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Existence and Non-Existence of Solutions to Some Degenerate Coercivity Quasilinear Elliptic Equations with Measure Data

Maoji RI[1], Xiangrui LI[1], Qiaoyu TIAN[1], Shuibo HUANG[1,2,*]
1. *School of Mathematics and Computer Science, Northwest Minzu University, Gansu* 730030*, P. R. China;*
2. *Key Laboratory of Streaming Data Computing Technologies and Application, Northwest Minzu University, Gansu* 730030*, P. R. China*

Abstract In this article, we study the existence and non-existence of weak solutions to the following quasilinear elliptic problem with principal part having degenerate coercivity and nonlinear term involving gradient,

$$\begin{cases} -\mathrm{div}\big(\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\big)+\frac{|u|^{p-2}u|\nabla u|^p}{(1+|u|)^{\theta p}}=\mu, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}$$

where $\Omega\subseteq\mathbb{R}^N$ $(N\geq 3)$ is a bounded smooth domain, $1<p<N$, $0\leq\theta<1$, μ is a Radon measure.

Keywords elliptic equation; degenerate coercivity; measures data; existence; non-existence

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1. Introduction and main results

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N\geq 3)$ with smooth boundary $\partial\Omega$, $1<p<N$ and μ be a Radon measure in Ω. In this paper, we mainly consider the existence and non-existence of solutions $u\in W_0^{1,p}(\Omega)$ to the problem

$$\begin{cases} -\mathrm{div}A(x,u,\nabla u)+g(x,u,\nabla u)=\mu, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}\tag{1.1}$$

where $A(x,t,\xi)\equiv A_i(x,t,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is the Carathéodory function, satisfying the following conditions: there exist positive constants c_0,c_1, such that

$$\langle A(x,t,\xi),\xi\rangle\geq\frac{c_0|\xi|^p}{(1+|t|)^{\theta(p-1)}},\tag{1.2}$$

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* Corresponding author
E-mail address: 614416238@qq.com (Maoji RI); lllxru@163.com (Xiangrui LI); tianqiaoyu2004@163.com (Qiaoyu TIAN); huangshuibo2008@163.com (Shuibo HUANG)

$$|A(x,t,\xi)| \le c_1(|\xi|^{p-1} + l(x)), \quad l \in L^{p'}(\Omega), \tag{1.3}$$

$$(A(x,t,\xi) - A(x,t,\xi^{'})) \cdot (\xi - \xi^{'}) > 0, \tag{1.4}$$

for almost every $x \in \Omega$, $t \in \mathbb{R}$ and $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$, where $0 \le \theta < 1$, $l \in L^{p'}(\Omega)$ is a non-negative function, p' is the conjugate Hölder exponent of p, $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is the Carathéodory function, such that the following assumptions hold,

$$|g(x,t,\xi)| \le b(|t|)(\frac{|\xi|^p}{(1+|t|)^{\theta p}} + d(x)), \tag{1.5}$$

$$g(x,t,\xi)\mathrm{sgn(t)} \ge \rho \frac{|\xi|^p}{(1+|t|)^{\theta p}}, \tag{1.6}$$

for almost every $x \in \Omega$, $t \in \mathbb{R}$, $|t| \ge \sigma$, $\xi \in \mathbb{R}^N$, where b is an increasing real valued positive continuous function, $d \in L^1(\Omega)$ is a non-negative function, ρ and σ are two positive real numbers.

The main features of problem (1.1) are the facts that the principal part has degenerate coercivity, the operator has lower order term, which also produce a lack of coercivity, and the right-hand side μ is a measure. Notice that, $\mathbb{A}(u) := -\mathrm{div}A(x,u,\nabla u)$ is well defined in $W_0^{1,p}(\Omega)$ when $\mathbb{A}$ satisfies (1.2). However, $\mathbb{A}$ is noncoercive in $W_0^{1,p}(\Omega)$ if u is large enough. Therefore, the standard Leray-Lions surjectivity theorem cannot be applied to problem (1.1) even in the case $f \in W^{-1,p'}(\Omega)$. Thus it is necessary to change the classical framework of the Sobolev spaces in order to prove existence results.

Nonlinear elliptic problems with measure data have been studied in a number of papers. Bénilan et al. [1] proved the existence and uniqueness of entropy solution to

$$\begin{cases} -\mathrm{div}A(x,\nabla u) = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $f \in L^1(\Omega)$. However, their method is confined to the case of an $L^1(\Omega)$ datum. In particular, the concept of entropy solution is meaningless if f is a Radon measure. The results in [1] were improved by Boccardo et al. [2], they considered a measure $f \in \mathcal{M}_0^\gamma(\Omega)$, proved that if γ is a real number such that $1 < \gamma < +\infty$, $f \in \mathcal{M}_b(\Omega)$, then $f \in L^1(\Omega) + W^{-1,\gamma'}(\Omega)$ if and only if $f \in \mathcal{M}_0^\gamma(\Omega)$.

Huang et al. [3] considered how the nonlinear term $|u|^{q-1}u$ and singular term $\frac{1}{(1+|u|)^{\theta(p-1)}}$ affect the existence of solution to the following degenerate coercivity elliptic problem,

$$\begin{cases} -\mathrm{div}(\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}) + |u|^{q-1}u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.7}$$

They obtained the stability of solution to (1.7) if

$$q > \frac{r(p-1)[1+\theta(p-1)]}{r-p},$$

where $f \in L^1_{\mathrm{loc}}(\Omega \backslash K)$, K is a compact subset in Ω with zero r-capacity ($p < r \le N$). We refer to [4–12] for some related results about existence and non-existence of solutions to elliptic equation with measure data.

There are many papers devoted to study the existence and regularity of solutions to quasilinear elliptic problem with gradient term. Boccardo et al. [13] showed that problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u|\nabla u|^p = \mu, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.8}$$

has solutions if $\mu \in \mathcal{M}_0^p(\Omega)$. Similar results for problem (1.8) with $p = 2$ and $\mu \in L^m(\Omega)$ ($1 \le m \le \frac{N}{2}$) were given by Boccardo [14]. Based on the results of [15–17], Huang et al. [18] investigated the existence of entropy solutions to a class of nonlinear elliptic problem whose prototype is

$$\begin{cases} -\mathrm{div}(\frac{|\nabla u|^{(p-2)}\nabla u + c(x)u^\gamma}{(1+|u|)^{\theta(p-1)}}) + \frac{b(x)|\nabla u|^\lambda}{(1+|u|)^{\theta(p-1)}} = \mu, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where μ is a diffuse measure with bounded variation on Ω, $2 - 1/N < p < N$, $0 < \gamma \le p - 1$, $0 < \lambda \le p - 1$, $c_0(x) \in L^{\frac{N}{p-1},r}(\Omega)$, $\frac{N}{p-1} \le r \le +\infty$, $b(x)$ belongs to some appropriate Lorentz spaces. For some other results see [19–23] and the references therein.

