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Nilpotent Structure of Generalized Semicommutative Rings

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Abstract We study the nilpotent structure of generalized semicommutative rings. The new concept of nilpotent α-semicommutative rings is defined and studied. This class of rings is closely related to many well-known concepts including semicommutative rings, α-semicommutative rings and weak α-rigid rings. An example is given to show that a nilpotent α-semicommutative ring need not be α-semicommutative. Various properties of this class of rings are investigated. Many known results related to various semicommutative properties of rings are generalized and unified.

Keywords nilpotent α-semicommutative rings; α-rigid rings; α-semicommutative rings; polynomial rings

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1. Introduction

Throughout this paper, R denotes an associative ring with identity and α denotes a nonzero non-identity endomorphism of R, unless specified otherwise. The set of all nilpotent elements in a ring R is denoted by $N(R)$. We denote by $T_n(R)$, $M_n(R)$ the $n \times n$ upper triangular matrix ring and the $n \times n$ full matrix ring over a ring R, respectively. Recall that a ring R is reduced if it has no nonzero nilpotent elements. A ring R is called an Armendariz ring if $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ such that $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. Note that every reduced ring is an Armendariz ring. According to [1], a ring R is α-compatible if for any $a, b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$. It is clear that this happens only when the endomorphism α is injective. Krempa [2] introduced the notion of an α-rigid ring. An endomorphism α of a ring R is said to be rigid if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$, while a ring R is said to be α-rigid if there exists a rigid endomorphism α of R. By [1, Lemma 2.2], R is α-rigid if and only if R is α-compatible and reduced.

There are many generalizations of reduced rings. A ring R is reversible if $ab = 0$ implies $ba = 0$ for any $a, b \in R$. A ring R is semicommutative if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. It is well-known that every reduced ring is reversible and every reversible ring is semicommutative. More generally, the semicommutative properties with respect to a ring endomorphism were further investigated in [3]. According to [3], a ring R is α-semicommutative

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if $ab = 0$ implies $aR\alpha(b) = 0$ for all $a, b \in R$. Note that the concept of α-semicommutative rings not only generalizes that of α-rigid rings, but also extends that of semicommutative rings. Recently, the nilpotent elements of a semicommutative ring was studied [4]. Recall that a ring R is nil-semicommutative if for every $a, b \in R$ with $ab \in N(R)$, then $arb \in N(R)$ for all $r \in R$. Further results on semicommutative rings and related topics can be found in [4–8].

In this paper, we continue to study the properties of generalized semicommutative rings. The nilpotent structure of this class of rings is investigated. The concept of nilpotent α-semicommutative rings is defined and studied. We show that a nilpotent α-semicommutative ring need not be α-semicommutative (Example 2.9). This class of rings is closely related to many known concepts, such as semicommutative rings, nil-semicommutative rings and weak α-rigid rings. We study various properties of nilpotent α-semicommutative rings. Firstly, we show that nilpotent α-semicommutative rings can be given by various ring extensions. If R is a nilpotent α-semicommutative ring, it is proved that the $n \times n$ upper triangular matrix ring $T_n(R)$ is nilpotent α-semicommutative (Proposition 2.6). If I is a nilpotent α-semicommutative ideal (as a ring without identity), it is proved that R/I is nilpotent α-semicommutative if and only if R is nilpotent α-semicommutative (Proposition 2.13). Secondly, we study the properties of polynomial rings over nilpotent α-semicommutative rings. Let R be an Armendariz ring. It is shown that if R is nilpotent α-semicommutative, then $R[x]$ is nilpotent α-semicommutative (Proposition 3.3). We also investigate the condition under which $R[x;\alpha]$ is nilpotent α-semicommutative. Let R be a nilpotent α-semicommutative ring and α an endomorphism of R. We prove that if R is an α-rigid ring, then $R[x;\alpha]$ is nilpotent α-semicommutative (Proposition 3.8).

2. Nilpotent elements in generalized semicommutative rings

In this section, we introduce and study the concept of nilpotent α-semicommutative rings. Observe that the notion of nilpotent α-semicommutative rings not only generalizes that of weak α-rigid rings (when α is a monomorphism of a ring R), but also extends that of α-semicommutative rings. Some examples to illustrate the concepts and results are also included.

We start with the following definition.

Definition 2.1 *Let R be a ring and α an endomorphism of R. We call R a nilpotent α-semicommutative ring if for any $a, b \in R$, $ab \in N(R)$ implies $ar\alpha(b) \in N(R)$ for all $r \in R$.*

It is easy to see that any subring S with $\alpha(S) \subseteq S$ of a nilpotent α-semicommutative ring is also nilpotent α-semicommutative. We shall show in Example 2.9 that there exists a nilpotent α-semicommutative ring R such that R is not α-semicommutative.

Note that nilpotent α-semicommutative rings are closely related to α-semicommutative rings, semicommutative rings, nil-semicommutative rings and α-rigid rings. The following remark reveals the relations between them.

Remark 2.2 Let R be a ring and α an endomorphism of R. Then

(1) If R is reduced, then the class of nilpotent α-semicommutative rings is precisely the class

of α-semicommutative rings.

