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Best Proximity Point Theorems for p-Proximal α-η-β-Quasi Contractions in Metric Spaces with w_0-Distance

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Abstract In this paper, we propose a new class of non-self mappings called p-proximal α-η-β-quasi contraction, and introduce the concepts of α-proximal admissible mapping with respect to η and (α, d) regular mapping with respect to η. Based on these new notions, we study the existence and uniqueness of best proximity point for this kind of new contractions in metric spaces with w_0-distance and obtain a new theorem, which generalize and complement the results in [Ayari, M. I. et al. Fixed Point Theory Appl., 2017, 2017: 16] and [Ayari, M. I. et al. Fixed Point Theory Appl., 2019, 2019: 7]. We give an example to show the validity of our main result. Moreover, we obtain several consequences concerning about best proximity point and common fixed point results for two mappings, and we present an application of a corollary to discuss the solutions to a class of systems of Volterra type integral equations.

Keywords best proximity point; p-proximal α-η-β quasi contraction; w_0-distance; α-proximal admissible mapping with respect to η; (α, d) regular mapping with respect to η

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1. Introduction

It was Banach who proposed the first fixed point result for a self mapping in the framework of a metric space in 1922 which was called Banach's contraction principle. From then on, many efforts have been devoted to studying fixed points in the setting of different spaces and for various classes of mappings [1–5]. The concept of a best proximity point for non-self mappings in a metric space has been put forward by Basha [6], and several best proximity point theorems have been derived by proposing sufficient conditions to guarantee the existence and uniqueness of best proximity points in recent years [6–12].

In 2012, the notions of α-ψ-contractive and α-admissible mappings have been introduced to assure the existence and uniqueness of fixed points in complete metric spaces [13]. α-admissible mappings and β-admissible mappings in Menger PM-spaces have been defined to obtain some fixed point results [14]. Wu et al. have introduced the new notions of α-admissible mappings with respect to η in single-valued and set-valued cases in Menger PM-spaces [15], in order to

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study the existence of fixed points under certain contractive conditions. Fixed point theorems for generalized α-ψ-contractive type mappings have also been formulated [16].

On the other hand, best proximity point problems for generalized α-ψ-contraction have been extensively investigated by many authors [17, 18]. A new kind of contraction called generalized α-β-proximal quasi contraction has been introduced and some new best proximity point results have been proved [19]. Recently, a new theorem on the existence and uniqueness of best proximity points for proximal β-quasi contractions for non-self mappings $S : M \to N$ and $T : N \to M$ has been presented [20], which generalize the results in [21]. As a consequence, an analogous result on proximal quasi contractions has been obtained which was first introduced by Jleli and Samet [22].

Recently, Kostić et al. [23] have put forward a new concept called w_0-distance, which is a special type of w-distance (for more details on w-distance [24–26]), and extend some theorems of Tchier et al. in [27] involving best proximity points and simulation functions.

In this paper, inspired by [19] and [20], we introduce the new concepts of p-proximal α-η-β-quasi contraction, α-proximal admissible mappings with respect to η and (α, d) regular mappings with respect to η, which are more general than the ones presented in [12,19,20]. We then establish some new best proximity point theorems in metric spaces with a w_0-distance, which extend and complement the results of [19] and [20] in metric spaces, and also generalize the main results in [22]. Some best proximity point and common fixed point results are also obtained as easy consequences. We provide an example and an application to illustrate the validity of the obtained results.

2. Preliminaries

Throughout this paper, let (M, N) be a pair of nonempty subsets of a metric space (X, d). We adopt the following notations:

$$d(M,N) := \inf\{d(m,n) : m \in M, n \in N\};$$
$$M_0 := \{m \in M : \text{there exists } n \in N \text{ such that } d(m,n) = d(M,N)\};$$
$$N_0 := \{n \in N : \text{there exists } m \in M \text{ such that } d(m,n) = d(M,N)\}.$$

Definition 2.1 ([6]) *Let $S : M \to N$ be a non-self mapping. An element $a^* \in M$ is said to be a best proximity point of S if $d(a^*, Sa^*) = d(M,N)$.*

Note that if $M = N$, a best proximity point of a non-self mapping reduces to a fixed point of a self mapping.

Definition 2.2 ([21]) *Let $S : M \to N$ and $T : N \to M$ be two non-self mappings. (S,T) is said to be a proximal cyclic contraction, if there exists a non-negative number $c < 1$ such that $d(u, Sa) = d(M,N)$ and $d(v, Tb) = d(M,N)$ imply $d(u,v) \le cd(a,b) + (1-c)d(M,N)$ for all $u, a \in M$ and $v, b \in N$.*

Definition 2.3 ([17]) *Let $T : M \to N$ be a non-self mapping and $\alpha : M \times M \to [0, \infty)$*

be a functional. We say that T is α-proximal admissible, if $\alpha(x_1, x_2) \geq 1$ and $d(u_1, Tx_1) = d(u_2, Tx_2) = d(M, N)$ imply $\alpha(u_1, u_2) \geq 1$ for all $x_1, x_2, u_1, u_2 \in M$.

Definition 2.4 ([28]) *Let $\beta \in (0, \infty)$. A β-comparison function is a map $\varphi : [0, \infty) \to [0, \infty)$ fulfilling the following properties:*

(1) φ is nondecreasing;

(2) $\lim_{n\to\infty} \varphi_\beta^n(t) = 0$ for all $t > 0$, where φ_β^n denotes the nth iteration of φ_β and $\varphi_\beta(t) = \varphi(\beta t)$;

(3) There exists $s \in (0, \infty)$ such that $\sum_{n=1}^{\infty} \varphi_\beta^n(s) < \infty$.

We denote by Φ_β the set of all β-comparison functions φ satisfying (1)–(3) in Definition 2.4. It is easy to see that such class of functions have the following property.

Remark 2.5 ([19]) Let $\alpha, \beta \in (0, \infty)$. If $\alpha < \beta$, then $\Phi_\beta \subset \Phi_\alpha$.

The next lemma is very useful in the proof of the main result of this paper.

Lemma 2.6 ([28]) *Let $\beta \in (0, \infty)$ and $\varphi \in \Phi_\beta$. Then*

(1) φ_β is nondecreasing;

(2) $\varphi_\beta(t) < t$ for all $t > 0$;

(3) $\sum_{n=1}^{\infty} \varphi_\beta^n(t) < \infty$ for all $t > 0$.

Definition 2.7 ([20]) *Let $\beta \in (0, \infty)$. A non-self mapping $T : M \to N$ is said to be a proximal β-quasi contraction if and only if there exist $\varphi \in \Phi_\beta$ and $\alpha_i > 0$ $(i = 0, 1, 2, 3, 4)$ such that*

$$d(u, v) \leq \varphi(\max\{\alpha_0 d(a, b), \alpha_1 d(a, u), \alpha_2 d(b, v), \alpha_3 d(a, v), \alpha_4 d(b, u)\})$$

for all $a, b, u, v \in M$ satisfying $d(u, Ta) = d(M, N)$ and $d(v, Tb) = d(M, N)$.