Based on the above research results, in this paper, we are interested in existence and non-existence of solutions to problem (1.1). We prove that there exists a solution $u \in W_0^{1,p}(\Omega)$ to problem (1.1) if and only if the measure μ does not charge the sets with zero p capacity in Ω. Furthermore, we show that if u_n are solutions to (1.1) with $\mu_n \in L^\infty(\Omega)$, then $u_n \to 0$ as $n \to \infty$.

In order to present the main results of this paper, several definitions need to be introduced.

Definition 1.1 *Let K be a compact subset of Ω, $r > 1$ is a real number. The r capacity of K respect to Ω is defined as*

$$\mathrm{cap}_r(K, \Omega) = \inf\Big\{ \int_\Omega |\nabla u|^r \mathrm{d}x : u \in C_0^\infty(\Omega),\ u \ge \chi_K \Big\},$$

where χ_K is the characteristic function of K.

We denote by $\mathcal{M}_b(\Omega)$ the space of all signed measures on Ω. Denote by $\mathcal{M}_0^\gamma(\Omega)$ the space of all measure $\mu \in \mathcal{M}_b(\Omega)$ such that $\mu(E) = 0$ for every set satisfying $\mathrm{cap}_\gamma(E, \Omega) = 0$.

If $\mu \in \mathcal{M}_b(\Omega)$, then $|\mu|$ is a bounded positive measure on Ω.

Let μ be a Radon measure, E is a Borel subset of Ω. The restriction of μ to E is the measure $\lambda = \mu \llcorner E$ defined by $\lambda(B) = \mu(E \cap B)$ for every Borel subset B of Ω. We say that λ is concentrated on a Borel set E if $\lambda = \lambda \llcorner E$.

Proposition 1.2 *Let $\mu \in \mathcal{M}_b(\Omega)$ and $1 < \gamma \le N$. Then μ can be decomposed in a unique way as $\mu_0 + \lambda$, where $\mu_0 \in \mathcal{M}_0^\gamma(\Omega), \lambda = \mu \llcorner E$ and $\mathrm{cap}_\gamma(E, \Omega) = 0$.*

Definition 1.3 *Let $g \in L^1(\Omega)$, a function $u \in W_0^{1,p}(\Omega)$ is a weak solution to Eq. (1.1), provided*

$$\int_\Omega A(x, u, \nabla u) \cdot \nabla \upsilon \mathrm{d}x + \int_\Omega g(x, u, \nabla u)\upsilon \mathrm{d}x = \int_\Omega \upsilon \mathrm{d}\mu, \tag{1.9}$$

for every $\upsilon \in C_0^\infty(\Omega)$.

For all $k > 0$, $s \in \mathbb{R}$, define $T_k(s) = \max(-k, \min\{k, s\})$, $G_k(s) = s - T_k(s)$.

Proposition 1.4 *Let $k > 0$ and $s \in \mathbb{R}$, then we have*

$$G_k(s) = \begin{cases} 0, & \text{if } |s| \le k, \\ s - k\,\mathrm{sgn}(s), & \text{if } |s| > k, \end{cases} \Rightarrow sG_k(s) \ge 0, \quad \forall\, s \in \mathbb{R},$$

and

$$T_k(s) = \begin{cases} s, & \text{if } |s| \le k, \\ k\,\mathrm{sgn}(s), & \text{if } |s| > k, \end{cases} \Rightarrow T_k(s) \le k, \quad \forall\, s \in \mathbb{R}.$$

Firstly, we consider the existence result for problem (1.1) when datum μ is regular.

Theorem 1.5 *Let $\mu \in \mathcal{M}_b(\Omega)$, $1 < p < N$ and (1.1)–(1.6) hold. Then there exists a weak solution $u \in W_0^{1,p}(\Omega)$ to problem (1.1) in the sense of (1.9) if and only if $\mu \in \mathcal{M}_0^p(\Omega)$.*

Remark 1.6 The result of Theorem 1.5 expands the result in [2, Theorem 2.1] in the sense that, if $\mu \in \mathcal{M}_0^p(\Omega)$, then there exists a function $u \in W_0^{1,p}(\Omega)$, such that

$$\mu = -\mathrm{div} A(x, u, \nabla u) + g(x, u, \nabla u)$$

with $g \in L^1(\Omega)$.

Now consider the non-existence of solution to problem (1.1).

Theorem 1.7 *Let $\lambda \in \mathcal{M}_b(\Omega)$ be concentrated on a set E such that $\mathrm{cap}_p(E, \Omega) = 0$, $\{u_n\}$ are weak solutions to*

$$\begin{cases} -\mathrm{div} A(x, u_n, \nabla u_n) + g(x, u_n, \nabla u_n) = f_n, & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega, \end{cases} \tag{1.10}$$

where $\{f_n\} \subset L^\infty(\Omega)$ are non-negative functions such that

$$\lim_{n \to +\infty} \int_\Omega f_n \varphi \mathrm{d}x = \int_\Omega \varphi \mathrm{d}\lambda, \quad \forall \varphi \in C(\bar{\Omega}). \tag{1.11}$$

Then there exists $k > 0$, such that $T_k(u_n) \to 0$ in $W_0^{1,p}(\Omega)$.

Moreover, $u_n \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$, and

$$\lim_{n \to +\infty} \int_\Omega g(x, u_n, \nabla u_n) \varphi \mathrm{d}x = \int_\Omega \varphi \mathrm{d}\lambda, \quad \forall \varphi \in C_0^1(\Omega).$$

Remark 1.8 A quite efficient way to prove the existence of a solution to nonlinear elliptic problems with measure data is to use an approximation method. The preceding theorem can be seen as a non-existence result for problem (1.1). More precisely, according to Proposition 1.2, given a measure $\mu \in \mathcal{M}_b(\Omega)$, it can be decomposed into $\mu_0 + \lambda$. Theorem 1.7 states that, suppose $\mu_0 = 0$, so that $\mu = \lambda$ is singular with respect to p-capacity, if we try to approximate the measure λ with f_n, which is bounded in $L^1(\Omega)$, then $u_n \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega)$.

The structure of this paper is as follows: Section 2 mainly gives a lemma and theorem which play an important role in the process of proof of the main theorem. The proofs of Theorems 1.5 and 1.7 are given in Section 3.

2. Preliminaries

In this paper, C denotes a constant and its value may change from line to line.

To prove the existence of solutions to problem (1.1), the following lemma and theorem are required.

Lemma 2.1 ([13, Lemma 2.4]) *Let $\varphi(t) = te^{\vartheta t^2}$ with $\vartheta = \frac{b^2}{4a^2}$. Then*

$$a\varphi'(t) - b|\varphi(t)| \geq \frac{a}{2}, \quad \forall t \in \mathbb{R}, \tag{2.1}$$

where a and b are two non-negative real numbers.

Theorem 2.2 *Let $f \in L^\infty(\Omega)$, $F \in (L^s(\Omega))^N$ with $s > \frac{N}{p-1}$. Then there exists a weak solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to the problem*

$$\begin{cases} -\mathrm{div}\, A(x,u,\nabla u) + g(x,u,\nabla u) = f - \mathrm{div}(F), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

Proof For simplicity, suppose $f = 0$. The case of $f \neq 0$, can be proved similarly.