(2) If R is α-rigid, then the class of nilpotent α-semicommutative rings is just the class of semicommutative rings.

Note that a nilpotent α-semicommutative ring need not be semicommutative (see Example 2.9). The following example shows that there exists a semicommutative ring R such that R is not nilpotent α-semicommutative for some endomorphism α of R.

Example 2.3 Let $\mathbb{Z}_2$ be the ring of integers modulo 2. Consider the ring $R = \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$ with the usual addition and multiplication. Then R is a semicommutative ring since R is commutative reduced. Let $\alpha : R \to R$ be defined by $\alpha((a, b)) = (b, a)$ for every $(a, b) \in R$. Then for $a = (1, 0), b = (0, 1) \in R$, we have

$$ab = (1, 0)(0, 1) = (0, 0) \in N(R).$$

However, for $c = (1, 1) \in R$, we have

$$ac\alpha(b) = (1, 0)(1, 1)(1, 0) = (1, 0)(1, 0) = (1, 0) \notin N(R).$$

This implies that R is not nilpotent α-semicommutative. Moreover, for $a = (1, 0) = b \in R$, we get $ab = (1, 0) \neq 0$. But for any $(c, d) \in R$, we have

$$(1, 0)(c, d)(0, 1) = (c, 0)(0, 1) = (0, 0) \in N(R).$$

The following proposition gives more examples of nilpotent α-semicommutative rings.

Proposition 2.4 *All α-rigid rings are nilpotent α-semicommutative.*

Proof Let R be an α-rigid ring and let $a, b \in R$ such that $ab \in N(R)$. In the following, we freely use the fact that every α-rigid ring is reduced and every reduced ring is reversible. Then there exists $n \in \mathbb{N}$ such that $0 = (ab)^n = ababab \cdots abab$. Then $aRbaRbaRb \cdots aRbaRb = 0$. Therefore, we have $aRbaRbaRb \cdots aRbaR\alpha(b) = 0$ since every α-rigid ring is α-compatible. This implies that $(aR\alpha(b))(aRbaRbaRb \cdots aRb) = 0$ since R is reversible. Continuing this process, we can eventually get $(aR\alpha(b))^n = 0$. This means $aR\alpha(b) \in N(R)$, and thus R is a nilpotent α-semicommutative ring. □

Proposition 2.5 *Let $\{R_i : i \in I\}$ be a family of rings. Then the direct sum $\bigoplus_{i \in I} R_i$ is nilpotent α-semicommutative if and only if each R_i is nilpotent α-semicommutative for all $i \in I$.*

Proof It suffices to prove the sufficiency. Let $a = (a_i), b = (b_i) \in \bigoplus_{i \in I} R_i$ such that $ab = (a_i b_i) \in N(\bigoplus_{i \in I} R_i)$. Then $a_i b_i \in N(R_i)$ for each $i \in I$, and thus $a_i r \alpha(b_i) \in N(R_i)$ for $r \in R$ since each R_i is nilpotent α-semicommutative for all $i \in I$. For any $r = (r_i) \in \bigoplus_{i \in I} R_i$, we have $ar\alpha(b) = (a_i r_i \alpha(b_i)) \in \bigoplus_{i \in I} R_i$. Note that there are only finitely $i \in I$ such that $a_i r_i \alpha(b_i) \neq 0$. Thus, $ar\alpha(b) \in N(\bigoplus_{i \in I} R_i)$, proving that $\bigoplus_{i \in I} R_i$ is nilpotent α-semicommutative. □

Note that if α is an endomorphism of a ring R, then the endomorphism α can be extended to the map $\overline{\alpha}$: $T_n(R) \longrightarrow T_n(R)$ as follows

$$\alpha\left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}\right) = \begin{pmatrix} \alpha(a_{11}) & \alpha(a_{12}) & \cdots & \alpha(a_{1n}) \\ 0 & \alpha(a_{22}) & \cdots & \alpha(a_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha(a_{nn}) \end{pmatrix}.$$

More generally, the next result gives one way to get more nilpotent α-semicommutative rings from old ones.

Proposition 2.6 *A ring R is nilpotent α-semicommutative if and only if $T_n(R)$ is nilpotent $\overline{\alpha}$-semicommutative.*

Proof It suffices to show that $T_n(R)$ is nilpotent $\bar{\alpha}$-semicommutative when R is a nilpotent α-semicommutative ring. Let $A = (a_{ij})$ and $B = (b_{ij}) \in T_n(R)$ such that $AB \in N(T_n(R))$. Then $a_{ii}b_{ii} \in N(R)$ for all a_{ii} and b_{ii}, where $1 \le i \le n$. Then we have $a_{ii}R\alpha(b_{ii}) \subseteq N(R)$ by the assumption. This implies that $A(T_n(R))\bar{\alpha}(B) \subseteq N(T_n(R))$, and the result follows. □

For a ring R and an (R,R)-bimodule M, the trivial extension of R by M is the ring $T(R,M) = R\bigoplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

For an endomorphism α of a ring R and the trivial extension $T(R,R)$ of R, the $\overline{\alpha} : T(R,R) \longrightarrow T(R,R)$ defined by

$$\overline{\alpha}\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} \alpha(a) & \alpha(b) \\ 0 & \alpha(a) \end{pmatrix}$$

is an endomorphism of $T(R,R)$. Since $(R,0)$ is isomorphic to R, we can identify the restriction of $\overline{\alpha}$ by $T(R,0)$ to α.

