,B)(1-|c_0|) \\
\gtrsim & \frac{[\mu(B)]^{\beta+\gamma-\frac{1}{p}+\frac{1}{s}}}{[\lambda(c_B,\varsigma r_B)]^{1-\gamma}} \mathrm{osc}_\beta(b,B)(1-\iota).
\end{aligned}$$

Now we turn to the D_2. By applying Lemma 3.2, we have

$$\mathrm{D}_2 \lesssim \Big(\int_{\varsigma' r_B \leq d(y,c_B) \leq \varsigma r_B} \frac{|b(y)-m_B(b)|^s [\mu(B)]^{s(1-\frac{1}{p})}}{[\lambda(y,d(y,c_B))]^{s(1-\gamma)}} \mathrm{d}\mu(y)\Big)^{\frac{1}{s}}$$

$$\lesssim \frac{[\mu(B)]^{1-\frac{1}{p}}}{[\lambda(c_B,\varsigma' r_B)]^{1-\gamma}}\Big(\int_{\varsigma' r_B\le d(y,c_B)\le \varsigma r_B}|b(y)-m_B(b)|^s[\frac{r_B}{d(c_B,y)}]^{\epsilon s}\mathrm{d}\mu(y)\Big)^{\frac{1}{s}}$$
$$\lesssim \frac{1}{(\varsigma')^{\epsilon}}\frac{[\mu(B)]^{1-\frac{1}{p}}}{[\lambda(c_B,\varsigma' r_B)]^{1-\gamma}}\Big(\int_{\varsigma' r_B\le d(y,c_B)\le \varsigma r_B}|b(y)-m_B(b)|^s\mathrm{d}\mu(y)\Big)^{\frac{1}{s}}$$
$$\lesssim \frac{1}{(\varsigma')^{\epsilon}}\frac{[\mu(B)]^{\gamma+\beta+\frac{1}{s}-\frac{1}{p}}}{[\lambda(c_B,\varsigma' r_B)]^{1-\gamma}}\mathrm{osc}_{\beta}(b,B),$$

Combining the estimate for D_1 and choosing $\varsigma\gg\varsigma'\gg 1$, we can deduce

$$\Big(\int_{\varsigma' r_B\le d(y,c_B)\le \varsigma r_B}|[b,I_\gamma]f(y)|^s\mathrm{d}\mu(y)\Big)^{\frac{1}{s}}$$
$$\gtrsim \frac{[\mu(B)]^{1+\beta-\frac{1}{p}+\frac{1}{s}}}{[\lambda(c_B,\varsigma r_B)]^{1-\gamma}}\mathrm{osc}_{\beta}(b,B)(1-\iota)-\frac{1}{(\varsigma')^{\epsilon}}\frac{[\mu(B)]^{1+\beta+\frac{1}{s}-\frac{1}{p}}}{[\lambda(c_B,\varsigma' r_B)]^{1-\gamma}}\mathrm{osc}_{\beta}(b,B)$$
$$\gtrsim \frac{[\mu(B)]^{\beta+\gamma-\frac{1}{p}+\frac{1}{s}}}{[\lambda(c_B,\varsigma r_B)]^{1-\gamma}}\mathrm{osc}_{\beta}(b,B)(1-\iota).$$

Hence, we complete the proof of the Lemma 3.4. □

The proof of the necessity for Theorem 1.9 is stated as follows.

Proof of Theorem 1.9 Assume that (2.2) fails. Let there be a sequence of balls such that $\lim_{j\to\infty}\mu(B_j)=\infty$ and

$$\mathrm{osc}_{\beta}(b,B_j)\ge\varepsilon>0, \tag{3.5}$$

here $\varepsilon\in(0,1)$ is not dependent on j with $j\in\mathbb{N}$. Set

$$c_j=m_{B_j}(\mathrm{sgn}(b-m_{B_j}(b))), \tag{3.6}$$

$$f_j=[\mu(B_j)]^{-\frac{1}{p}}(\mathrm{sgn}(b-m_{B_j}(b))-c_j)\chi_j. \tag{3.7}$$

According to the condition (3.5), it is not difficult to find that

$$\sup_{j\in\mathbb{N}}\log|c_j|<0.$$

By choosing a subsequence, we may assume that $r_{B_{j+1}}\ge\delta r_{B_j}$ for all $j\in\mathbb{N}$, where $\delta>0$ is determined at the end of the proof.