Definition 2.8 ([17]) *Let $T : M \to N$ be a non-self mapping and $\alpha : M \times M \to [0, \infty)$ be a functional. T is said to be (α, d) regular, if for all (x, y) such that $0 \leq \alpha(x, y) < 1$, there exists $u_0 \in M_0$ such that*

$$\alpha(x, u_0) \geq 1 \text{ and } \alpha(y, u_0) \geq 1.$$

Definition 2.9 ([23]) *Let X be a metric space with metric d. Then a function $p : X \times X \to [0, \infty)$ is called a w_0-distance on X if the following are satisfied:*

(P_1) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$;

(P_2) For any $x \in X$, functions $p(x, \cdot), p(\cdot, x) : X \to [0, \infty)$ are lower semi-continuous;

(P_3) For any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Remark 2.10 ([23]) Note that the concept of w_0-distance proposed in Definition 2.9 is more general than the standard concept of a metric, but less general than the one of a w-distance.

Recall that a real-valued function f defined on a metric space X is said to be lower semi-continuous at a point $x_0 \in X$ if either $\liminf_{x_n\to x_0} f(x_n) = \infty$ or $f(x_0) \leq \liminf_{x_n\to x_0} f(x_n)$, whenever $x_n \in X$ and $x_n \to x_0$.

Remark 2.11 ([23]) Let (X, d) be a metric space, $p : X \times X \to [0, \infty)$ a w_0-distance on X, and let M and N be two non-empty subsets of X (which need not be equal). Also, for every $x, y \in X$ let $\mu(x, y) := \max\{p(x, y), p(y, x)\}$. It is easy to know that the function $\mu : X \times X \to [0, \infty)$ has the following properties (for all $x, y, z \in X$)

(1) $\mu(x, y) = 0 \Rightarrow x = y$;

(2) $\mu(x, y) = \mu(y, x)$, i.e., μ is symmetric;

(3) $\mu(x, y) \leq \mu(x, z) + \mu(z, y)$, i.e., μ satisfies the triangle inequality.

The following lemma will also play an important role in proving our main result, which has been given in [26].

Lemma 2.12 ([26]) *Let X be a metric space d and let p be a w-distance on X, let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to 0, and let $x, y, z \in X$. Then the following hold:*

(i) If $p(x_n, y\} \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$;

(ii) If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then y_n converges to z;

(iii) If $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;

(iv) If $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

3. Main results

In this section, we shall present and prove the main results of this paper. Now, we start this section by introducing some new concepts. We first propose the following notion.

Definition 3.1 *Let M and N be two non-empty closed subsets of a complete metric space (X, d), $T : M \to N$ be a non-self mapping and $\alpha, \eta : M \times M \to [0, \infty)$ be two functionals. We say that T is α-proximal admissible with respect to η, if $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ and $d(u_1, Tx_1) = d(u_2, Tx_2) = d(M, N)$ imply $\alpha(u_1, u_2) \geq \eta(u_1, u_2)$ for all $x_1, x_2, u_1, u_2 \in M$.*

Note that Definition 3.1 reduces to Definition 2.3 in Section 2 when $\eta(x, y) = 1$ for all $x, y \in M$. So the concept of α-proximal admissible mapping with respect to η (Definition 3.1) is more general than the one of α-proximal admissible mapping (Definition 2.3).

Motivated by Definitions 2.7 and 2.8, we give the following two definitions, which will be used in the proof of the main results.

Definition 3.2 *Let M and N be two non-empty subsets of a complete metric space (X, d) with a w_0-distance p, and $\beta \in (0, \infty)$. A non-self mapping $T : M \to N$ is said to be a p-proximal α-η-β-quasi contraction, if there exist $\alpha, \eta : M \times M \to [0, \infty)$, $\varphi \in \Phi_\beta$ and $\alpha_i > 0$ $(i = 0, 1, 2, 3, 4)$, such that*

$$\alpha(a, b)\mu(u, v) \leq \eta(a, b)\varphi(\max\{\alpha_0\mu(a, b), \alpha_1\mu(a, u), \alpha_2\mu(b, v), \alpha_3\mu(a, v), \alpha_4\mu(b, u)\})$$

for all $a, b, u, v \in M$ satisfying $d(u, Ta) = d(M, N)$ and $d(v, Tb) = d(M, N)$.

Definition 3.3 *Let M and N be two non-empty subsets of a complete metric space (X, d),*

$T : M \to N$ be a non-self mapping, and $\alpha, \eta : M \times M \to [0, \infty)$ be two functionals. T is said to be (α, d) regular with respect to η, if for all (x, y) such that $0 \le \alpha(x, y) < \eta(x, y)$, there exists $u_0 \in M_0$ such that

$$\alpha(x, u_0) \ge \eta(x, u_0) \quad \text{and} \quad \alpha(y, u_0) \ge \eta(y, u_0).$$

Note that Definition 3.3 reduces to [17, Definition 16] when $\eta(x, y) = 1$ for all $x, y \in M$. We are now ready to give the main result of this paper.

Theorem 3.4 *Let (M, N) be a pair of non-empty closed subsets of a complete metric space (X, d) with a w_0-distance p, such that M_0 and N_0 are non-empty. Let $\alpha, \eta : M \times M \to [0, \infty)$ and $\alpha', \eta' : N \times N \to [0, \infty)$ be four functionals. Let $S : M \to N$ and $T : N \to M$ be two non-self mappings satisfying the following conditions:*

(C_1) $S(M_0) \subset N_0$ and $T(N_0) \subset M_0$;

(C_2) S is α-proximal admissible with respect to η, and T is α'-proximal admissible with respect to η';

(C_3) S is (α, d) regular with respect to η, and T is (α', d) regular with respect to η';

(C_4) There exist elements $a_0, a_1 \in M$ such that $d(a_1, Sa_0) = d(M, N)$ and $\alpha(a_0, a_1) \ge \eta(a_0, a_1)$, there exist elements $b_0, b_1 \in N$ such that $d(b_1, Tb_0) = d(M, N)$ and $\alpha'(b_0, b_1) \ge \eta'(b_0, b_1)$;

(C_5) There exist $\beta_1, \beta_2 \ge \max\{\alpha_0, 2\alpha_1, 2\alpha_2, 2\alpha_3, 2\alpha_4\}$ such that S is a p-proximal α-η-β_1-quasi contraction (say, $\psi \in \Phi_{\beta_1}$) and T is a p-proximal α'-η'-β_2-quasi contraction (say, $\phi \in \Phi_{\beta_2}$);

(C_6) (S, T) is a proximal cyclic contraction;

(C_7) If $\{a_n\}$ is a sequence in M such that $\alpha(a_n, a_{n+1}) \ge \eta(a_n, a_{n+1})$ and $\lim_{n\to\infty} a_n = a_* \in M$, then there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\alpha(a_{n_k}, a_*) \ge \eta(a_{n_k}, a_*)$ for all k; if $\{b_n\}$ is a sequence in N such that $\alpha'(b_n, b_{n+1}) \ge \eta'(b_n, b_{n+1})$ and $\lim_{n\to\infty} b_n = b_* \in N$, then there exists a subsequence $\{b_{m_k}\}$ of $\{b_n\}$ such that $\alpha'(b_{m_k}, b_*) \ge \eta'(b_{m_k}, b_*)$ for all k.