Corollary 2.7 *Let α be an endomorphism of a ring R. Then R is a nilpotent α-semicommutative ring if and only if $T(R,R)$ is a nilpotent $\bar{\alpha}$-semicommutative ring.*

Corollary 2.8 *A ring R is a nil-semicommutative ring if and only if $T(R,R)$ is nil-semicommutative.*

Now we are in a position to give an example to show that there exists a nilpotent α-semicommutative ring R such that $T(R,R)$ is nilpotent $\bar{\alpha}$-semicommutative, but $T(R,R)$ is not $\bar{\alpha}$-semicommutative.

Example 2.9 Let $\mathbb{Z}$ be the ring of integers and let $R = \left\{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) | a, b \in \mathbb{Z}\right\}$. Then we have the following implications:

(1) R is nilpotent α-semicommutative for some endomorphism α of R. In fact, let $\alpha : R \to R$

be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

Now we claim that R is nilpotent α-semicommutative. In fact, for any

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \quad B = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \in R,$$

if $AB \in N(R)$, then it is clear $ac \in N(\mathbb{Z})$. Moreover, for any element $\left(\begin{smallmatrix} h & k \\ 0 & h \end{smallmatrix}\right) \in R$, we have the following implication

$$AR\alpha(B) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} h & k \\ 0 & h \end{pmatrix} \alpha\left(\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} ahc & * \\ 0 & ahc \end{pmatrix} \in N(R).$$

Therefore, R is a nilpotent α-semicommutative ring.

(2) By Corollary 2.7, $T(R,R)$ is nilpotent $\bar{\alpha}$-semicommutative.

(3) $T(R,R)$ is not $\bar{\alpha}$-semicommutative. In fact, for

$$A = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}, \quad B = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in T(R,R),$$

we have $AB = 0$. However, if we let

$$C = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \in T(R,R),$$

then we have

$$0 \neq AC\bar{\alpha}(B) = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in AT(R,R)\bar{\alpha}(B).$$

This implies that $T(R,R)$ is not $\bar{\alpha}$-semicommutative. Moreover, it can be easily checked that $T(R,R)$ is not semicommutative.

More generally, let σ be an endomorphism of a ring R. We consider the following subring of the upper triangular matrix ring $T_n(R)$:

$$T(R,n,\sigma) := \left\{ \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_1 \\ 0 & 0 & \cdots & 0 & a_0 \end{pmatrix} \middle| a_i \in R \right\}, \text{ with } n \geq 2.$$

We can denote the elements of $T(R,n,\sigma)$ by $(a_0,\dots,a_{n-1})$. Then $T(R,n,\sigma)$ is a ring with addition point-wise and multiplication given by

$$(a_0,\dots,a_{n-1})(b_0,\dots,b_{n-1})=(a_0b_0,a_0*b_1+a_1*b_0,\dots,a_0*b_{n-1}+\cdots+a_{n-1}*b_0),$$

with $a_i*b_i=a_i\sigma^i(b_j)$ for each i,j. Let α and σ be endomorphisms of a ring R such that $\alpha\sigma=\sigma\alpha$. Then $\overline{\alpha}:T(R,n,\sigma)\to T(R,n,\sigma)$, given by $\overline{\alpha}((a_{ij}))=(\alpha(a_{ij}))$ is an endomorphism of $T(R,n,\sigma)$.

Proposition 2.10 *Let α and σ be endomorphisms of a ring R with $\alpha\sigma=\sigma\alpha$. Then R is nilpotent α-semicommutative if and only if $T(R,n,\sigma)$ is nilpotent $\bar{\alpha}$-semicommutative.*

Proof Notice that $N(T(R,n,\sigma))=(N(R),R,\dots,R)$. So the proof is similar to that of Proposition 2.6. □

Based on Proposition 2.6, one may suspect that if R is nilpotent α-semicommutative, then every n by n full matrix ring $M_n(R)$ over R is nilpotent $\bar{\alpha}$-semicommutative, where $n\geq 2$. However, the following example eliminates the possibility.

Example 2.11 Let R be a nilpotent α-semicommutative ring. Consider $S=M_2(R)$ and an endomorphism α of S defined by

$$\alpha\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)=\begin{pmatrix} a & -b\\ -c & d\end{pmatrix}.$$

For

$$A=\begin{pmatrix} 1 & 0\\ -1 & 0\end{pmatrix},\ B=\begin{pmatrix} 1 & 1\\ 0 & 0\end{pmatrix},\ C=\begin{pmatrix} 1 & -1\\ -1 & 1\end{pmatrix}\in S,$$

we have $AB\in N(S)$. But we have

$$AC\alpha(B)=\begin{pmatrix} 1 & -1\\ -1 & 1\end{pmatrix}\notin N(S).$$

Therefore, $M_2(R)$ is not nilpotent α-semicommutative.