Suppose that $j,k\in\mathbb{N}$ with $j>k$. Then we have

$$\|[b,I_\gamma]f_j-[b,I_\gamma]f_k\|_{M^s_t(\mu)}$$
$$=\sup_{B(c_{B_j},\varsigma r_{B_j})}[\mu(B(c_{B_j},\varsigma r_{B_j}))]^{\frac{1}{s}-\frac{1}{t}}\Big(\int_{B(c_{B_j},\varsigma r_{B_j})}\big|[b,I_\gamma]f_j(x)-[b,I_\gamma]f_k(x)\big|^t\mathrm{d}\mu(x)\Big)^{\frac{1}{t}}$$
$$\ge[\mu(B(c_{B_j},\varsigma r_{B_j}))]^{\frac{1}{s}-\frac{1}{t}}\Big(\int_{B(c_{B_j},\varsigma r_{B_j})}\big|[b,I_\gamma]f_j(x)-[b,I_\gamma]f_k(x)\big|^t\mathrm{d}\mu(x)\Big)^{\frac{1}{t}}$$
$$\ge[\mu(B(c_B,\varsigma r_{B_j}))]^{\frac{1}{s}-\frac{1}{t}}\Big(\int_{B(c_{B_j},\varsigma r_{B_j})}|[b,I_\gamma]f_j(x)|^t\mathrm{d}\mu(x)\Big)^{\frac{1}{t}}-$$
$$[\mu(B(c_{B_j},\varsigma r_{B_j}))]^{\frac{1}{s}-\frac{1}{t}}\Big(\int_{B(c_{B_j},\varsigma r_{B_j})}|[b,I_\gamma]f_k(x)|^t\mathrm{d}\mu(x)\Big)^{\frac{1}{t}}$$

$$:= \mathrm{E}_1 + \mathrm{E}_2.$$

By applying the Lemma 3.4, we can get

$$\begin{aligned}\mathrm{E}_1 \geq&[\mu(B(c_{B_j},\varsigma r_{B_j}))]^{\frac{1}{t}-\frac{1}{s}}\Big(\int_{\varsigma' r_{B_j}<d(c_{B_j},x)<\varsigma r_{B_j}}|[b,I_\gamma]f_j(x)|^s\mathrm{d}\mu(x)\Big)^{\frac{1}{s}}\\ \gtrsim&\frac{[\mu(B_j)]^{\beta+\gamma-\frac{1}{p}+\frac{1}{s}}}{[\lambda(c_{B_j},\varsigma r_{B_j})]^{1-\gamma}}\mathrm{osc}_\beta(b,B_j)(1-\sup_{j\in\mathbb{N}}|c_j|)\\ \gtrsim&\frac{\mathrm{osc}_\beta(b,B_j)}{[\lambda(c_{B_j},\varsigma r_{B_j})]^{1-\gamma}}(1-\sup_{j\in\mathbb{N}}|c_j|),\end{aligned}$$

where we use $\frac{1}{s}=\frac{1}{q}-(\beta+\gamma)$.

For E_2, from Hölder inequality, it follows that

$$\begin{aligned}\mathrm{E}_2 \leq&[\mu(B(c_{B_j},\varsigma r_{B_j}))]^{\frac{1}{s}-\frac{1}{t}}\Big\{\Big(\int_{B(c_{B_j},\varsigma r_{B_j})}|[b,I_\gamma]f_k(x)|^{t\times\frac{s}{t}}\mathrm{d}\mu(x)\Big)^{\frac{t}{s}}\Big(\int_{B(c_{B_j},\varsigma r_{B_j})}\mathrm{d}\mu(x)\Big)^{1-\frac{t}{s}}\Big\}^{\frac{1}{t}}\\ =&\Big(\int_{B(c_{B_j},\varsigma r_{B_j})}|[b,I_\gamma]f_k(x)|^s\mathrm{d}\mu(x)\Big)^{\frac{1}{s}}.\end{aligned}$$