Let $n \in \mathbb{N}$ and

$$g_n(x,t,\xi) = \frac{g(x,t,\xi)}{1 + \frac{1}{n}|g(x,t,\xi)|}.$$

Then $g_n(x,t,\xi)$ is bounded and satisfies (1.5). Thanks to (1.6), we have

$$g_n(x,t,\xi)\mathrm{sgn(t)} \geq 0, \tag{2.3}$$

for almost every $x \in \Omega$, $\xi \in \mathbb{R}^N$ and $t \in \mathbb{R}$ with $|t| \geq \sigma$.

Since g_n is bounded, by [24], there exists a weak solution $u_n \in W_0^{1,p}(\Omega)$ to

$$\begin{cases} -\mathrm{div}\, A(x,u_n,\nabla u_n) + g_n(x,u_n,\nabla u_n) = -\mathrm{div}(F), & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega. \end{cases} \tag{2.4}$$

As proved in [25], if $\{u_n\}$ is bounded in $L^\infty(\Omega)$, then there exists a subsequence of u_n, still denoted by u_n, which converges to a solution to (2.2) in $W_0^{1,p}(\Omega)$. Hence, we only need to estimate $\|u_n\|_{L^\infty(\Omega)}$.

To do this, choosing $\int_0^{G_k(u_n)} \frac{1}{(1+k+|t|)^\theta}\mathrm{d}t$ as a test function in (2.4) with $k \geq \sigma$, we obtain

$$\begin{aligned} &\int_\Omega A(x,u_n,\nabla u_n) \cdot \frac{\nabla G_k(u_n)}{(1+k+|G_k(u_n)|)^\theta}\mathrm{d}x + \int_\Omega g_n(x,u_n,\nabla u_n) \int_0^{G_k(u_n)} \frac{1}{(1+k+|t|)^\theta}\mathrm{d}t\mathrm{d}x \\ &= \int_\Omega F \cdot \frac{\nabla G_k(u_n)}{(1+k+|G_k(u_n)|)^\theta}\mathrm{d}x. \end{aligned}$$

On the one hand, from (1.2) it follows that,

$$\int_\Omega A(x,u_n,\nabla u_n) \cdot \frac{\nabla G_k(u_n)}{(1+k+|G_k(u_n)|)^\theta}\mathrm{d}x \geq c_0 \int_\Omega \frac{|\nabla G_k(u_n)|^p}{(1+k+|G_k(u_n)|)^{\theta p}}\mathrm{d}x. \tag{2.5}$$

By Proposition 1.4, $G_k(u_n)u_n \geq 0$ and $G_k \neq 0$ only where $x \in \{x \in \Omega : |u_n(x)| \geq k\}$, then (2.3) implies

$$g_n(x,u_n,\nabla u_n) \int_0^{G_k(u_n)} \frac{1}{(1+k+|t|)^\theta}\mathrm{d}t \geq g_n(x,u_n,\nabla u_n) \frac{G_k(u_n)}{(1+k+|G_k(u_n)|)^\theta} \geq 0. \tag{2.6}$$

By the Young inequality, we get

$$\int_\Omega F\cdot\frac{\nabla G_k(u_n)}{(1+k+|G_k(u_n)|)^\theta}\mathrm{d}x\le C\int_{A_{k,n}}|F|^{p'}\mathrm{d}x+\frac{c_0}{2}\int_\Omega\frac{|\nabla G_k(u_n)|^p}{(1+k+|G_k(u_n)|)^{\theta p}}\mathrm{d}x,\tag{2.7}$$

where $A_{k,n}=\{x\in\Omega:|u_n(x)|\ge k\}$. Combining (2.5)–(2.7), we have

$$\int_\Omega\frac{|\nabla G_k(u_n)|^p}{(1+k+|G_k(u_n)|)^{\theta p}}\mathrm{d}x\le C\int_{A_{k,n}}|F|^{p'}\mathrm{d}x.\tag{2.8}$$

Since $|F|\in L^s(\Omega)$ with $s>p'$, using the Hölder inequality,

$$\int_{A_{k,n}}|F|^{p'}\mathrm{d}x\le\|F\|^{p'}_{L^s(\Omega)}|A_{k,n}|^{1-\frac{p'}{s}}.\tag{2.9}$$

On the other hand, we have

$$\Big(\int_\Omega|(1+k+|G_k(u_n)|)^{1-\theta}|^{p^*}\mathrm{d}x\Big)^{\frac{p}{p^*}}\le C\int_\Omega|\frac{\nabla G_k(u_n)}{(1+k+|G_k(u_n)|)^\theta}|^p\mathrm{d}x,\tag{2.10}$$

in fact, by the Sobolev embedding,

$$\Big(\int_\Omega|\eta(x)|^{p^*}\mathrm{d}x\Big)^{\frac{p}{p^*}}\le C\int_\Omega|\nabla\eta(x)|^p\mathrm{d}x,\ p^*=\frac{Np}{N-p},\quad 1<p<N,\ \forall\,\eta\in W_0^{1,p}(\Omega),$$

for

$$\eta(x)=(1+k+|G_k(u_n)|)^{1-\theta}\Rightarrow\nabla\eta(x)=(1-\theta)\frac{\nabla G_k(u_n)}{(1+k+|G_k(u_n)|)^\theta}.$$

According to (2.8)–(2.10), we obtain

$$\Big(\int_\Omega|(1+k+|G_k(u_n)|)^{1-\theta}|^{p^*}\mathrm{d}x\Big)^{\frac{p}{p^*}}\le C|A_{k,n}|^{1-\frac{p'}{s}}.\tag{2.11}$$

Next, choosing $h>k$ and using the fact that $|G_k(u_n)|\ge h-k$ where $x\in A_{h,n}\subset A_{k,n}$, we have

$$(h+1)^{(1-\theta)p}|A_{h,n}|^{\frac{p}{p^*}}\le\Big(\int_{A_{k,n}}|(1+k+|G_k(u_n)|)^{1-\theta}|^{p^*}\mathrm{d}x\Big)^{\frac{p}{p^*}}.\tag{2.12}$$

By (2.11) and (2.12), we obtain

$$(h+1)^{(1-\theta)p}|A_{h,n}|^{\frac{p}{p^*}}\le C|A_{k,n}|^{1-\frac{p'}{s}},$$

for every $h>k\ge\sigma$ and combining with

$$(h-k)^{(1-\theta)p}<h^{(1-\theta)p}<(h+1)^{(1-\theta)p},$$

we get

$$|A_{h,n}|\le\frac{C}{(h-k)^{(1-\theta)p^*}}|A_{k,n}|^{\frac{p^*}{p}(1-\frac{p'}{s})}.$$

Since $s>\frac{N}{p-1}$ and $0\le\theta<1$, observe that

$$\frac{p^*}{p}(1-\frac{p'}{s})>1,\quad(1-\theta)p^*>0.$$

According to [26, Lemma 4.1], there exists a constant M which depends on n, such that $|A_{k,n}|=0$, for every $k\ge\sigma+M$. This fact shows that $\|u_n\|_{L^\infty(\Omega)}\le\sigma+M$. $\square$

3. Proofs of main results

In the process of proving Theorems 1.5 and 1.7, denote by ε_δ and $\varepsilon_{n,\delta}$, respectively, any function, such that $\lim_{\delta\to 0^+}\varepsilon_\delta = 0$, $\lim_{\delta\to 0^+}\lim_{n\to+\infty}\varepsilon_{n,\delta} = 0$.