According to [9], R is weak α-rigid if $a\alpha(a)\in N(R)$ if and only if $a\in N(R)$. The next proposition gives the relation of a weak α-rigid ring and a nilpotent α-semicommutative ring.

Proposition 2.12 *Let R be any ring with an endomorphism α. Then*

(1) If α is a monomorphism, then each nilpotent α-semicommutative ring is weak α-rigid.

(2) If R is reduced, then each weak α-rigid ring is nilpotent α-semicommutative.

Proof (1) Assume that α is a monomorphism and R is a nilpotent α-semicommutative ring. On one hand, if $a\alpha(a)\in N(R)$ for $a\in R$, then $\alpha(a)a\in N(R)$ and thus $\alpha(a)R\alpha(a)\subseteq N(R)$. Then $\alpha(a^2)\in N(R)$. This implies that there exists $k\in\mathbb{N}$ such that $\alpha(a^{2k})=0$. Since α is a monomorphism, we have $a\in N(R)$. On the other hand, if $a\in N(R)$, then $a^2\in N(R)$ and $aR\alpha(a)\subseteq N(R)$ by the assumption. In particular, we have $a\alpha(a)\in N(R)$.

(2) If R is reduced and $ab\in N(R)$, then $ba\in N(R)$. It follows that $ba=0$ since R is reduced.

This implies that $ar\alpha(b)\alpha(ar\alpha(b)) = ar\alpha(ba)\alpha(r)\alpha^2(b) = 0 \in N(R)$ for any $r \in R$. Since R is weak α-rigid, we get $ar\alpha(b) \in N(R)$. $\square$

Let α be an endomorphism of R. Recall that an ideal I of R is called an α-ideal if $\alpha(I) \subseteq I$. Note that if I is an α-ideal of R, then $\bar{\alpha} : R/I \to R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ for $a \in R$ is an endomorphism of the factor ring R/I.

Proposition 2.13 *If I is an α-ideal of a ring R such that $I \subseteq N(R)$, then R/I is nilpotent $\bar{\alpha}$-semicommutative if and only if R is nilpotent α-semicommutative.*

Proof Assume that R is nilpotent α-semicommutative. Let $\bar{a} = a + I, \bar{b} = b + I \in R/I$ such that $\bar{a}\bar{b} \in N(R/I)$. Then there exists a positive integer n such that $(ab)^n \in I$. This implies that $ab \in N(R)$ since $I \subseteq N(R)$ by the assumption. Since R is nilpotent α-semicommutative, we get $aR\alpha(b) \subseteq N(R)$, and thus $\bar{a}\bar{R}\bar{\alpha}(\bar{b}) \subseteq N(R/I)$. This shows that R/I is nilpotent $\bar{\alpha}$-semicommutative.

Conversely, assume that R/I is nilpotent $\bar{\alpha}$-semicommutative. Let $a, b \in R$ such that $ab \in N(R)$. Then we have $\bar{a}\bar{b} \in N(R/I)$. This implies that $\bar{a}\bar{R}\bar{\alpha}(\bar{b}) \subseteq N(R/I)$ since R/I is a nilpotent $\bar{\alpha}$-semicommutative ring. Then there exists a positive integer s such that $(aR\alpha(b))^s \subseteq I$. Therefore, we get $(aR\alpha(b))^s \subseteq N(R)$, as desired. $\square$

Recall from [10] that a ring R is said to be nil-Armendariz if whenever two polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n, g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) \in N(R)[x]$, then $a_ib_j \in N(R)$ for all i, j. Note that if R is nil-Armendariz, then $N(R)$ is a subring of R by [10, Theorem 3.2].

Proposition 2.14 *Let R be a nil-Armendariz ring. If $e^2 = e$ is a central idempotent of R such that $\alpha(e) = e$, then the following statements are equivalent:*

(1) R is a nilpotent α-semicommutative ring.

(2) eRe is nilpotent α-semicommutative for every $e^2 = e$.

(3) eR and $(1 - e)R$ are nilpotent α-semicommutative for each $e^2 = e \in R$.

Proof (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are trivial since any subring S with $\alpha(S) \subseteq S$ of a nilpotent α-semicommutative ring is also nilpotent α-semicommutative. It suffices to prove (3) $\Rightarrow$ (1). Let $a, b \in R$ such that $ab \in N(R)$. Then $eaeb \in N(R)$ and $(1 - e)a(1 - e)b \in N(R)$. Since eR and $(1 - e)R$ are nilpotent α-semicommutative, we get

$$ear\alpha(eb) \in N(R), \quad (1 - e)ar\alpha[(1 - e)b] \in N(R).$$

Since R is a nil-Armendariz ring, $N(R)$ is a subring of R. This implies that

$$ear\alpha(eb) + (1 - e)ar\alpha[(1 - e)b] = ear\alpha(b) + (1 - e)ar\alpha(b) = ar\alpha(b) \in N(R).$$

Therefore, R is nilpotent α-semicommutative. $\square$

Let R be a ring and Δ be a multiplicatively closed subset of R consisting of central regular elements. Let $\Delta^{-1}R = \{u^{-1}a | u \in \Delta, a \in R\}$, then $\Delta^{-1}R$ is a ring. Then we have the following result.