(C_8) One of the following two assertions holds:

(i) ψ and ϕ are continuous;

(ii) $\beta_1, \beta_2 > \max\{\alpha_0, 2\alpha_1, 2\alpha_2, \alpha_3, \alpha_4\}$.

Then S has a unique best proximity point $a_* \in M$ and T has a unique best proximity point $b_* \in N$. Moreover, the best proximity points satisfy $d(a_*, b_*) = d(M, N)$.

Proof By condition (C_4), there exist $a_0, a_1 \in M$ such that

$$d(a_1, Sa_0) = d(M, N) \quad \text{and} \quad \alpha(a_0, a_1) \ge \eta(a_0, a_1).$$

Since $S(M_0) \subset N_0$, there exists $a_2 \in M_0$ such that $d(a_2, Sa_1) = d(M, N)$. Since S is α-proximal admissible with respect to η, noting that $\alpha(a_0, a_1) \ge \eta(a_0, a_1)$ and $d(a_1, Sa_0) = d(a_2, Sa_1) = d(M, N)$, we get

$$d(a_2, Sa_1) = d(M, N) \quad \text{and} \quad \alpha(a_1, a_2) \ge \eta(a_1, a_2).$$

Continuing this process, for $a_n \in M_0$, we can find $a_{n+1} \in M_0$ such that

$$d(a_{n+1}, Sa_n) = d(M, N) \quad \text{and} \quad \alpha(a_n, a_{n+1}) \ge \eta(a_n, a_{n+1}) \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}. \tag{3.1}$$

If $a_n = a_{n+1}$, we get $d(a_n, Sa_n) = d(M, N)$, i.e., a_n is a best proximity point of S, which completes the proof. □

Hence, we can assume that $\mu(a_{n-1}, a_n) > 0$ for all $n \in \mathbb{N}$. Since S is p-proximal α-η-β_1-quasi contraction for $\psi \in \Phi_{\beta_1}$, and $d(a_{n+1}, Sa_n) = d(a_n, Sa_{n-1}) = d(M, N)$, then by Definition 3.2 we have

$$\begin{aligned}
&\alpha(a_n, a_{n-1})\mu(a_{n+1}, a_n) \\
&\quad \le \eta(a_n, a_{n-1})\psi(\max\{\alpha_0\mu(a_n, a_{n-1}), \alpha_1\mu(a_n, a_{n+1}), \alpha_2\mu(a_n, a_{n-1}), \\
&\qquad \alpha_3\mu(a_n, a_n), \alpha_4\mu(a_{n+1}, a_{n-1})\}).
\end{aligned} \tag{3.2}$$

Note that from (3.1) and (3.2), we have

$$\begin{aligned}
&\mu(a_{n+1}, a_n) \\
&\quad \le \psi(\max\{\alpha_0\mu(a_n, a_{n-1}), \alpha_1\mu(a_n, a_{n+1}), \alpha_2\mu(a_n, a_{n-1}), \alpha_3\mu(a_n, a_n), \alpha_4\mu(a_{n+1}, a_{n-1})\}) \\
&\quad \le \psi(\max\{\alpha_0\mu(a_n, a_{n-1}), \alpha_1\mu(a_n, a_{n+1}), \alpha_2\mu(a_n, a_{n-1}), 2\alpha_3\mu(a_{n+1}, a_n), \\
&\qquad \alpha_4\mu(a_{n+1}, a_n) + \alpha_4\mu(a_{n-1}, a_n)\}) \\
&\quad \le \psi(\max\{\alpha_0\mu(a_n, a_{n-1}), \alpha_1\mu(a_n, a_{n+1}), \alpha_2\mu(a_n, a_{n-1}), 2\alpha_3\mu(a_{n+1}, a_n), \\
&\qquad 2\alpha_4 \max\{\mu(a_{n+1}, a_n), \mu(a_{n-1}, a_n)\}\}) \\
&\quad \le \psi(\beta_1 \max\{\mu(a_{n+1}, a_n), \mu(a_{n-1}, a_n)\}) \\
&\quad = \psi_{\beta_1}(\max\{\mu(a_{n+1}, a_n), \mu(a_{n-1}, a_n)\}).
\end{aligned} \tag{3.3}$$

Now, if $\max\{\mu(a_{n+1}, a_n), \mu(a_{n-1}, a_n)\} = \mu(a_{n+1}, a_n)$, then by Lemma 2.6, it follows from (3.3) that

$$\mu(a_{n+1}, a_n) \le \psi_{\beta_1}(\mu(a_{n+1}, a_n)) < \mu(a_{n+1}, a_n),$$

which is a contradiction. Thus, $\max\{\mu(a_{n+1}, a_n), \mu(a_{n-1}, a_n)\} = \mu(a_{n-1}, a_n)$, and we have

$$\mu(a_{n+1}, a_n) \le \psi_{\beta_1}(\mu(a_{n-1}, a_n)).$$

By applying induction on n and from Lemma 2.6, we obtain that

$$\mu(a_{n+1}, a_n) \le \psi_{\beta_1}^n(\mu(a_1, a_0)) < \infty, \quad \forall n \ge 1. \tag{3.4}$$

Using the triangle inequality and (3.4), for integers $n < m$, we get

$$\mu(a_n, a_m) \le \sum_{k=n}^{m-1} \mu(a_k, a_{k+1}) \le \sum_{k=n}^{m-1} \psi_{\beta_1}^k(\mu(a_1, a_0)),$$

which implies that $\mu(a_n, a_m) \to 0$ as $n, m \to \infty$. From Remark 2.11, we get $\lim_{n,m\to\infty} p(a_n, a_m) = 0$ and $\lim_{n,m\to\infty} p(a_m, a_n) = 0$. i.e., for any $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$, such that for all $m, n > N_1$, we have $p(a_n, a_{n+1}) \le \epsilon$ and $p(a_m, a_{n+1}) \le \epsilon$. By (P_3) of Definition 2.9, we obtain

$$d(a_n, a_m) \le \epsilon \quad \text{when} \quad m, n > N_1.$$

Thus, the sequence $\{a_n\}$ is a Cauchy sequence in M. Noting that M is a closed subset of a complete metric space (X, d), the sequence $\{a_n\}$ converges to some element $a_* \in M$.