3.1. Proof of Theorem 1.5

We give the proof of existence result for problem (1.1) provided the datum μ is regular.

Proof Suppose there exists a weak solution $u \in W_0^{1,p}(\Omega)$ to problem (1.1) with $g \in L^1(\Omega)$, since $A(x,t,\xi) \in (L^{p'}(\Omega))^N$ by (1.3), then $\mu \in L^1(\Omega) + W^{-1,p'}(\Omega)$. Hence, $\mu \in \mathcal{M}_0^p(\Omega)$ by [2, Theorem 2.1].

On the other hand, suppose $\mu \in \mathcal{M}_0^p(\Omega)$. Thanks to [2, Theorem 2.1], μ can be decomposed as $f - \operatorname{div}(F)$ with $f \in L^1(\Omega)$ and $F \in (L^{p'}(\Omega))^N$.

Assume that $\{f_n\} \in L^\infty(\Omega)$ converges to f strongly in $L^1(\Omega)$, $\{F_n\} \in (L^\infty(\Omega))^N$ converges to F strongly in $(L^{p'}(\Omega))^N$. Then according to Theorem 2.2, there exists a weak solution $u_n \in W_0^{1,p}(\Omega)$ to

$$\begin{cases} -\operatorname{div}A(x,u_n,\nabla u_n) + g(x,u_n,\nabla u_n) = f_n - \operatorname{div}(F_n), & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Choosing $\varphi_\sigma := \varphi(\psi(T_\sigma(u_n)))$ as a test function in (3.1) with $\psi(x) = \int_0^x \frac{1}{(1+|t|)^\theta}\mathrm{d}t$, where $\varphi(s)$ appears in Lemma 2.1 with $a = \frac{c_0}{2}$ and $b = b(\sigma)$ ($b(s)$ is given by (1.5)), we get

$$\int_\Omega (A(x,u_n,\nabla u_n)\cdot\nabla\varphi_\sigma + g(x,u_n,\nabla u_n)\varphi_\sigma)\mathrm{d}x = \int_\Omega (f_n\varphi_\sigma - F_n\cdot\nabla\varphi_\sigma)\mathrm{d}x. \tag{3.2}$$

Next we calculate

$$\begin{aligned} \nabla\varphi_\sigma &= \varphi'_\sigma \nabla\psi(T_\sigma(u_n)), \\ \nabla\psi(T_\sigma(u_n)) &= \frac{\partial\psi}{\partial T_\sigma}\nabla T_\sigma(u_n) = \frac{1}{(1+|T_\sigma(u_n)|)^\theta}\nabla T_\sigma(u_n), \end{aligned} \tag{3.3}$$

where $\varphi'_\sigma := \varphi'(\psi(T_\sigma(u_n)))$. Now we present $\Omega = (\Omega\cap\{x\in\Omega : |u_n(x)| > \sigma\}) \cup (\Omega\cap\{x\in\Omega : |u_n(x)| \le \sigma\})$. By Proposition 1.4, we have

$$\frac{\partial T_\sigma(u_n)}{\partial x_i} = \begin{cases} \frac{\partial u_n}{\partial x_i}, & |u_n| \le \sigma \\ 0, & |u_n| > \sigma, \end{cases} \quad i = 1,\ldots,N. \tag{3.4}$$

Then from (3.3) it follows that

$$\nabla\varphi_\sigma = \frac{\varphi'_\sigma}{(1+|T_\sigma(u_n)|)^\theta}\nabla T_\sigma(u_n) = \frac{\varphi'_\sigma}{(1+|T_\sigma(u_n)|)^\theta}\begin{cases} \nabla u_n, & |u_n| \le \sigma, \\ 0, & |u_n| > \sigma. \end{cases} \tag{3.5}$$

Therefore, by (1.2), we obtain

$$\begin{aligned} \int_\Omega A(x,u_n,\nabla u_n)\cdot\nabla\varphi_\sigma &= \int_{\Omega\cap\{|u_n|\le\sigma\}} \frac{\varphi'_\sigma}{(1+|u_n|)^\theta} A(x,u_n,T_\sigma(\nabla u_n))\cdot\nabla T_\sigma(\nabla u_n) \\ &\ge c_0 \int_{\Omega\cap\{|u_n|\le\sigma\}} \frac{|\nabla T_\sigma(u_n)|^p}{(1+|u_n|)^{\theta p}}\varphi'_\sigma \mathrm{d}x = c_0\int_\Omega \frac{|\nabla T_\sigma(u_n)|^p}{(1+|u_n|)^{\theta p}}\varphi'_\sigma\mathrm{d}x. \end{aligned} \tag{3.6}$$

Proposition 3.1 *Let* $\psi(x)=\int_0^x \frac{1}{(1+|t|)^\theta}\mathrm{d}t$. *Then*

$$\psi(T_\sigma(u_n))\le\sigma. \tag{3.7}$$

Proof By Proposition 1.4,

$$T_\sigma(u_n)=\begin{cases} u_n, & |u_n|\le\sigma,\\ \sigma, & |u_n|>\sigma,\end{cases}\Rightarrow T_\sigma(u_n)\le\sigma,\ \ \forall u_n\in\mathbb{R}. \tag{3.8}$$

If $|u_n|>\sigma$, it follows that

$$\psi(T_\sigma(u_n))=\psi(\sigma)=\int_0^\sigma\frac{1}{(1+|t|)^\theta}\mathrm{d}t\le\sigma\ \ \text{and}\ \ \psi(T_\sigma(u_n))\ge 0,$$

i.e., (3.7) is true.

Now, let $|u_n|\le\sigma$. Then we have

$$\psi(T_\sigma(u_n))=\psi(u_n)=\int_0^{u_n}\frac{1}{(1+|t|)^\theta}\mathrm{d}t\le u_n\le\sigma\ \ \text{and}\ \ \psi(T_\sigma(u_n))\ge 0\ \ \text{if}\ u_n\ge 0,$$

It remains for us to consider the case $-\sigma\le u_n<0$. In this case we derive

$$\begin{aligned}\psi(T_\sigma(u_n))=\psi(u_n)&=\int_0^{u_n}\frac{1}{(1+|t|)^\theta}\mathrm{d}t=\int_{u_n}^0\frac{d(1-t)}{(1-t)^\theta}\\ &=\int_{1-u_n}^1\tau^{-\theta}\mathrm{d}\tau=\frac{1-(1-u_n)^{1-\theta}}{1-\theta}\\ &\ge\frac{1-(1+\sigma)^{1-\theta}}{1-\theta}\ge-\sigma,\end{aligned}$$

by virtue of the well known inequality $x^\alpha-1\le\alpha(x-1)$, $x>0$, $0<\alpha<1$. Thus, (3.7) is proved. □