Combining the E_1 and E_2, we obtain

$$\begin{aligned}&\|[b,I_\gamma]f_j-[b,I_\gamma]f_k\|_{M^s_t(\mu)}\\ &\geq[\mu(B(c_B,\varsigma r_{B_j}))]^{\frac{1}{s}-\frac{1}{t}}\Big(\int_{B(c_{B_j},\varsigma r_{B_j})\backslash B(c_{B_j},\varsigma' r_{B_j})}|[b,I_\gamma]f_j(x)|^t\mathrm{d}\mu(x)\Big)^{\frac{1}{t}}-\\ &\quad[\mu(B(c_{B_j},\varsigma r_{B_j}))]^{\frac{1}{s}-\frac{1}{t}}\Big(\int_{B(c_{B_j},\varsigma r_{B_j})\backslash B(c_{B_j},\varsigma' r_{B_j})}|[b,I_\gamma]f_k(x)|^s\mathrm{d}\mu(x)\Big)^{\frac{1}{s}}\\ &\geq[\mu(B(c_B,\varsigma r_{B_j}))]^{\frac{1}{s}-\frac{1}{t}}\Big(\int_{B(c_{B_j},\varsigma r_{B_j})\backslash B(c_{B_j},\varsigma' r_{B_j})}|[b,I_\gamma]f_j(x)|^t\mathrm{d}\mu(x)\Big)^{\frac{1}{t}}-\\ &\quad\Big(\int_{\mathcal{X}\backslash B(c_{B_j},\varsigma' r_{B_j})}|[b,I_\gamma]f_k(x)|^s\mathrm{d}\mu(x)\Big)^{\frac{1}{s}}\\ &\gtrsim\frac{\mathrm{osc}_\beta(b,B_j)}{[\lambda(c_{B_j},\varsigma r_{B_j})]^{1-\gamma}}-\frac{\omega}{[\lambda(c_{B_j},\varsigma r_{B_j})]^{1-\gamma}}-\Big(\int_{\mathcal{X}\backslash B(c_{B_j},\varsigma' r_{B_j})}|[b,I_\gamma]f_k(x)|^s\mathrm{d}\mu(x)\Big)^{\frac{1}{s}},\end{aligned}\tag{3.8}$$

for some constant ω. By (1.3) and $|f_k(x)|\leq 2[\mu(B_k)]^{-\frac{1}{p}}\chi_{B_k}(x)$, we have

$$\begin{aligned}|[b,I_\gamma]f_k(x)|\leq&\int_{\mathcal{X}}\frac{|b(x)-b(y)|}{[\lambda(x,d(x,y))]^{1-\gamma}}|f_k(y)|\mathrm{d}\mu(y)\\ \leq&C[\mu(B_k)]^{-\frac{1}{p}}\|b\|_{\mathrm{Lip}_\beta(\mu)}\int_{B_k}\frac{\mathrm{d}\mu(y)}{[\lambda(x,d(x,y))]^{1-\gamma-\beta}}\\ \leq&C[\mu(B_k)]^{-\frac{1}{p}}\|b\|_{\mathrm{Lip}_\beta(\mu)}I_{\beta+\gamma}(\chi_{B_k})(x).\end{aligned}$$

Here, if $x\in 6B_k$, then

$$\begin{aligned}I_{\beta+\gamma}(\chi_{B_k})(x)\leq&\int_{B(x,7r_{B_k})}\frac{\mathrm{d}\mu(z)}{[\lambda(x,d(x,z))]^{1-(\beta+\gamma)}}\\ \leq&\sum_{k=1}^{\infty}\int_{B(x,2^{-k+1}\times(7r_{B_k}))\backslash B(x,2^{-k}\times(7r_{B_k}))}\frac{\mathrm{d}\mu(z)}{[\lambda(x,d(x,z))]^{1-(\beta+\gamma)}}\end{aligned}$$

$$\leq C\sum_{k=1}^{\infty}\frac{\mu(B(x,2^{-k+1}\times(7r_{B_k})))}{[\lambda(x,2^{-k}\times(7r_{B_k}))]^{1-(\beta+\gamma)}}$$
$$\leq C[\lambda(x,r_{B_k})]^{\beta+\gamma}\Big(\sum_{k=1}^{\infty}\frac{1}{[C(2^k)]^{\beta+\gamma}}\Big)$$
$$\leq C[\lambda(x,r_{B_k})]^{\beta+\gamma}. \tag{3.9}$$