Since $T(N_0) \subset M_0$, by using a similar argument as above, there exists a sequence $\{b_n\} \subset N_0$ such that $d(b_{n+1}, Tb_n) = d(M, N)$ and $\alpha'(b_n, b_{n+1}) \geq \eta'(b_n, b_{n+1})$ for each $n \in \mathbb{N}\cup\{0\}$. Since T is a p-proximal α'-η'-β_2-quasi contraction for $\phi \in \Phi_{\beta_2}$ and $d(b_{n+1}, Tb_n) = d(b_n, Tb_{n-1}) = d(M, N)$, we deduce from Definition 3.2 that

$$\begin{aligned}
&\alpha'(b_n, b_{n-1})\mu(b_{n+1}, b_n) \\
&\quad \leq \eta'(b_n, b_{n-1})\phi(\max\{\alpha_0\mu(b_n, b_{n-1}), \alpha_1\mu(b_n, b_{n+1}), \alpha_2\mu(b_n, b_{n-1}), \alpha_3\mu(b_n, b_n), \alpha_4\mu(b_{n+1}, b_{n-1})\}).
\end{aligned}$$

Noting that $\alpha'(b_n, b_{n+1}) \geq \eta'(b_n, b_{n+1})$, we get

$$\begin{aligned}
&\mu(b_{n+1}, b_n) \\
&\quad \leq \phi(\max\{\alpha_0\mu(b_n, b_{n-1}), \alpha_1\mu(b_n, b_{n+1}), \alpha_2\mu(b_n, b_{n-1}), \alpha_3\mu(b_n, b_n), \alpha_4\mu(b_{n+1}, b_{n-1})\}) \\
&\quad \leq \phi(\max\{\alpha_0\mu(b_n, b_{n-1}), \alpha_1\mu(b_n, b_{n+1}), \alpha_2\mu(b_n, b_{n-1}), 2\alpha_3\mu(b_{n+1}, b_n), \\
&\qquad \alpha_4\mu(b_{n+1}, b_n) + \alpha_4\mu(b_{n-1}, b_n)\}) \\
&\quad \leq \phi(\max\{\alpha_0\mu(b_n, b_{n-1}), \alpha_1\mu(b_n, b_{n+1}), \alpha_2\mu(b_n, b_{n-1}), 2\alpha_3\mu(b_{n+1}, b_n), \\
&\qquad 2\alpha_4 \max\{\mu(b_{n+1}, b_n), \mu(b_{n-1}, b_n)\}\}) \\
&\quad \leq \phi(\beta_2 \max\{\mu(b_{n+1}, b_n), \mu(b_{n-1}, b_n)\}) \\
&\quad = \phi_{\beta_2}(\max\{\mu(b_{n+1}, b_n), \mu(b_{n-1}, b_n)\}).
\end{aligned}$$

Similarly, we deduce that $\{b_n\}$ converges to some $b_* \in N$.

Now we prove that a_* and b_* are best proximal points of S and T, respectively. By condition (C_6), the pair (S, T) is a proximal cyclic contraction, so we have

$$d(a_{n+1}, b_{n+1}) \leq cd(a_n, b_n) + (1 - c)d(M, N), \quad 0 \leq c < 1. \tag{3.5}$$

Taking the limit as $n \to \infty$, it immediately follows from (3.5) that $d(a_*, b_*) \leq cd(a_*, b_*) + (1 - c)d(M, N)$, which yields that

$$d(a_*, b_*) \leq d(M, N). \tag{3.6}$$

Combining the fact that $d(M, N) \leq d(a_*, b_*)$ and (3.6), we get $d(a_*, b_*) = d(M, N)$. Thus, we conclude that $a_* \in M_0$ and $b_* \in N_0$.

Since $S(M_0) \subset N_0$ and $T(N_0) \subset M_0$, there exist $u \in M$ and $v \in N$ such that

$$d(u, Sa_*) = d(v, Tb_*) = d(M, N). \tag{3.7}$$

On the other hand, since S is p-proximal α-η-β_1-quasi contraction and $\psi \in \Phi_{\beta_1}$, by Definition 3.2 and (3.7), we deduce that

$$\begin{aligned}
&\alpha(a_{n_k}, a_*)\mu(a_{n_k+1}, u) \\
&\quad \leq \eta(a_{n_k}, a_*)\psi(\max\{\alpha_0\mu(a_{n_k}, a_*), \alpha_1\mu(a_{n_k}, a_{n_k+1}), \alpha_2\mu(a_*, u), \alpha_3\mu(a_{n_k}, u), \alpha_4\mu(a_*, a_{n_k+1})\}).
\end{aligned}$$

From (C_7), we get $\alpha(a_{n_k}, a_*) \geq \eta(a_{n_k}, a_*)$. Hence

$$\begin{aligned}
&\mu(a_{n_k+1}, u) \\
&\quad \leq \psi(\max\{\alpha_0\mu(a_{n_k}, a_*), \alpha_1\mu(a_{n_k}, a_{n_k+1}), \alpha_2\mu(a_*, u), \alpha_3\mu(a_{n_k}, u), \alpha_4\mu(a_*, a_{n_k+1})\}).
\end{aligned} \tag{3.8}$$

Now, denote

$$\rho = \mu(a_*, u)$$

and

$$A_{n_k} = \max\{\alpha_0\mu(a_{n_k}, a_*), \alpha_1\mu(a_{n_k}, a_{n_k+1}), \alpha_2\mu(a_*, u), \alpha_3\mu(a_{n_k}, u), \alpha_4\mu(a_*, a_{n_k+1})\}.$$

From the above argument, we know that $\lim_{n,m\to\infty} p(a_n, a_m) = 0$ and $\lim_{n,m\to\infty} p(a_m, a_n) = 0$. This means that for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$, such that $p(a_n, a_m) < \epsilon$ for all $m > n \geq N_\epsilon$. For a fixed $n \in \mathbb{N}$, the function $p(a_n, \cdot)$ is lower semi-continuous. So $p(a_n, a_*) \leq \liminf_{m\to\infty} p(a_n, a_m) < \epsilon$, and thus

$$\lim_{n\to\infty} p(a_n, a_*) = 0. \tag{3.9}$$

Similarly, we can derive that $\lim_{n\to\infty} p(a_*, a_n) = 0$, which combined with (3.9) yields

$$\lim_{n\to\infty} \mu(a_*, a_n) = 0.$$

Noting that

$$\mu(a_n, u) \leq \mu(a_n, a_*) + \mu(a_*, u)$$

and

$$\mu(a_*, u) \leq \mu(a_n, u) + \mu(a_n, a_*),$$

we get $\lim\limits_{n\to\infty} \mu(a_n, u) = \mu(a_*, u)$. Thus,

$$\lim_{k\to\infty} A_{n_k} = \max\{\alpha_2, \alpha_3\}\rho. \tag{3.10}$$

Now, we show that $\rho = 0$. Suppose that $\rho > 0$. From Definition 3.2 and (C_5), we get $\beta_1 > \max\{\alpha_2, \alpha_3\}$. Then there exist $\epsilon > 0$ and $K \in \mathbb{N}$, such that for all $k > K$, we have

$$A_{n_k} < (\max\{\alpha_2, \alpha_3\} + \epsilon)\rho, \quad \text{and} \quad \beta_1 > \max\{\alpha_2, \alpha_3\} + \epsilon.$$