Further, by Lemma 2.1, function $\varphi(t)$ is increasing function, therefore from (3.7) it follows that

$$\varphi_\sigma=\varphi(\psi(T_\sigma(u_n)))\le\varphi(\sigma)\Rightarrow\int_\Omega f_n\varphi_\sigma\mathrm{d}x\le\varphi(\sigma)\int_\Omega|f_n|\mathrm{d}x. \tag{3.9}$$

Next, by the Young inequality,

$$F_n\cdot\nabla\psi(T_\sigma(u_n))\le\frac{\varepsilon^p}{p}|\nabla\psi(T_\sigma(u_n))|^p+\frac{1}{p'\varepsilon^{p'}}|F_n|^{p'},\ \ \forall\,\varepsilon>0,\ \frac{1}{p}+\frac{1}{p'}=1.$$

Now we choose $\varepsilon=(\frac{pc_0}{2})^{\frac{1}{p}}$. Then

$$\frac{1}{p'}\varepsilon^{-p'}=\frac{p-1}{p}(\frac{2}{pc_0})^{\frac{1}{p-1}}\le 1,\ \ \text{if}\ \ c_0\ge\frac{2}{p}(\frac{1}{p-1})^{p-1}.$$

From $\varphi''(t)=2\vartheta t(3+2\vartheta t^2)e^{\vartheta t^2}>0$ for $t>0$, we obtain that $\varphi'(t)$ is an increasing function. Therefore, by (3.7), $\varphi'_\sigma=\varphi'(\psi(T_\sigma(u_n)))\le\varphi'(\sigma)$. From above inequalities, we derive

$$\begin{aligned}F_n\cdot\nabla\varphi_\sigma&=\varphi'_\sigma(F_n\cdot\nabla\psi(T_\sigma(u_n)))\\ &\le\varphi'(\sigma)|F_n|^{p'}+\frac{c_0}{2}|\nabla\psi(T_\sigma(u_n))|^p\varphi'_\sigma\\ &\le\varphi'(\sigma)|F_n|^{p'}+\frac{c_0}{2}\frac{|\nabla T_\sigma(u_n)|^p}{(1+|u_n|)^{\theta p}}\varphi'_\sigma,\end{aligned} \tag{3.10}$$

here we have used (3.3) and (3.8).

At last, from (3.2), (3.6), (3.9) and (3.10), we obtain

$$\begin{aligned} c_0 \int_\Omega \frac{|\nabla T_\sigma(u_n)|^p}{(1+|u_n|)^{\theta p}} \varphi'_\sigma \mathrm{d}x + \int_\Omega g(x,u_n,\nabla u_n)\varphi_\sigma \mathrm{d}x \\ \le \varphi(\sigma)\int_\Omega |f_n|\mathrm{d}x + \varphi'(\sigma)\int_\Omega |F_n|^{p'}\mathrm{d}x + \frac{c_0}{2}\int_\Omega \frac{|\nabla T_\sigma(u_n)|^p}{(1+|u_n|)^{\theta p}}\varphi'_\sigma \mathrm{d}x, \end{aligned}$$

that is

$$\frac{c_0}{2}\int_\Omega \frac{|\nabla T_\sigma(u_n)|^p}{(1+|u_n|)^{\theta p}}\varphi'_\sigma \mathrm{d}x + \int_\Omega g(x,u_n,\nabla u_n)\varphi_\sigma \mathrm{d}x \le \varphi(\sigma)\|f_n\|_{L^1(\Omega)} + \varphi'(\sigma)||F_n|||^{p'}_{L^{p'}(\Omega)}. \tag{3.11}$$

Note that

$$\int_\Omega g(x,u_n,\nabla u_n)\varphi_\sigma \mathrm{d}x = \int_{\{|u_n|<\sigma\}} g(x,u_n,\nabla u_n)\varphi_\sigma \mathrm{d}x + \int_{\{|u_n|\ge\sigma\}} g(x,u_n,\nabla u_n)\varphi_\sigma \mathrm{d}x.$$

Using (1.5), we have

$$\Big|\int_{\{|u_n|<\sigma\}} g(x,u_n,\nabla u_n)\varphi_\sigma \mathrm{d}x\Big| \le b(\sigma)\int_\Omega \big(\frac{|\nabla T_\sigma(u_n)|^p}{(1+|u_n|)^{\theta p}}|\varphi_\sigma| + d(x)\varphi(\sigma)\big)\mathrm{d}x, \tag{3.12}$$

from (1.6) it follows that

$$\int_{\{|u_n|\ge\sigma\}} g(x,u_n,\nabla u_n)\varphi_\sigma \mathrm{d}x \ge \rho\int_{\{|u_n|\ge\sigma\}} \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta p}}\varphi_\sigma \mathrm{d}x. \tag{3.13}$$

By (3.8), for $|u_n| > \sigma$

$$\varphi_\sigma = \varphi(\psi(\sigma)), \quad \psi(\sigma) = \int_0^\sigma \frac{1}{(1+|t|)^\theta}\mathrm{d}t \ge \frac{\sigma}{(1+\sigma)^\theta},$$

because $\varphi(t)$ is the increasing function. From (3.11)–(3.13), we obtain

$$\begin{aligned} \int_\Omega \frac{|\nabla T_\sigma(u_n)|^p}{(1+|u_n|)^{\theta p}}[\frac{c_0}{2}\varphi'_\sigma - b(\sigma)|\varphi_\sigma|]\mathrm{d}x + \rho\varphi(\frac{\sigma}{(1+\sigma)^\theta})\int_{\{|u_n|\ge\sigma\}} \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta p}}\mathrm{d}x \\ \le \varphi(\sigma)(\|f_n\|_{L^1(\Omega)} + b(\sigma)\|d\|_{L^1(\Omega)}) + \varphi'(\sigma)\||F_n|\|^{p'}_{L^{p'}(\Omega)}. \end{aligned} \tag{3.14}$$

This fact, together with (2.1), implies that

$$\int_\Omega \frac{|\nabla T_\sigma(u_n)|^p}{(1+|u_n|)^{\theta p}}\mathrm{d}x + \int_{\{|u_n|\ge\sigma\}} \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta p}}\mathrm{d}x \le C(1 + \|f_n\|_{L^1(\Omega)} + \||F_n|\|^{p'}_{L^{p'}(\Omega)}).$$

Since $\|u_n\|_{L^\infty(\Omega)} \le \sigma + M$ and Ω is a bounded domain, we have

$$\int_\Omega |\nabla T_\sigma(u_n)|^p \mathrm{d}x + \int_{\{|u_n|\ge\sigma\}} |\nabla u_n|^p \mathrm{d}x \le C(1 + \|f_n\|_{L^1(\Omega)} + \||F_n|\|^{p'}_{L^{p'}(\Omega)}).$$

This proves that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Hence, there exists a function $u \in W_0^{1,p}(\Omega)$ and a subsequence, still denoted by $\{u_n\}$, which converges to u weakly in $W_0^{1,p}(\Omega)$ and a.e. in Ω.