On the other hand, if $x\in\mathcal{X}\setminus 6B_k$ and $z\in B_k$, then $\lambda(x,d(x,z))\sim\lambda(x,d(x,c_{B_k}))$. Thus, we can deduce that

$$I_{\gamma+\beta}(\chi_{B_k})(x)=\int_{B_k}\frac{\mathrm{d}\mu(z)}{[\lambda(x,d(x,z))]^{1-\gamma-\beta}}\leq C\frac{\mu(B_k)}{[\lambda(x,d(x,c_{B_k}))]^{1-\gamma-\beta}}.$$

Combining (3.9) and $x\in\mathcal{X}$, we have

$$\begin{aligned}|[b,I_\gamma]f_k(x)|\leq& C[\mu(B_k)]^{-\frac{1}{p}}\|b\|_{\mathrm{Lip}_\beta(\mu)}I_{\beta+\gamma}(\chi_{B_k})(x)\\ \leq& C[\mu(B_k)]^{-\frac{1}{p}}[\lambda(x,r_{B_k})]^{\beta+\gamma}\|b\|_{\mathrm{Lip}_\beta(\mu)}\big\{1+\frac{\mu(B_k)[\lambda(x,d(x,c_{B_k}))]^{\gamma+\beta}}{\lambda(x,d(x,c_{B_k}))[\lambda(x,r_{B_k})]^{\beta+\gamma}}\big\}\\ \leq& C[\lambda(x,r_{B_k})]^{\beta+\gamma-\frac{1}{p}}\|b\|_{\mathrm{Lip}_\beta(\mu)}\big(1+\frac{\lambda(x,r_{B_k})}{\lambda(x,d(x,c_{B_k}))}\big)^{1-(\gamma+\beta)}.\end{aligned} \tag{3.10}$$

Further, we obtain that

$$\begin{aligned}&\Big(\int_{\mathcal{X}\setminus B(c_{B_j},\varsigma' r_{B_j})}|[b,I_\gamma]f_k(x)|^s\mathrm{d}\mu(x)\Big)^{\frac{1}{s}}\\ &\lesssim\|b\|_{\mathrm{Lip}_\beta(\mu)}\Big\{\int_{\mathcal{X}\setminus B(c_{B_j},\varsigma' r_{B_j})}[\lambda(x,r_{B_k})]^{(\beta+\gamma-\frac{1}{p})s}\big(1+\frac{\lambda(x,r_{B_k})}{\lambda(x,d(x,c_{B_k}))}\big)^{[1-(\gamma+\beta)]s}\mathrm{d}\mu(x)\Big\}^{\frac{1}{s}}\\ &\lesssim\|b\|_{\mathrm{Lip}_\beta(\mu)}\Big\{\sum_{\ell=1}^{\infty}\int_{6^\ell B(c_{B_j},\varsigma' r_{B_j})\setminus 6^{\ell-1}B(c_{B_j},\varsigma' r_{B_j})}[\lambda(x,r_{B_k})]^{(\beta+\gamma-\frac{1}{p})s}\times\\ &\quad\big(1+\frac{\lambda(x,r_{B_k})}{\lambda(x,d(x,c_{B_j}))}\big)^{[1-(\gamma+\beta)]s}\mathrm{d}\mu(x)\Big\}^{\frac{1}{s}}\\ &\lesssim\|b\|_{\mathrm{Lip}_\beta(\mu)}\Big\{\sum_{\ell=1}^{\infty}\frac{1}{[C(6^{\ell-1})]^{1-(\gamma+\beta)}}\frac{[\lambda(c_{B_j},r_{B_k})]^{1-\frac{1}{p}}}{[\lambda(c_{B_j},\varsigma' r_{B_j})]^{1-\frac{1}{s}-(\gamma+\beta)}}\Big\}\\ &\lesssim\|b\|_{\mathrm{Lip}_\beta(\mu)}\big\{\frac{[\lambda(c_{B_j},\varsigma' r_{B_j})]^{1-\frac{1}{p}+\beta+\gamma}}{[\lambda(c_{B_j},\varsigma' r_{B_j})]^{1-\frac{1}{s}}}\big\}\\ &\lesssim\|b\|_{\mathrm{Lip}_\beta(\mu)}\big[\frac{\lambda(c_{B_j},\varsigma' r_{B_k})}{\lambda(c_{B_j},\delta\varsigma' r_{B_k})}\big]^{1-\frac{1}{s}}.\end{aligned} \tag{3.11}$$