Therefore, it follows from (3.8) that

$$\mu(a_{n_k+1}, u) \leq \psi(A_{n_k}) \leq \psi((\max\{\alpha_2, \alpha_3\} + \epsilon)\rho) = \psi_{\beta_1}\left(\frac{\max\{\alpha_2, \alpha_3\} + \epsilon}{\beta_1}\rho\right).$$

Since $\psi \in \Phi_{\beta_1}$, by Lemma 2.6 we get

$$\mu(a_{n_k+1}, u) < \frac{\max\{\alpha_2, \alpha_3\} + \epsilon}{\beta_1}\rho < \rho,$$

which is also a contradiction. Therefore, we have $\mu(a_*, u) = \rho = 0$. From Remark 2.11 we obtain that $a_* = u$, and so from (3.7) we get $d(a_*, Sa_*) = d(M, N)$, i.e., a_* is a best proximity point of S. A similar argument shows that $v = b_*$ and hence by (3.7), we know that $d(b_*, Tb_*) = d(M, N)$, i.e., b_* is a best proximity point of the non-self mapping T.

Next, we prove the uniqueness of the best proximity point. Suppose that a_* and x are two distinct best proximity points of T. Then $s = \mu(a_*, x) > 0$. Consider the following two cases:

Case 1. $\alpha(a_*, x) \geq \eta(a_*, x)$. Since S is p-proximal α-η-β_1-quasi contraction, by Definition 3.2 we get

$$\alpha(a_*, x)\mu(a_*, x) \leq \eta(a_*, x)\psi(\max\{\alpha_0\mu(a_*, x), \alpha_1\mu(x, x), \alpha_2\mu(a_*, a_*), \alpha_3\mu(a_*, x), \alpha_4\mu(a_*, x)\}).$$

Using the condition that $\alpha(a_*, x) \geq \eta(a_*, x)$, we have

$$\begin{aligned}\mu(a_*, x) &\leq \psi(\max\{\alpha_0\mu(a_*, x), \alpha_1\mu(x, x), \alpha_2\mu(a_*, a_*), \alpha_3\mu(a_*, x), \alpha_4\mu(a_*, x)\}) \\ &\leq \psi(\beta_1\mu(a_*, x)) = \psi_{\beta_1}(\mu(a_*, x)) < \mu(a_*, x),\end{aligned}$$

which is a contradiction. So $s = 0$ and thus $a_* = x$.

Case 2. $\alpha(a_*, x) < \eta(a_*, x)$. Since S is (α, d) regular with respect to η, there exists $u_0 \in M_0$ such that $\alpha(a_*, u_0) \geq \eta(a_*, u_0)$ and $\alpha(x, u_0) \geq \eta(x, u_0)$. On the one hand, $S(M_0) \subset N_0$, so there exists $u_1 \in M_0$ such that $d(u_1, Su_0) = d(M, N)$. On the other hand, since S is α-proximal admissible with respect to η, by using $\alpha(a_*, u_0) \geq \eta(a_*, u_0)$ and $d(a_*, Sa_*) = d(u_1, Su_0) = d(M, N)$, we get $\alpha(a_*, u_1) \geq \eta(a_*, u_1)$. In a similar fashion, we can find $u_n \in M_0$ such that

$$d(u_{n+1}, Su_n) = d(M, N) \quad \text{and} \quad \alpha(a_*, u_n) \geq \eta(a_*, u_n), \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\},$$

and we can also prove that $\{u_n\}$ is a Cauchy sequence in M. Assume that $\{u_n\}$ converges to $u_* \in M$. Using the fact that $d(a_*, Sa_*) = d(u_{n+1}, Su_n) = d(M, N)$ and S is p-proximal α-η-β_1-quasi contraction, by Definition 3.2 we get

$$\begin{aligned}&\alpha(a_*, u_n)\mu(a_*, u_{n+1}) \\ &\quad \leq \eta(a_*, u_n)\psi(\max\{\alpha_0\mu(a_*, u_n), \alpha_1\mu(a_*, a_*), \alpha_2\mu(u_n, u_{n+1}), \alpha_3\mu(a_*, u_{n+1}), \alpha_4\mu(u_n, a_*)\}).\end{aligned}$$

By induction, we know that $\alpha(a_*, u_n) \geq \eta(a_*, u_n)$. Therefore, we have

$$\begin{aligned}&\mu(a_*, u_{n+1}) \\ &\quad \leq \psi(\max\{\alpha_0\mu(a_*, u_n), \alpha_1\mu(a_*, a_*), \alpha_2\mu(u_n, u_{n+1}), \alpha_3\mu(a_*, u_{n+1}), \alpha_4\mu(u_n, a_*)\}).\end{aligned} \tag{3.11}$$

Next, denote $\varrho = \mu(a_*, u_*) > 0$ and

$$U_n = \max\{\alpha_0\mu(a_*, u_n), \alpha_1\mu(a_*, a_*), \alpha_2\mu(u_n, u_{n+1}), \alpha_3\mu(a_*, u_{n+1}), \alpha_4\mu(u_n, a_*)\}.$$

In a similar fashion we get $\lim_{n\to\infty}\mu(a_n, u) = \mu(a_*, u)$, and $\lim_{n\to\infty}\mu(a_*, u_n) = \mu(a_*, u_*)$. Therefore, using triangle inequality we get

$$\lim_{n\to\infty} U_n \leq \max\{\alpha_0, 2\alpha_1, 2\alpha_2, \alpha_3, \alpha_4\}\varrho. \tag{3.12}$$

First, consider the case where the assertion (i) of (C_8) is satisfied, that is, ψ is continuous. Then taking the limit as $n \to \infty$ in (3.11) and using (3.12) and Lemma 2.6, we obtain

$$\varrho \leq \psi(\beta_1\varrho) = \psi_{\beta_1}(\varrho) < \varrho,$$

which is a contradiction. Now, assume that assertion (ii) of (C_8) holds. Then there exist $\epsilon > 0$ and $N_2 \in \mathbb{N}$, such that for all $n > N_2$, we have

$$U_n < (\max\{\alpha_0, 2\alpha_1, 2\alpha_2, \alpha_3, \alpha_4\} + \epsilon)\varrho \quad \text{and} \quad \beta_1 > \max\{\alpha_0, 2\alpha_1, 2\alpha_2, \alpha_3, \alpha_4\} + \epsilon.$$