Next, we will prove that $u_n \to u$ in $W_0^{1,p}(\Omega)$. Firstly we prove

$$\lim_{h\to+\infty}\sup_{n\in N}\int_{\{|u_n|\ge k\}} |\nabla u_n|^p \mathrm{d}x = 0. \tag{3.15}$$

Taking $T_1(u_n - T_{k-1}(u_n))$ with $k > \sigma + 1$ as a test function in (3.1), we have

$$\int_\Omega A(x,u_n,\nabla u_n)\cdot\nabla T_1(u_n - T_{k-1}(u_n))\mathrm{d}x + \int_\Omega g(x,u_n,\nabla u_n)T_1(u_n - T_{k-1}(u_n))\mathrm{d}x$$

$$= \int_\Omega f_n T_1(u_n - T_{k-1}(u_n))\mathrm{d}x + \int_\Omega F_n \cdot \nabla T_1(u_n - T_{k-1}(u_n))\mathrm{d}x.$$

Note that $\nabla T_1(u_n - T_{k-1}(u_n)) = \nabla u_n$ if $k-1 \le |u_n| \le k$, and is zero elsewhere. In addition, using the fact that $T_1(u_n - T_{k-1}(u_n))u_n \ge 0$ if $|u_n| > \sigma$ and is zero if $|u_n| \le \sigma$, by (1.6), we get

$$g(x,u_n,\nabla u_n)T_1(u_n - T_{k-1}(u_n))\mathrm{d}x \ge |g(x,u_n,\nabla u_n)|\chi_{\{|u_n|\ge k\}}. \tag{3.16}$$

By (1.2) and using $\|u_n\|_{L^\infty(\Omega)} \le \sigma + M$ again, we have

$$\begin{aligned}&\int_\Omega A(x,u_n,\nabla u_n)\cdot \nabla T_k(u_n - T_h(u_n))\mathrm{d}x\\ &\ge c_0 \int_{\{k-1\le |u_n|\le k\}} \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}}\mathrm{d}x\\ &\ge C\int_{\{k-1\le |u_n|\le k\}} |\nabla u_n|^p \mathrm{d}x.\end{aligned} \tag{3.17}$$

By the Young inequality, we can write

$$\int_\Omega F_n\cdot \nabla T_1(u_n - T_{k-1}(u_n))\mathrm{d}x \le \int_{\{k-1\le |u_n|\le k\}} |F_n|^{p'}\mathrm{d}x + \frac{C}{2}\int_{\{k-1\le |u_n|\le k\}} |\nabla u_n|^p\mathrm{d}x. \tag{3.18}$$

Combining (3.16)–(3.18) and dropping positive terms, we obtain

$$\int_{\{|u_n|\ge k\}} |g(x,u_n,\nabla u_n)|\mathrm{d}x \le \int_{\{|u_n|>k-1\}} |f_n|\mathrm{d}x + C\int_{\{k-1\le |u_n|\le k\}} |F_n|^{p'}\mathrm{d}x. \tag{3.19}$$

Since $\{u_n\}$ is bounded in $L^1(\Omega)$, we have

$$\lim_{h\to+\infty}\sup_{n\in N} |(\{|u_n|\ge k-1\})| = 0.$$

Moreover, f_n and $|F_n|$ are strongly compact in $L^1(\Omega)$ and $L^{p'}(\Omega)$, respectively. Thus

$$\lim_{h\to+\infty}\sup_{n\in N}\Big(\int_{\{|u_n|>k-1\}} |f_n|\mathrm{d}x + C\int_{\{k-1\le |u_n|\le k\}} |F_n|^{p'}\mathrm{d}x\Big) = 0. \tag{3.20}$$

By (3.19) and (3.20), using the fact that $k\ge\sigma$ and (1.6), we can get (3.15).

In the following, we prove $T_k(u_n)\to T_k(u)$ in $W_0^{1,p}(\Omega)$ for every $k\ge\sigma$.

Let $k\ge\sigma$, choose $\varphi(T_k(u_n)-T_k(u))$ as a test function in (3.1), then

$$\int_\Omega A(x,u_n,\nabla u_n)\cdot\nabla(T_k(u_n)-T_k(u))\varphi'(T_k(u_n)-T_k(u))\mathrm{d}x+ \tag{A}$$

$$\int_\Omega g(x,u_n,\nabla u_n)\varphi(T_k(u_n)-T_k(u))\mathrm{d}x \tag{B}$$

$$=\int_\Omega f_n\varphi(T_k(u_n)-T_k(u))\mathrm{d}x+ \tag{C}$$

$$\int_\Omega F_n\cdot\nabla(T_k(u_n)-T_k(u))\varphi'(T_k(u_n)-T_k(u))\mathrm{d}x. \tag{D}$$

In the following, for simplicity of notation, denote $\varphi_n := \varphi(T_k(u_n)-T_k(u))$, $\varphi_n' := \varphi'(T_k(u_n) - T_k(u))$.

According to Lemma 2.1, we find

$$\lim_{n\to+\infty}\varphi_n = \varphi(0) = 0,\quad \lim_{n\to+\infty}\varphi_n' = \varphi'(0) = 1. \tag{3.21}$$

First, A can be decomposed as

$$\begin{aligned}(\mathrm{A}) = &\int_\Omega A(x,u_n,\nabla G_k(u_n))\cdot\nabla(T_k(u_n)-T_k(u))\varphi_n'\mathrm{d}x+\\ &\int_\Omega A(x,u_n,\nabla T_k(u_n))\cdot\nabla(T_k(u_n)-T_k(u))\varphi_n'\mathrm{d}x.\end{aligned} \tag{3.22}$$

Due to $\nabla T_k(u_n)=0$ where $\nabla G_k(u_n)\neq 0$, there are

$$\begin{aligned}&\int_\Omega A(x,u_n,\nabla G_k(u_n))\cdot\nabla(T_k(u_n)-T_k(u))\varphi_n'\mathrm{d}x\\ &=-\int_\Omega A(x,u_n,\nabla G_k(u_n))\cdot\nabla T_k(u)\varphi_n'\mathrm{d}x.\end{aligned} \tag{3.23}$$

Since $\nabla T_k(u)=0$ if $x\in\{x\in\Omega:|u(x)|\geq k\}$, we have $\nabla T_k(u)\chi_{\{|u|\geq k\}}\to 0$ a.e. in Ω. Using the fact that $\nabla T_k(u)\in(L^p(\Omega))^N$, we obtain

$$\nabla T_k(u)\chi_{\{|u|\geq k\}}\to 0,\quad \text{strongly in}\quad (L^p(\Omega))^N. \tag{3.24}$$