Proposition 2.15 *Let α be an endomorphism of a ring R. Then R is nilpotent α-semicommutative if and only if $\Delta^{-1}R$ is nilpotent α-semicommutative.*

Proof It suffices to prove that if R is nilpotent α-semicommutative, then $\Delta^{-1}R$ is nilpotent α-semicommutative. Let $\delta = u^{-1}a$, $\beta = v^{-1}b$ and $\gamma = w^{-1}c \in \Delta^{-1}R$ with $\delta\beta \in N(\Delta^{-1}R)$. Then $\delta\beta = u^{-1}av^{-1}b = (vu)^{-1}(ab) \in N(R)$. Therefore, $ab \in N(R)$ since Δ is contained in the center of R. Since R is nilpotent α-semicommutative, we have $ac\alpha(b) \in N(R)$. Then we deduce that

$$\delta\gamma\alpha(\beta) = (u^{-1}a)(w^{-1}c)(\alpha(v)^{-1}\alpha(b)) = (wu)^{-1}(ac)(\alpha(v)^{-1}\alpha(b)) = (\alpha(v)(wu))^{-1}(ac\alpha(b)),$$

which implies that $\delta\gamma\alpha(\beta) \in N(\Delta^{-1}R)$. □

The ring of Laurent polynomials in x, with coefficients in a ring R, consists of all formal sum $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are integers. We denote it by $R[x; x^{-1}]$. The map $\bar{\alpha} : R[x, x^{-1}] \to R[x, x^{-1}]$ defined by $\bar{\alpha}(\sum_{i=k}^{n} a_i x^i) = \sum_{i=k}^{n} \alpha(a_i)x^i$ extends α and also is an endomorphism of $R[x, x^{-1}]$. Multiplication is subject to $xr = \alpha(r)x$ and $rx^{-1} = x^{-1}\alpha(r)$.

Corollary 2.16 *Let α be an endomorphism of a ring R. Then $R[x]$ is nilpotent α-semicommutative if and only if $R[x; x^{-1}]$ is nilpotent α-semicommutative.*

Proof It is easy to prove the necessity since $R[x]$ is a subring of $R[x; x^{-1}]$. Let $\Delta = \{1, x, x^2, \ldots\}$. Then Δ is a multiplicatively closed subset of $R[x]$. Since $R[x; x^{-1}] = \Delta^{-1}R[x]$, we conclude that $R[x; x^{-1}]$ is nilpotent α-semicommutative by Poposition 2.15. □

3. Polynomial extensions of nilpotent α-semicommutative rings

In this section, we study various polynomial extensions of nilpotent α-semicommutative rings. The relation between nilpotent α-semicommutative rings and weak α-skew Armendariz is also investigated. First we give the following.

Lemma 3.1 *Let R be a nilpotent α-semicommutative ring. If $ab \in N(R)$, then $aR\alpha^m(b) \subseteq N(R)$ and $bR\alpha^n(a) \subseteq N(R)$ for any positive integers m, n.*

Proof Let $ab \in N(R)$. On the one hand, since R is nilpotent α-semicommutative, $aR\alpha(b) \subseteq N(R)$. In particular, we have $a\alpha(b) \in N(R)$. By using again the nilpotent α-semicommutative condition, we have $aR\alpha^2(b) \subseteq N(R)$. Continuing this process, we get $aR\alpha^m(b) \subseteq N(R)$ for some positive integer m. On the other hand, since $ab \in N(R)$, we get $ba \in N(R)$. Then $bR\alpha(a) \subseteq N(R)$ since R is nilpotent α-semicommutative. In particular, we have $b\alpha(a) \in N(R)$ and thus $bR\alpha^2(a) \subseteq N(R)$ since R is nilpotent α-semicommutative. Continuing this process, we have $bR\alpha^n(a) \subseteq N(R)$ for some positive integer n. □

Let R be a ring and α an endomorphism of R. Recall from [11] that a ring R is α-skew Armendariz if for any $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha]$ with $f(x)g(x) = 0$, then $a_i\alpha^i(b_j) = 0$ for all i and j. More generally, a ring R is weak α-skew Armendariz [12] if

$f(x)=\sum_{i=0}^m a_ix^i$ and $g(x)=\sum_{j=0}^n b_jx^j \in R[x;\alpha]$ satisfy $f(x)g(x)=0$, then $a_i\alpha^i(b_j)\in N(R)$ for all $0\le i\le m$ and $0\le j\le n$.

Theorem 3.2 *Let R be a semicommutative ring. If R is nilpotent α-semicommutative, then R is weak α-skew Armendariz.*

Proof Let $f(x)=a_0+a_1x+\cdots+a_mx^m$, $g(x)=b_0+b_1x+\cdots+b_nx^n\in R[x;\alpha]$. Then

$$f(x)g(x)=\sum_{k=0}^{m+n}(\sum_{i+j=k}a_i\alpha^i(b_j))x^k=0.$$

Then we have the following equations:

$$\sum_{i+j=k}a_i\alpha^i(b_j)=0,\quad k=0,1,\ldots,m+n. \tag{$\natural$}$$

We will show that $a_i\alpha^i(b_j)\in N(R)$ by induction on $i+j$.