Therefore, it follows from (3.11) that

$$\begin{aligned}\mu(a_*, u_{n+1}) &\leq \psi(U_n) \leq \psi((\max\{\alpha_0, 2\alpha_1, 2\alpha_2, \alpha_3, \alpha_4\} + \epsilon)\varrho) \\ &= \psi_{\beta_1}\left(\frac{\max\{\alpha_0, 2\alpha_1, 2\alpha_2, \alpha_3, \alpha_4\} + \epsilon}{\beta_1}\varrho\right).\end{aligned}$$

Since $\psi \in \Phi_{\beta_1}$, by Lemma 2.6 we get

$$\mu(u_{n+1}, a_*) < \frac{\max\{\alpha_0, 2\alpha_1, 2\alpha_2, \alpha_3, \alpha_4\} + \epsilon}{\beta_1} \varrho < \varrho.$$

Letting $n \to \infty$ in the above inequality yields

$$\varrho \leq \frac{\max\{\alpha_0, 2\alpha_1, 2\alpha_2, \alpha_3, \alpha_4\} + \epsilon}{\beta_1} < \varrho,$$

which is a contradiction as well. Thus, we get $\varrho = \mu(a_*, u_*) = 0$. Analogously, we can prove that $x = u_*$. So, we get $a_* = x$. Therefore, the best proximity point a_* of S is unique. Similarly, the best proximity point of T is also unique. This completes the proof. $\square$

Example 3.5 Consider the space $X = \mathbb{R}^2$ endowed with the metric

$$d(x, y) = |x_1 - x_2| + |y_1 - y_2|, \quad \text{for all } \ x = (x_1, y_1), y = (x_2, y_2) \in X.$$

Then (X, d) is a complete metric space. Define p by

$$p(x, y) = |x_1 - x_2| + y_1 + y_2, \quad \text{for all } \ x = (x_1, y_1), y = (x_2, y_2) \in X.$$

Then p is a w_0-distance and we define $\mu(x, y) = \max\{p(x, y), p(y, x)\}$ for $x = (x_1, y_1), y = (x_2, y_2) \in X$.

Let us define

$$M := \{(0, \theta) \in X, 0 \leq \theta \leq 1\} \text{ and } \ N := \{(1, \xi) \in X, 0 \leq \xi \leq 1\}.$$

Clearly, (M, N) is a pair of closed subsets of (X, d) with $M_0 = M$, $N_0 = N$ and we have $d(M, N) = 1$.

Define non-self mappings $S : M \to N$ and $T : N \to M$ by

$$S(0, \theta) = (1, f(\theta)) \ \text{ and } \ T(1, \xi) = (0, g(\xi)),$$

where

$$f(\theta) = \frac{\theta}{4} \ \text{ for all } \ \theta \in [0, 1], \text{ and } \ g(\xi) = \frac{\xi}{5} \ \text{ for all } \ \xi \in [0, 1].$$

Then it is easy to see that $S(M_0) \subset S(M) \subset N = N_0$ and $T(N_0) \subset T(N) \subset M = M_0$. Therefore, (C_1) holds. Let

$$\alpha(x, y) = 3, \eta(x, y) = 2 \ \text{ for all } \ x = (x_1, y_1), y = (x_2, y_2) \in M,$$

and

$$\alpha'(x, y) = 3, \eta'(x, y) = 2 \ \text{ for all } \ x = (x_1, y_1), y = (x_2, y_2) \in N.$$

Then it is obvious that (C_2), (C_3) and (C_4) and (C_7) hold.

Let $\psi(t) = \frac{t}{8}$, $\phi(t) = \frac{t}{10}$, $\beta_1 = \beta_2 = 6$, $\alpha_0 = 4, \alpha_i = 1$ $(i = 1, 2, 3, 4)$. It is obvious that ψ and ϕ are both continuous, so (C_8) holds. Noting that $\psi_{\beta_1}(t) = \psi(\beta_1 t) = \frac{3}{4}t$ and $\phi_{\beta_2}(t) = \phi(\beta_2 t) = \frac{3}{5}t$, it can be shown that $\psi(t) = \frac{1}{8}t \in \Phi_6$ and $\phi(t) = \frac{1}{10}t \in \Phi_6$.

Now we shall prove that S is a p-proximal α-η-β_1-quasi contraction. In fact, let

$$u = (0, \theta), \ v = (0, \xi), \ a = (0, \gamma), \ b = (0, \delta) \in M$$

such that

$$d(u, Sa) = d(v, Sb) = d(M, N) = 1.$$

It is easy to show that

$$d(u, Sa) = 1 \Longleftrightarrow \theta = f(\gamma)$$

and

$$d(v, Tb) = 1 \Longleftrightarrow \xi = f(\delta).$$

Thus

$$\begin{aligned}\alpha(a,b)\mu(u,v) &= 3(\theta+\xi) = \frac{3}{4}(\gamma+\delta) = \frac{3}{4}\mu(a,b) \le \eta(a,b)\psi(\alpha_0\mu(a,b))\\ &\le \eta(a,b)\psi(\max\{\alpha_0\mu(a,b), \alpha_1\mu(a,u), \alpha_2\mu(b,v), \alpha_3\mu(a,v), \alpha_4\mu(b,v)\}).\end{aligned}$$

So S is a p-proximal α-η-β_1-quasi contraction.

Similarly, for $a, b, u, v \in N$, $u = (1, \theta)$, $v = (1, \xi)$, $a = (1, \gamma)$, $b = (1, \delta) \in N$ and $d(u, Ta) = d(M, N) = 1 = d(v, Tb)$ implies $\theta = g(\gamma)$, $\xi = g(\delta)$.

$$\begin{aligned}\alpha'(a,b)\mu(u,v) &= 3(\theta+\xi) = \frac{3}{5}(\gamma+\delta) = \frac{3}{5}\mu(a,b) \le \eta'(a,b)\phi(\alpha_0\mu(a,b))\\ &\le \eta'(a,b)\phi(\max\{\alpha_0\mu(a,b), \alpha_1\mu(a,u), \alpha_2\mu(b,v), \alpha_3\mu(a,v), \alpha_4\mu(b,v)\}).\end{aligned}$$

So T is a p-proximal α'-η'-β_2-quasi contraction. Therefore, (C_5) holds.

Since $d(u, Sa) = d(M, N)$ and $d(v, Tb) = d(M, N)$ implies that $u = (0, \theta)$, $a = (0, \gamma) \in M$ and $v = (1, \xi)$, $b = (1, \delta) \in N$, respectively, we have $\theta = f(\gamma) = \frac{\gamma}{4}$ and $\xi = g(\delta) = \frac{\delta}{5}$, and thus

$$\begin{aligned}d(u,v) &= 1 + |\theta - \xi| = 1 + |\frac{\gamma}{4} - \frac{\delta}{5}|\\ &\le \frac{1}{4}[1 + |\gamma - \delta|] + 1 - \frac{1}{4} = \frac{1}{4}d(a,b) + (1 - \frac{1}{4})d(M,N).\end{aligned}$$

So (S, T) is a proximal cyclic contraction, which means that (C_6) holds.