Thus, we have

$$\begin{aligned}&\|[b,I_\gamma]f_j-[b,I_\gamma]f_k\|_{M_t^s(\mu)}\\ &\gtrsim G=:\frac{\mathrm{osc}_\beta(b,B_j)}{[\lambda(c_{B_j},\varsigma r_{B_j})]^{1-\gamma}}-\frac{\omega}{[\lambda(c_{B_j},\varsigma r_{B_j})]^{1-\gamma}}-\Big(\int_{\mathcal{X}\setminus B(c_{B_j},\varsigma' r_{B_j})}|[b,I_\gamma]f_k(x)|^s\mathrm{d}\mu(x)\Big)^{\frac{1}{s}}\\ &\gtrsim\frac{\mathrm{osc}_\beta(b,B_j)}{[\lambda(c_{B_j},\varsigma r_{B_j})]^{1-\gamma}}-\frac{\omega}{[\lambda(c_{B_j},\varsigma r_{B_j})]^{1-\gamma}}-\|b\|_{\mathrm{Lip}_\beta(\mu)}\big[\frac{\lambda(c_{B_j},\varsigma' r_{B_k})}{\lambda(c_{B_j},\delta\varsigma' r_{B_k})}\big]^{1-\frac{1}{s}},\end{aligned} \tag{3.12}$$

for some ω independent of ς', ς and δ. If we choose $\varsigma\gg\varsigma'\gg 1$, then $\delta\gg 1$ such that $G>0$, we see that the sequence $\{[b, I_\gamma]f_k\}$ does not converge in $\mathcal{M}_t^s(\mu)$.

Now we assume (2.3) fails. Then there is a sequence of balls $\{B_j\}_{j=1}^\infty$ such that $\lim_{j\to\infty}\mu(B_j) = 0$ and (3.5) holds. With an argument similar to that used in the first part of this proof, by passing to a subsequence, we may assume that $r_{B_{j+1}} \le \delta r_{B_j}$ with $j \in \mathbb{N}$, where $\delta > 0$.

Notice that

$$\begin{aligned}
&\Big(\int_{B(c_{B_j},\varsigma r_{B_j})} |[b, I_\gamma]f_k(x)|^s \mathrm{d}\mu(x)\Big)^{\frac{1}{s}} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(\mu)}\Big(\int_{B(c_{B_j},\varsigma r_{B_j})} [\lambda(x, r_{B_k})]^{(\beta+\gamma-\frac{1}{p})s}\mathrm{d}\mu(x)\Big)^{\frac{1}{s}} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(\mu)}[\lambda(c_{B_j}, r_{B_k})]^{\beta+\gamma-\frac{1}{p}}[B(c_{B_j},\varsigma r_{B_j})]^{\frac{1}{s}} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(\mu)}\Big[\frac{\lambda(c_{B_j},\varsigma\delta r_{B_k})}{\lambda(c_{B_j}, r_{B_k})}\Big]^{\frac{1}{p}-\gamma-\beta}.
\end{aligned}$$

Thus, we have (3.12), and we see that the sequence $\{[b, I_\gamma]f_j\}$ never converges in $M_t^s(\mu)$. Thus, combining the above estimates and Lemma 2.3, we can show that $b \in \mathrm{Lip}_\beta(\mu)$. □

4. Sufficiency of Theorem 1.9

In this section, we will mainly state the proof of the sufficiency for Theorem 1.9.