If $i+j=0$, then $0=a_0b_0\in N(R)$. Now suppose that k is a positive integer such that $a_i\alpha^i(b_j)\in N(R)$ when $i+j<k$. We claim that $a_i\alpha^i(b_j)\in N(R)$ when $i+j=k$. Since $a_i\alpha^i(b_j)\in N(R)$ when $i+j<k$, then we have $a_ir\alpha^k(b_0)\in N(R)$ by Lemma 3.1 for any $i<k$ since R is nilpotent α-semicommutative. Multiplying the coefficient of x^k in ($\natural$) from right side by $\alpha^k(b_0)$, we have

$$a_0b_k\alpha^k(b_0)+a_1\alpha(b_{k-1})\alpha^k(b_0)+a_2\alpha^2(b_{k-2})\alpha^k(b_0)+\cdots+a_k\alpha^k(b_0)\alpha^k(b_0)=0.$$

Then we have

$$a_k\alpha^k(b_0)\alpha^k(b_0)=-(a_0b_k\alpha^k(b_0)+a_1\alpha(b_{k-1})\alpha^k(b_0)+\cdots+a_{k-1}\alpha^{k-1}(b_1)\alpha^k(b_0)).$$

Since R is semicommutative, $N(R)$ is an ideal of R. This implies that $a_k\alpha^k(b_0)\alpha^k(b_0)\in N(R)$, and hence $a_k\alpha^k(b_0)\in N(R)$. Multiplying the coefficient of x^{k-1} in ($\natural$) from the right side by $\alpha^{k-1}(b_0)$, and in a similar way as above, we can get $a_{k-1}\alpha^{k-1}(b_1)\in N(R)$. Continuing this process, we have $a_i\alpha^i(b_j)\in N(R)$ when $i+j=k$. Therefore, $a_i\alpha^i(b_j)\in N(R)$ for each i,j. This shows that R is a weak α-skew Armendariz ring. □

Let α be an endomorphism of a ring R. Then the map $\bar{\alpha}:R[x]\to R[x]$ defined by $\bar{\alpha}(\sum_{i=0}^m a_ix^i)=\sum_{i=0}^m\alpha(a_i)x^i$ is an extension of α to $R[x]$. It was shown in [5] that the polynomial rings over semicommutative rings need not be semicommutative. However, we have the following.

Proposition 3.3 *Let R be an Armendariz ring. If R is nilpotent α-semicommutative, then $R[x]$ is nilpotent α-semicommutative.*

Proof Since R is an Armendariz ring, $N(R)$ is a subring (without 1) of R by [10, Corollary 3.3], and $R[x]$ is also Armendariz by [13, Theorem 2]. Hence $N(R)[x]=N(R[x])$ by [10, Proposition 2.7 and Theorem 5.3]. Let $f(x)=\sum_{i=0}^m a_ix^i$, $g(x)=\sum_{j=0}^n b_jx^j\in R[x]$ such that $f(x)g(x)\in N(R[x])$, then $a_ib_j\in N(R)$ for all i and j by [13, Proposition 1] since R is Armendariz. Note that for any $h(x)=\sum_{k=0}^s c_kx^k\in R[x]$, each coefficient of $f(x)h(x)\alpha(g(x))$ has the form of

$\sum a_i c_k \alpha(b_j)$. Since R is nilpotent α-semicommutative, $a_i b_j \in N(R)$ implies $a_i c_k \alpha(b_j) \in N(R)$. This means $\sum a_i c_k \alpha(b_j) \in N(R)$. It follows that $f(x)h(x)\alpha(g(x)) \in N(R)[x] = N(R[x])$. $\square$

Note that if a ring R is an α-compatible ring, then for any $a, b \in R$, $ab = 0$ if and only if $a\alpha^n(b) = 0$. Using this fact, we give the following

Lemma 3.4 *Let R be an α-compatible ring. If $a_1, a_2, \ldots, a_n$ are some elements in R, then $a_1 a_2 \cdots a_n \in N(R)$ if and only if $\alpha^{k_1}(a_1)\alpha^{k_2}(a_2)\cdots\alpha^{k_n}(a_n) \in N(R)$ for arbitrary positive integers $k_1, k_2, \ldots, k_n$.*

Proof Note that if R is an α-compatible ring, then α is a monomorphism. On the one hand, assume that $(a_1 a_2 \cdots a_n)^k = (a_1 a_2 \cdots a_n)\cdots(a_1 a_2 \cdots a_n) = 0$ for $a_1, a_2, \ldots, a_n \in R$, then

$$\alpha^{k_1}((a_1 a_2 \cdots a_n)\cdots(a_1 a_2 \cdots a_n)) = 0.$$

This implies that $\alpha^{k_1}(a_1)\alpha^{k_1}((a_2 a_3 \cdots a_n)\cdots(a_1 a_2 \cdots a_n)) = 0$. Since R is α-compatible, we get $\alpha^{k_1}(a_1)(a_2 a_3 \cdots a_n)\cdots(a_1 a_2 \cdots a_n) = 0$. Since α^k is a momonorphism for any nonnegative integer k by the α-compatible condition, we have

$$\alpha^{k_1}(a_1)\alpha^{k_2}(a_2)(a_3 \cdots a_n)\cdots(a_1 a_2 \cdots a_n) = 0.$$