By Theorem 3.4, S has a unique best proximity point a_* in M and T has a unique best proximity point b_* in N, and $d(a_*, b_*) = d(M, N)$. In our example, $a_* = (0, 0) \in M$ and $b_* = (1, 0) \in N$ are the unique best proximity points of S and T, respectively, and $d(a_*, b_*) = d((0, 0), (1, 0)) = 1 = d(M, N)$. Note that our result Theorem 3.4 can solve the problem in this example, but the results in other literatures before (e.g., Theorem 4.1 in [22], Theorem 3.2 in [12] and Theorem 3.1 in [19]) cannot solve it.

4. Consequent results

In this section, we derive some results as consequences of Theorem 3.4. First, setting $\alpha(x, y) = \eta(x, y) = 1$ for all $x, y \in M$ in Definition 3.2, we get the following definition.

Definition 4.1 *Let M and N be two non-empty subsets of a complete metric space (X, d) with a w_0-distance p and $\beta \in (0, \infty)$. A non-self mapping $T : M \to N$ is said to be a p-proximal β-quasi contraction, if there exist $\varphi \in \Phi_\beta$ and $\alpha_i > 0$ $(i = 0, 1, 2, 3, 4)$ such that*

$$\mu(u, v) \le \varphi(\max\{\alpha_0\mu(a, b), \alpha_1\mu(a, u), \alpha_2\mu(b, v), \alpha_3\mu(a, v), \alpha_4\mu(b, u)\})$$

for all $a, b, u, v \in M$ satisfying $d(u, Ta) = d(M, N)$ and $d(v, Tb) = d(M, N)$.

Based on this definition, we can get the following corollary which follows immediately from Theorem 3.4 by taking $\alpha(x, y) = \eta(x, y) = 1$ for all $x, y \in M$ and $\alpha'(s, t) = \eta'(s, t) = 1$ for all $s, t \in N$.

Corollary 4.2 *Let (M, N) be a pair of non-empty closed subsets of a complete metric space (X, d) with a w_0-distance p, such that M_0 and N_0 are non-empty. Let $S : M \to N$ and $T : N \to M$ be two non-self mappings satisfying the following conditions:*

(D_1) $S(M_0) \subset N_0$ and $T(N_0) \subset M_0$;

(D_2) There exist $\beta_1, \beta_2 \geq \max\{\alpha_0, 2\alpha_1, 2\alpha_2, 2\alpha_3, 2\alpha_4\}$ such that S is a p-proximal β_1-quasi contraction (say, $\psi \in \Phi_{\beta_1}$) and T is a p-proximal β_2-quasi contraction (say, $\phi \in \Phi_{\beta_2}$);

(D_3) (S, T) is a proximal cyclic contraction;

(D_4) One of the following two assertions holds:

(i) ψ and ϕ are continuous;

(ii) $\beta_1, \beta_2 > \max\{\alpha_0, 2\alpha_1, 2\alpha_2, \alpha_3, \alpha_4\}$.

Then S has a unique best proximity point $a_* \in M$ and T has a unique best proximity point $b_* \in N$. Moreover, the best proximity points satisfy $d(a_*, b_*) = d(M, N)$.

A special case of Definition 4.1 is the following one.

Definition 4.3 *Let M and N be two non-empty subsets of a complete metric space (X, d) with a w_0-distance p. A non-self mapping $T : M \to N$ is said to be p-proximal quasi contraction, if there exists $q \in [0, 1)$ and $\alpha_i > 0$ $(i = 0, 1, 2, 3, 4)$ such that*

$$\mu(u, v) \leq q \max\{\mu(a, b), \mu(a, u), \mu(b, v), \mu(a, v), \mu(b, u)\}$$

for all $a, b, u, v \in M$ satisfying $d(u, Ta) = d(M, N)$ and $d(v, Tb) = d(M, N)$.

By taking $\alpha_0 = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2}$ and $\psi(t) = \phi(t) = qt$ for $q \in [0, 1)$, we obtain the following result, which is a generalization in [21, Corollary 3.3] to a metric space with a w_0-distance.

Corollary 4.4 *Let (M, N) be a pair of non-empty closed subsets of a complete metric space (X, d) with a w_0-distance p, such that M_0 and N_0 are non-empty. Let $S : M \to N$ and $T : N \to M$ be two non-self mappings satisfying the following conditions:*

(E_1) $S(M_0) \subset N_0$ and $T(N_0) \subset M_0$;

(E_2) S and T are p-proximal quasi contractions;

(E_3) (S, T) is a proximal cyclic contraction.

Then S has a unique best proximity point $a_* \in M$ and T has a unique best proximity point $b_* \in N$. Moreover, the best proximity points satisfy $d(a_*, b_*) = d(M, N)$.

In particular, setting $M = N = X$ in Definitions 4.1 and 4.3, respectively, we have the following two definitions.

Definition 4.5 *Let (X, d) be a complete metric space with a w_0-distance p and $\beta \in (0, \infty)$. A self mapping $T : X \to X$ is said to be a p-β-quasi contraction, if there exist $\varphi \in \Phi_\beta$ and*

$\alpha_i > 0$ $(i = 0, 1, 2, 3, 4)$ *such that*

$$\mu(Ta, Tb) \leq \varphi(\max\{\alpha_0\mu(a, b), \alpha_1\mu(a, Ta), \alpha_2\mu(b, Tb), \alpha_3\mu(a, Tb), \alpha_4\mu(b, Ta)\})$$

for all $a, b \in X$.

Definition 4.6 *Let* (X, d) *be a complete metric space with a* w_0*-distance* p*. A self mapping* $T : X \to X$ *is said to be a* p*-quasi contraction, if there exists* $q \in [0, 1)$ *such that*

$$\mu(Ta, Tb) \leq q \max\{\mu(a, b), \mu(a, Ta), \mu(b, Tb), \mu(a, Tb), \mu(b, Ta)\}$$

for all $a, b \in X$.

By taking $M = N = X$ in Corollary 4.2, we obtain the following result, which is a common fixed point result for two self mappings.

Corollary 4.7 *Let* (X, d) *be a complete metric space with a* w_0*-distance* p*. Let* $S, T : X \to X$ *be two self mappings satisfying the following conditions:*

(F_1) There exist $\beta_1, \beta_2 \geq \max\{\alpha_0, 2\alpha_1, 2\alpha_2, 2\alpha_3, 2\alpha_4\}$ *such that* S *is* p*-*β_1*-quasi contraction (say,* $\psi \in \Phi_{\beta_1}$*) and* T *is* p*-*β_2*-quasi contraction (say,* $\phi \in \Phi_{\beta_2}$*);*

(F_2) For all $a, b \in X$, $d(Sa, Tb) \leq cd(a, b)$ *for some* $c \in (0, 1)$*;*

(F_3) One of the following two assertions holds:

(i) ψ *and* ϕ *are continuous;*

(ii) $\beta_1, \beta_2 > \max\{\alpha_0, 2\alpha_1, 2\alpha_2, \alpha_3, \alpha_4\}$.