Combining (3.23) and (3.24) with the fact that $A(x,u_n,\nabla G_k(u_n))$ is bounded in $(L^{p'}(\Omega))^N$, we get

$$\int_\Omega A(x,u_n,\nabla G_k(u_n))\cdot\nabla(T_k(u_n)-T_k(u))\varphi_n'\mathrm{d}x=\varepsilon_n \tag{3.25}$$

and

$$\begin{aligned}&\int_\Omega A(x,u_n,\nabla T_k(u_n))\cdot\nabla(T_k(u_n)-T_k(u))\varphi_n'\mathrm{d}x\\ &=\int_\Omega[A(x,u_n,\nabla T_k(u_n))-A(x,u_n,\nabla T_k(u))]\cdot\nabla(T_k(u_n)-T_k(u))\varphi_n'\mathrm{d}x+\\ &\quad\int_\Omega A(x,u_n,\nabla T_k(u))\cdot\nabla(T_k(u_n)-T_k(u))\varphi_n'\mathrm{d}x.\end{aligned} \tag{3.26}$$

Since $T_k(u_n)\rightharpoonup T_k(u)$ in $W_0^{1,p}(\Omega)$, we have

$$\int_\Omega A(x,u_n,\nabla T_k(u))\cdot\nabla(T_k(u_n)-T_k(u))\varphi_n'\mathrm{d}x=\varepsilon_n. \tag{3.27}$$

Combining (3.25)–(3.27) with (3.22), we find

$$(\mathrm{A})=\int_\Omega[A(x,u_n,\nabla T_k(u_n))-A(x,u_n,\nabla T_k(u))]\cdot\nabla(T_k(u_n)-T_k(u))\varphi_n'\mathrm{d}x+\varepsilon_n. \tag{3.28}$$

Next, decompose (B) into

$$(\mathrm{B})=\int_{\{|u_n|\geq k\}}g(x,u_n,\nabla u_n)\varphi_n\mathrm{d}x+\int_{\{|u_n|<k\}}g(x,u_n,\nabla u_n)\varphi_n\mathrm{d}x. \tag{3.29}$$

According to (2.1), we deduce that $\varphi(t)t\geq 0$. Using the fact that $k-T_k(u)\geq 0$ and $-k-T_k(u)\leq 0$ with $k\geq\sigma$, for $x\in\{x\in\Omega:u_n(x)\geq k\}$, then we have

$$\varphi_n=\varphi(k-T_k(u))\geq 0.$$

For $x\in\{x\in\Omega:u_n(x)\leq -k\}$, we get

$$\varphi_n=\varphi(-k-T_k(u))\leq 0.$$

Thus, from (1.6) it follows that

$$\int_{\{|u_n|\geq k\}} g(x,u_n,\nabla u_n)\varphi_n \mathrm{d}x \geq 0. \tag{3.30}$$

Using (1.5), we get

$$\Big|\int_{\{|u_n|<k\}} g(x,u_n,\nabla u_n)\varphi_n \mathrm{d}x\Big| \leq b(k)\int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+|u_n|)^{\theta p}}|\varphi_n|\mathrm{d}x + b(k)\int_\Omega d(x)|\varphi_n|\mathrm{d}x, \tag{3.31}$$

Since $d\in L^1(\Omega)$, and by (3.21), we have

$$\int_\Omega d(x)|\varphi_n|\mathrm{d}x = \varepsilon_n. \tag{3.32}$$

Combining (3.31) and (3.32) with (1.2), we obtain

$$\Big|\int_{\{|u_n|<k\}} g(x,u_n,\nabla u_n)\varphi_n \mathrm{d}x\Big| \leq \frac{b(k)}{c_0}\int_\Omega A(x,u_n,\nabla T_k(u_n))\cdot\nabla T_k(u_n)|\varphi_n|\mathrm{d}x + \varepsilon_n. \tag{3.33}$$

Note that

$$\int_\Omega A(x,u,\nabla T_k(u))\cdot\nabla(T_k(u_n)-T_k(u))|\varphi_n|\mathrm{d}x = \varepsilon_n \tag{3.34}$$

and

$$\int_\Omega A(x,u_n,\nabla T_k(u_n))\cdot\nabla T_k(u)|\varphi_n|\mathrm{d}x = \varepsilon_n. \tag{3.35}$$

It follows from (3.33)–(3.35), that

$$\begin{aligned}&\Big|\int_{\{|u_n|<k\}} g(x,u_n,\nabla u_n)\varphi_n \mathrm{d}x\Big| \\ &\quad\leq \frac{b(k)}{c_0}\int_\Omega [A(x,u_n,\nabla T_k(u_n))-A(x,u_n,\nabla T_k(u))]\cdot\nabla(T_k(u_n)-T_k(u))|\varphi_n|\mathrm{d}x.\end{aligned} \tag{3.36}$$

By (3.29), (3.30) and (3.36), we get

$$(\mathrm{B}) \geq -\frac{b(k)}{c_0}\int_\Omega [A(x,u_n,\nabla T_k(u_n))-A(x,u_n,\nabla T_k(u))]\cdot\nabla(T_k(u_n)-T_k(u))|\varphi_n|\mathrm{d}x. \tag{3.37}$$

For (C) and (D), since f_n and F_n are strongly compact in $L^1(\Omega)$ and $(L^{p'}(\Omega))^N$, respectively, $T_k(u_n)\rightharpoonup T_k(u)$ in $W_0^{1,p}(\Omega)$, by (3.21), we obtain

$$(\mathrm{C}) = \varepsilon_n, \quad (\mathrm{D}) = \varepsilon_n. \tag{3.38}$$

According to (3.28), (3.37) and (3.38), we get

$$\int_\Omega [A(x,u_n,\nabla T_k(u_n))-A(x,u_n,\nabla T_k(u))]\cdot\nabla(T_k(u_n)-T_k(u))[\varphi_n' - \frac{b(k)}{c_0}|\varphi_n|]\mathrm{d}x = \varepsilon_n. \tag{3.39}$$

Combining (3.39) with Lemma 2.1, we have

$$\int_\Omega [A(x,u_n,\nabla T_k(u_n))-A(x,u_n,\nabla T_k(u))]\cdot\nabla(T_k(u_n)-T_k(u))\mathrm{d}x = \varepsilon_n.$$

This shows that $T_k(u_n)\to T_k(u)$ in $W_0^{1,p}(\Omega)$.

Let $E\subset\Omega$ be a measurable subset. Then

$$\int_E |\nabla u_n|^p\mathrm{d}x = \int_{E\cap\{|u_n|\leq k\}}|\nabla u_n|^p\mathrm{d}x + \int_{E\cap\{|u_n|>k\}}|\nabla u_n|^p\mathrm{d}x. \tag{3.40}$$

Let $\varepsilon > 0$ be fixed. Since

$$\int_{E\cap\{|u_n|>k\}} |\nabla u_n|^p \mathrm{d}x \le \int_{\{|u_n|>k\}} |\nabla u_n|^p \mathrm{d}x,$$

(3.15) implies that there exists a $k \ge \sigma$, such that

$$\int_{E\cap\{|u_n|>k\}} |\nabla u_n|^p \mathrm{d}x \le \frac{\varepsilon}{2}, \quad \forall\, n \in \mathbb{N}. \tag{3.41}$$

For fixed k, due to

$$\int_{E\cap\{|u_n|\le k\}} |\nabla u_n|^p \mathrm{d}x \le \int_E |\nabla T_k(u_n)|^p \mathrm{d}x,$$

the strong compactness of $T_k(u_n)$ in $W_0^{1,p}(\Omega)$ implies, there exists $\delta > 0$, such that

$$\int_{E\cap\{|u_n|\le k\}} |\nabla u_n|^p \mathrm{d}x \le \frac{\varepsilon}{2}, \quad \forall\, n \in \mathbb{N}, \tag{3.42}$$

if $|E| < \delta$.