Proof Without loss of generality, we may assume that a function $a \in C_c^\infty(\mu)$. Using the endpoint estimate of the commutator $[b, T](f)$ generated by the operator $T(f)$ as in (2.6) and $b \in \mathrm{Lip}_\beta(\mu)$, we see that

$$\|[b, I_\gamma](f)\|_{M_q^p(\mu)\to M_t^s(\mu)} \le C\|b\|_{\mathrm{Lip}_\beta(\mu)}. \tag{4.1}$$

If $a \in \mathrm{Lip}_\beta(\mu)$, then there exists a sequence $\{b_j\}_{j=1}^\infty \subset C_c^\infty(\mu)$-functions such that

$$\|a - b_j\|_{\mathrm{Lip}_\beta(\mu)} \le \frac{1}{j}$$

with $j \in \mathbb{N}$. Via (4.1), we can obtain

$$\|[b_j, I_\gamma](f) - [a, I_\gamma](f)\|_{M_q^p(\mu)\to M_t^s(\mu)} \le C\|b_j - a\|_{\mathrm{Lip}_\beta(\mu)} \le \frac{C}{j}.$$

Thus, once we prove that $[b_j, I_\alpha]$ is a compact, it will follow that $[a, I_\alpha]$ is compact, too. Thus, we can set $a \in C_c^\infty(\mu)$.

Let

$$[a, I_\gamma]_\varepsilon f(x) = \int_{\varepsilon<d(x,y)} \frac{a(x) - a(y)}{[\lambda(x, d(x, y))]^{1-\gamma}} f(y)\mathrm{d}\mu(y), \quad x \in \mathcal{X}.$$

Then, for any $x \in \mathcal{X}$, we have

$$\begin{aligned}
&|[a, I_\gamma]_\varepsilon f(x) - [a, I_\gamma]f(x)| \\
&\le \int_{d(x,y)\le\varepsilon} \frac{|a(x) - a(y)|}{[\lambda(x, d(x, y))]^{1-\gamma}}|f(y)|\mathrm{d}\mu(y) \\
&\le C\varepsilon\|a'\|_{L^\infty(\mu)} I_\gamma(|f|)(x),
\end{aligned}$$

where a' represents the derivative of the function a. Thus, under the condition for the norm topology of $B(M_q^p(\mu), M_t^s(\mu))$, the following equation

$$\lim_{\varepsilon\downarrow 0}[a, I_\gamma]_\varepsilon = [a, I_\gamma]$$

holds.

Next, let us assume

$$[a, I_\gamma]_\varepsilon^R f(x) = \int_{\varepsilon<d(x,y)<R} \frac{a(x)-a(y)}{[\lambda(x, d(x,y))]^{1-\gamma}} f(y)\mathrm{d}\mu(y), \quad x \in \mathcal{X}.$$

Suppose that the support of function a is a given ball B_0, c_{B_0} and r_{B_0} are the center and radius of the ball B_0, respectively. Then, by applying the Hölder inequality, Definition 1.6 and (1.1), we have

$$\begin{aligned}
&\int_{d(x,y)\geq R} \frac{|f(y)|}{[\lambda(x, d(x,y))]^{1-\gamma}}\mathrm{d}\mu(y)\\
&\leq \sum_{k=1}^{\infty}\int_{6^{k-1}R\leq d(x,y)\leq 6^k R} \frac{|f(y)|}{[\lambda(x, d(x,y))]^{1-\gamma}}\mathrm{d}\mu(y)\\
&\leq C\sum_{k=1}^{\infty}\frac{1}{[\lambda(x, 6^{k-1}R)]^{1-\gamma}}\Big(\int_{d(x,y)\leq 6^k R}|f(y)|^q\mathrm{d}\mu(y)\Big)^{\frac{1}{q}}[\mu(B(x, 6^k R))]^{1-\frac{1}{q}}\\
&\leq C\|f\|_{M_q^p(\mu)}[\lambda(x,R)]^{-\frac{1}{p}+\gamma}\sum_{k=1}^{\infty}\frac{1}{[C(6^{k-1})]^{\frac{1}{q}-\gamma}}\\
&\leq C\|f\|_{M_q^p(\mu)}[\lambda(x,R)]^{\gamma-\frac{1}{p}},
\end{aligned}$$