Continuing this process, eventually we can get $(\alpha^{k_1}(a_1)\alpha^{k_2}(a_2)\cdots\alpha^{k_n}(a_n))^k = 0$. On the other hand, if $\alpha^{k_1}(a_1)\alpha^{k_2}(a_2)\cdots\alpha^{k_n}(a_n) \in N(R)$, then it can be proved similarly that $a_1 a_2 \cdots a_n \in N(R)$. $\square$

Corollary 3.5 *Let R be an α-compatible ring. If $a_1, a_2, \ldots, a_n \in R$, then $a_1 a_2 \cdots a_n = 0$ if and only if $\alpha^{k_1}(a_1)\alpha^{k_2}(a_2)\cdots\alpha^{k_n}(a_n) = 0$ for arbitrary positive integers $k_1, k_2, \ldots, k_n$.*

We next explore the relation of a nilpotent skew polynomial $f(x)$ in $R[x;\alpha]$ and the nilpotency of its coefficients.

Lemma 3.6 *Let R be an α-rigid ring and let $f(x) = a_0 + a_1 x + \cdots a_n x^n \in R[x;\alpha]$. Then $f(x) \in N(R[x;\alpha])$ if and only if each $a_i \in N(R)$ for all $0 \le i \le n$.*

Proof Suppose $f(x) \in N(R[x;\alpha])$, i.e., there exists $k \in \mathbb{N}$ such that $f^k(x) = 0$. Then we have $a_n \alpha^n(a_n)\cdots\alpha^{(k-1)n}(a_n) = 0$. Since every α-rigid ring is an α-compatible ring, by [1, Lemma 2.1] we have

$$\alpha^n(a_n)\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n) = \alpha^n(a_n^2)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n) = 0.$$

Therefore, we get $a_n^2\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n) = 0$ by [1, Lemma 2.1] again. Then

$$\alpha^{2n}(a_n^2)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n) = 0,$$

and thus $\alpha^{2n}(a_n^3)\alpha^{3n}(a_n)\cdots\alpha^{(k-1)n}(a_n) = 0$. Continuing this process, we can get $a_n^k = 0$. This implies that $[a_0 + a_1 x + \cdots a_{n-1}x^{n-1}]^k \in N(R)[x;\alpha]$. Since R is reduced (and hence semicommutative), $N(R)$ is an ideal of R. It follows that

$$a_{n-1}\alpha^{n-1}(a_{n-1})\cdots\alpha^{(k-1)(n-1)}(a_{n-1}) \in N(R).$$

Then $a_{n-1} \in N(R)$ by a similar discussion as above. By induction, we can eventually show that all $a_i \in N(R)$ for each $0 \leq i \leq n$.

Conversely, assume that all the coefficients $a_i \in N(R)$ of $f(x)$ for $i = 0, 1, \ldots, n$. Then there exists $m_i \in \mathbb{N}$ such that $a_i^{m_i} = 0$ for each a_i. We claim that $f(x) \in N(R[x;\alpha])$. In fact, let $t = \sum m_i + 1$. Then we have

$$\begin{aligned}(f(x))^t =&(a_0 + a_1x + \cdots a_nx^n)^t \\ =&\sum [a_0^{i_{0_1}}(a_1x)^{i_{1_1}} \cdots (a_nx^n)^{i_{n_1}}][a_0^{i_{0_2}}(a_1x)^{i_{1_2}} \cdots (a_nx^n)^{i_{n_2}}] \cdots [a_0^{i_{0_t}}(a_1x)^{i_{1_t}} \cdots (a_nx^n)^{i_{n_t}}],\end{aligned}$$

where $\sum_{h=0}^{n} i_{h_s} = 1$ for each $s \in \{1, 2, \ldots, t\}$, and $i_{h_s} = 0$ or $i_{h_s} = 1$ for all s. Note that each coefficient of $f^t(x)$ is a sum of the elements

$$[(\alpha^{v_{0_1}}(a_0))^{i_{0_1}} \cdots (\alpha^{v_{n_1}}(a_n))^{i_{n_1}}] \cdots [(\alpha^{v_{0_t}}(a_0))^{i_{0_t}} \cdots (\alpha^{v_{n_t}}(a_n))^{i_{n_t}}]$$

such that $i_{0_s} + i_{1_s} + \cdots + i_{n_s} = 1$, where $s \in \{1, 2, \ldots, t\}$. Clearly, there exists $a_j \in \{a_0, a_1, \ldots, a_n\}$ such that $i_{j_1} + i_{j_2} + \cdots + i_{j_t} \geq m_j$. Since $a_j^{m_j} = 0$ by assumption, this implies that $a_j^{i_{j_1} + i_{j_2} + \cdots + i_{j_t}} = 0$. By [1, Lemma 2.1], we have

$$((\alpha^{v_{j_1}}(a_j))^{i_{j_1}}((\alpha^{v_{j_2}}(a_j))^{i_{j_2}} \cdots ((\alpha^{v_{j_t}}(a_j))^{i_{j_t}}) = 0.$$