Then S *and* T *have a unique common fixed point in* X.

By taking $M = N = X$ in Corollary 4.4, we obtain the following common fixed point result.

Corollary 4.8 *Let* (X, d) *be a complete metric space with a* w_0*-distance* p*. Let* $S, T : X \to X$ *be two self mappings satisfying the following conditions:*

(G_1) S *and* T *are* p*-quasi contractions;*

(G_2) For all $a, b \in X$, $d(Sa, Tb) \leq cd(a, b)$ *for some* $c \in (0, 1)$.

Then S *and* T *have a unique common fixed point in* X.

5. An application

In this section, we apply Corollary 4.8 to discuss the solutions to a class of system of Volterra type integral equations.

Let $X = C([0, K]), \mathbb{R})$ be the Banach space of all continuous functions defined on $[0, K]$ endowed with the norm (called Bielecki norm, see [29])

$$\|x\|_B = \max_{t \in [0, K]} |x(t)|e^{-Lt}, \quad x \in X, \ L > 0.$$

The induced metric is

$$d_B(x, y) = \max_{t \in [0, K]} |x(t) - y(t)|e^{-Lt}, \quad x, y \in X, \ L > 0.$$

It is easy to see that (X, d_B) is a complete metric space. Furthermore, let

$$p(x,y)=\max_{t\in[0,K]}|x(t)+y(t)|e^{-Lt},\quad x,y\in X,\ L>0.$$

Then p is a w_0-distance and $\mu(x,y)=p(x,y)$.

Now consider the following system of Volterra type integral equations:

$$\begin{cases} x_1(t)=\int_0^t \Omega_1(t,s,x_1(s))\mathrm{d}s,\\ x_2(t)=\int_0^t \Omega_2(t,s,x_2(s))\mathrm{d}s. \end{cases} \tag{5.1}$$

for all $t\in[0,K]$, where $K>0$, $x\in X$ and $\Omega\in C([0,K]\times[0,K]\times X,\mathbb{R})$. We discuss the existence and uniqueness of solutions to (5.1).

Theorem 5.1 *Let (X, d_B) be the complete metric space defined above, and $\Omega_i\in C([0,K]\times[0,K]\times X,\mathbb{R})$ $(i=1,2)$ be the functions satisfying the following conditions:*

(i) $\|\Omega_i\|_\infty=\sup_{t,s\in[0,K],x\in C([0,K],\mathbb{R})}|\Omega(t,s,x(s))|<\infty$;

(ii) There exists $L>0$, such that for all $a,b\in X$ and all $t,s\in[0,K]$ with $0\le e^{-LK}\le 1$, we have

$$|\Omega_i(t,s,a(s))|+|\Omega_i(t,s,b(s))|\le L|a(s)+b(s)|,\quad i=1,2;$$

(iii) There exists $L>0$, such that for all $a,b\in X$ and all $t,s\in[0,K]$, we have

$$|\Omega_1(t,s,a(s))-\Omega_2(t,s,b(s))|\le L|a(s)-b(s)|,$$

then (5.1) has a unique solution in X.

Proof Define $S,T: X\to X$ by $Sx_1(t)=\int_0^t\Omega_1(t,s,x_1(s))\mathrm{d}s$ and $Tx_2(t)=\int_0^t\Omega_2(t,s,x_2(s))\mathrm{d}s$, respectively. Then (5.1) has a unique solution in X is equivalent to the fact that S and T have a unique fixed point in X.

Now, by (ii), for all $a,b\in X$, we have

$$\begin{aligned}\mu(Sa,Sb)&=\max_{t\in[0.K]}\Big|\int_0^t\Omega_1(t,s,a(s))\mathrm{d}s+\int_0^t\Omega_1(t,s,b(s))\mathrm{d}s\Big|e^{-Lt}\\ &\le\max_{t\in[0.K]}\int_0^t|\Omega_1(t,s,a(s))+\Omega_1(t,s,b(s))|e^{L(s-t)}e^{-Ls}\mathrm{d}s\\ &\le L\mu(a,b)\max_{t\in[0.K]}\int_0^t e^{L(s-t)}\mathrm{d}s\le(1-e^{-LK})\mu(a,b).\end{aligned}$$

Similarly, we have $\mu(Ta,Tb)\le(1-e^{-LK})\mu(a,b)$.

Noting that $0\le e^{-LK}\le1$, we conclude that there exists $q=1-e^{-LK}\in[0,1)$, such that

$$\mu(Sa,Sb)\le q\mu(a,b)\le q\max\{\mu(a,b),\mu(a,Sa),\mu(b,Sb),\mu(a,Sb),\mu(b,Sa)\},$$

and

$$\mu(Ta,Tb)\le q\mu(a,b)\le q\max\{\mu(a,b),\mu(a,Ta),\mu(b,Tb),\mu(a,Tb),\mu(b,Ta)\},$$

which implies that S and T are p-quasi contractions, i.e., (G_1) holds.

On the other hand, by (iii), for all $a,b\in X$, we have

$$d_B(Sa,Tb)=\max_{t\in[0.K]}\Big|\int_0^t\Omega_1(t,s,a(s))\mathrm{d}s-\int_0^t\Omega_2(t,s,b(s))\mathrm{d}s\Big|e^{-Lt}$$

$$\leq \max_{t\in[0.K]} \int_0^t |\Omega_1(t,s,a(s)) - \Omega_2(t,s,b(s))| e^{L(s-t)} e^{-Ls} \mathrm{d}s$$
$$\leq Ld(a,b) \max_{t\in[0.K]} \int_0^t e^{L(s-t)} \mathrm{d}s \leq (1-e^{-LK}) d(a,b).$$

So there exists $c = 1 - e^{-LK} \in (0,1)$, such that $d(Sa,Tb) \leq cd(a,b)$ for all $a, b \in X$, which implies that (G_2) holds. It follows from Corollary 4.8 that S and T have a unique common fixed point in X. Therefore, (5.1) has a unique solution in X. This completes the proof. □

6. Conclusions

We have studied best proximity point problems for a new class of contractions which is more general than previous ones in the framework of a metric space with w_0-distance. By proposing the notions of p-proximal α-η-β-quasi contraction, α-proximal admissible mappings with respect to η and (α, d) regular mappings with respect to η, we have proved the existence and uniqueness of best proximity points of the mappings and have obtained many consequent results, which shed some new light on the study of best proximity point as well as fixed point problems in a metric space with w_0-distance. It would also be interesting to investigate best proximity point problems in the framework of other spaces for other kinds of mappings. These problems deserve studying in the future.

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