further, we can obtain

$$\begin{aligned}
&|[a, I_\gamma]_\varepsilon f(x) - [a, I_\gamma]_\varepsilon^R f(x)|\\
&\leq \int_{d(x,y)\geq R}\frac{|a(x)-a(y)|}{[\lambda(x, d(x,y))]^{1-\gamma}}|f(y)|\mathrm{d}\mu(y)\\
&\leq C\|f\|_{M_q^p(\mu)}\|a\|_{L^\infty(\mu)}\frac{\chi_{B_0}(x)}{[\lambda(x,R)]^{\frac{1}{p}-\gamma}} + \|a\|_{L^\infty(\mu)}\int_{d(x,y)\geq R}\frac{\chi_{B_0}(y)}{[\lambda(x, d(x,y))]^{1-\gamma}}|f(y)|\mathrm{d}\mu(y).
\end{aligned}$$

Notice that

$$\begin{aligned}
&\Big[\int_{\mathcal{X}}\Big(\int_{d(x,y)\geq R}\frac{\chi_{B_0}(y)}{[\lambda(x, d(x,y))]^{1-\gamma}}|f(y)|\mathrm{d}\mu(y)\Big)^s\mathrm{d}\mu(x)\Big]^{\frac{1}{s}}\\
&\leq \int_{B_0}|f(y)|\Big(\sum_{k=1}^{\infty}\int_{6^{k-1}R\leq d(x,y)\leq 6^k R}\frac{1}{[\lambda(x, d(x,y))]^{(1-\gamma)s}}\mathrm{d}\mu(x)\Big)^{\frac{1}{s}}\mathrm{d}\mu(y)\\
&\leq C\int_{B_0}|f(y)|\Big(\sum_{k=1}^{\infty}\frac{\mu(B(c_{B_0}, 6^k R))}{[\lambda(c_{B_0}, 6^{k-1}R)]^{(1-\gamma)s}}\Big)^{\frac{1}{s}}\mathrm{d}\mu(y)\\
&\leq C\sum_{k=1}^{\infty}\frac{[\mu(B(c_{B_0}, 6^k R))]^{\frac{1}{s}}}{[\lambda(c_{B_0}, 6^{k-1}R)]^{1-\gamma}}\Big(\int_{B_0}|f(y)|^q\mathrm{d}\mu(y)\Big)^{\frac{1}{q}}[\mu(B_0)]^{1-\frac{1}{q}}\\
&\leq C\|f\|_{M_q^p(\mu)}\Big(\sum_{k=1}^{\infty}\frac{[\mu(B(c_{B_0}, 6^k R))]^{\frac{1}{s}}}{[\lambda(c_{B_0}, 6^{k-1}R)]^{1-\gamma}}\Big)[\mu(B_0)]^{1-\frac{1}{p}}
\end{aligned}$$

$$\leq C\|f\|_{M_q^p(\mu)}\Big(\sum_{k=1}^{\infty}\frac{[\lambda(c_{B_0},6^kR)]^{\frac{1}{p}}}{[\lambda(c_{B_0},6^{k-1}R)]^{1+\beta}}\Big)[\lambda(c_{B_0},r_{B_0})]^{1-\frac{1}{p}}$$
$$\leq \frac{C}{[\lambda(c_{B_0},R)]^{\frac{1}{p'}+\beta}}\|f\|_{M_q^p(\mu)},$$

which yields

$$\|[a,I_\gamma]_\varepsilon-[a,I_\gamma]_\varepsilon^R\|_{M_q^p(\mu)\to M_t^s(\mu)}=o(R^{-\tau})$$

for some $\tau>0$. Therefore, we only need to show that $[a,I_\gamma]_\varepsilon^R$ is compact. The integral kernel of K_ε^R is defined by

$$K_\varepsilon^R(x,y)=\frac{a(x)-a(y)}{[\lambda(x,d(x,y))]^{1-\gamma}}\chi_{\{\varepsilon<d(x,y)<R\}}(x,y),\ \text{for all}\ x,y\in\mathcal{X},$$

and K_ε^R is in $L_c^\infty(\mu)$. Thus, via the Lemma 2.4, we see that $[a,I_\gamma]_\varepsilon^R$ is compact. □

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