Therefore, each coefficient of $f^t(x)$ is zero. This implies that $f^t(x) = 0$, as desired. $\Box$

Let α be an endomorphism of a ring R. If $N(R)$ is an ideal of R, then the endomorphism $\bar{\alpha}$ of the ring $R/N(R)$ can be induced by α via $a + N(R) \to \alpha(a) + N(R)$. Moreover, it is easy to see that the map defined by

$$a_0 + \cdots a_nx^n \to (a_0 + N(R)) + \cdots + (a_n + N(R))x^n$$

is a ring homomorphism from $R[x;\alpha]$ to $R/N(R)[x;\bar{\alpha}]$. It is clear that we have the isomorphism $R[x;\alpha]/N(R)[x;\alpha] \cong R/N(R)[x;\bar{\alpha}]$.

Lemma 3.7 *Let R be a nilpotent α-semicommutative ring and α an endomorphism of R. If R is α-rigid, then $N(R[x;\alpha]) = N(R)[x;\alpha]$.*

Proof Clearly, R is semicommutative since R is nilpotent α-semicommutative and α-rigid. Then $N(R)$ is an ideal of R. Note that $R/N(R)$ is $\bar{\alpha}$-rigid since R is an α-rigid ring. On the one hand, we claim that $N(R[x;\alpha]) \subseteq N(R)[x;\alpha]$. In fact, let $f(x) = \sum_{i=0}^{n} a_ix^i \in N(R[x;\alpha])$, then there is a positive k such that $f^k(x) = 0$. Then $(\bar{f}(x))^k = \bar{0}$ in $R/N(R)[x;\bar{\alpha}]$. Since every α-rigid ring is α-skew Armendariz, $R/N(R)[x;\alpha]$ is $\overline{\alpha}$-skew Armendariz. This implies that $\bar{a}_i\bar{\alpha}^i(\bar{a}_i) \cdots \bar{\alpha}^{(k-1)i}(\bar{a}_i) = \bar{0}$ for all integers $i = 1, 2, \ldots, n$. Then $\bar{a}_i^k = \bar{0}$ by Corollary 3.5, and thus $a_i \in N(R)$ for each i. On the other hand, it is clear $N(R)[x;\alpha] \subseteq N(R[x;\alpha])$ by Lemma 3.6. Therefore, we get $N(R[x;\alpha]) = N(R)[x;\alpha]$. $\Box$

We conclude this section by the following proposition, which gives the condition under which $R[x;\alpha]$ is nilpotent α-semicommutative.

Proposition 3.8 *Let R be a nilpotent α-semicommutative ring and α an endomorphism of R. If R is α-rigid, then $R[x;\alpha]$ is nilpotent α-semicommutative.*

Proof Let $f(x)=\sum_{i=0}^n a_ix^i$, $g(x)=\sum_{j=0}^m b_jx^j\in R[x;\alpha]$ such that $f(x)g(x)\in N(R[x;\alpha])$. Then there is a positive t such that $(f(x)g(x))^t=0$. Since R is semicommutative, $N(R)$ is an ideal of R. This implies that $R/N(R)[x;\bar{\alpha}]$ is $\bar{\alpha}$-skew Armendariz by the proof of Lemma 3.7. Therefore, we get $\bar{a}_i\bar{\alpha}^i(\bar{b}_j)\cdots\bar{\alpha}^{ki+(k-1)j}(\bar{b}_j)=\bar{0}$. It follows that $(\bar{a}_i\bar{b}_j)^k=\bar{0}$ by Corollary 3.5. So we have $a_ib_j\in N(R)$. For any $h(x)=\sum_{k=0}^l c_kx^k\in R[x;\alpha]$, we have $a_ic_k\alpha(b_j)\in N(R)$. Since each coefficient of $f(x)h(x)\alpha(g(x))$ has the form of $\sum_{p=0}^{n+k+m} a_i\alpha^i(c_k)\alpha^{i+k+1}(b_j)$. Then $\sum_{p=0}^{n+k+m} a_i\alpha^i(c_k)\alpha^{i+k+1}(b_j)\in N(R)$ by Lemma 3.4. Therefore, $f(x)h(x)\alpha(g(x))\in N(R)[x;\alpha]=N(R[x;\alpha])$ by Lemma 3.7. $\square$

Recall that a ring R is weakly semicommutative if for any $a,b\in R$, $ab=0$ implies $arb\in N(R)$ for any $r\in R$. In particular, we have the following corollary.

Corollary 3.9 *Let R be a semicommutative ring and α be an endomorphism of R. If R is α-compatible, then $R[x;\alpha]$ is weakly semicommutative.*

Corollary 3.10 *Let R be a semicommutative ring and α an endomorphism of R. If R is α-rigid, then $R[x;\alpha]$ is nil-semicommutative.*

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