By (3.40)–(3.42), there exists $\delta > 0$, such that

$$\int_E |\nabla u_n|^p \mathrm{d}x \le \varepsilon, \quad \forall\, n \in \mathbb{N},$$

for every $\varepsilon > 0$ if $|E| < \delta$.

This fact shows that $\{|\nabla u_n|^p\}$ is equi-integrable. Then there exists a subsequence, still denoted by u_n, such that ∇u_n almost everywhere converges to ∇u and u_n converges to u strongly in $W_0^{1,p}(\Omega)$.

In order to pass to the limit to problem (3.1), we need to prove $g(x, u_n, \nabla u_n) \to g(x, u, \nabla u)$ in $L^1(\Omega)$.

Since $g(x, u_n, \nabla u_n)$ almost everywhere converges to $g(x, u, \nabla u)$ in Ω, we only prove the equi-integrability of $\{|g(x, u_n, \nabla u_n)|\}$.

Using the above method, let $E \subset \Omega$ be a measurable subset, we have

$$\begin{aligned}\int_E |g(x, u_n, \nabla u_n)| \mathrm{d}x = &\int_{E\cap\{|u_n|\le k\}} |g(x, u_n, \nabla u_n)| \mathrm{d}x + \\ &\int_{E\cap\{|u_n|>k\}} |g(x, u_n, \nabla u_n)| \mathrm{d}x.\end{aligned} \tag{3.43}$$

Let $\varepsilon > 0$ be fixed. Using the fact that $\nabla T_k(u_n) = \nabla u_n$ if $|u_n| \le k$ and (1.5), we get

$$\begin{aligned}\int_{E\cap\{|u_n|\le k\}} |g(x, u_n, \nabla u_n)| \mathrm{d}x &\le b(k) \int_E \Big(\frac{|\nabla T_k(u_n)|^p}{(1+|u_n|)^{\theta p}} + d(x)\Big) \mathrm{d}x \\ &\le b(k) \int_E \Big(\frac{|\nabla T_k(u_n)|^p}{(1-k)^{\theta p}} + d(x)\Big) \mathrm{d}x.\end{aligned}$$

Due to $d \in L^1(\Omega)$ and the fact that $T_k(u_n)$ is strongly compact in $W_0^{1,p}(\Omega)$, then there exists $\delta > 0$, such that

$$\int_{E\cap\{|u_n|\le k\}} |g(x, u_n, \nabla u_n)| \mathrm{d}x \le \frac{\varepsilon}{2}, \quad \forall n \in \mathbb{N}, \tag{3.44}$$

for every $\varepsilon > 0$ if $|E| < \delta$.

Notice that

$$\int_{E\cap\{|u_n|>k\}}|g(x,u_n,\nabla u_n)|\mathrm{d}x\le\int_{\{|u_n|>k\}}|g(x,u_n,\nabla u_n)|\mathrm{d}x.$$

Using (3.15) and (1.5), there exists $k\ge\sigma$, such that

$$\int_{E\cap\{|u_n|>k\}}|g(x,u_n,\nabla u_n)|\mathrm{d}x\le\frac{\varepsilon}{2},\quad\forall n\in\mathbb{N}. \tag{3.45}$$

By (3.43)–(3.45), we show that $\{|g(x,u_n,\nabla u_n)|\}$ is equi-integrable. Hence, we can get (1.9) by taking the limit of (3.1). □

3.2. Proof of Theorem 1.7

Before giving the proof of Theorem 1.7, we need to construct a suitable collection of cut-off function.

Lemma 3.2 ([13, Lemma 3.3]) *Let $\lambda\in\mathcal{M}_b(\Omega)$ be a non-negative measure concentrated on a set E and $cap_p(E,\Omega)=0$. Then there exists a $\{\psi_\delta\}\in C_0^\infty(\Omega)$, such that*

$$\int_\Omega|\nabla\psi_\delta|^p\mathrm{d}x=\varepsilon_\delta,\quad 0\le\psi_\delta\le 1,\quad\int_\Omega(1-\psi_\delta)\mathrm{d}\lambda=\varepsilon_\delta, \tag{3.46}$$

for every $\delta>0$.

In the following, we give the proof of Theorem 1.7.

Proof Since f_n are non-negative, u_n are also non-negative by (1.6). Due to that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, there exists a subsequence $\{u_n\}$, a function $u\in W_0^{1,p}(\Omega)$ and $G\in(L^{p'}(\Omega))^N$, such that $u_n\rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and a.e. in Ω, $A(x,u_n,\nabla u_n)\rightharpoonup G$ in $(L^{p'}(\Omega))^N$.

Since b is a continuous function, there exists $k>0$, such that

$$b(k)k\le\frac{c_0}{2}. \tag{3.47}$$

Choosing $v=(k-T_k(u_n))\psi_\delta$ as a test function in (1.10), since $v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, we have

$$-\int_\Omega A(x,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\psi_\delta\mathrm{d}x+ \tag{A}$$

$$\int_\Omega[A(x,u_n,\nabla u_n)\cdot\nabla\psi_\delta](k-T_k(u_n))\mathrm{d}x+ \tag{B}$$

$$\int_\Omega g(x,u_n,\nabla u_n)(k-T_k(u_n))\psi_\delta\mathrm{d}x \tag{C}$$

$$=\int_\Omega f_n(k-T_k(u_n))\psi_\delta\mathrm{d}x. \tag{D}$$

By (1.2), we get

$$(\mathrm{A})\le -c_0\int_\Omega\frac{|\nabla T_k(u_n)|^p}{(1+u_n)^{\theta(p-1)}}\psi_\delta\mathrm{d}x.$$

Since $k-T_k(u_n)\to k-T_k(u)$ a.e. in Ω, we have that $\nabla\psi_\delta(k-T_k(u_n))\to\nabla\psi_\delta(k-T_k(u))$ in $(L^p(\Omega))^N$. Then by (3.46), we find

$$(\mathrm{B})=\int_\Omega G\cdot\nabla\psi_\delta(k-T_k(u))\mathrm{d}x+\varepsilon_n=\varepsilon_{n,\delta}.$$

Using (1.5), we have that

$$|(\mathrm{C})| \leq \int_{\{0\leq u_n\leq k\}} b(k)(k-T_k(u_n))\psi_\delta(\frac{|\nabla T_k(u_n)|^p}{(1+u_n)^{\theta p}}+d(x))\mathrm{d}x.$$

According to (3.47), we obtain

$$\int_{\{0\leq u_n\leq k\}} b(k)(k-T_k(u_n))\psi_\delta d(x)\mathrm{d}x \leq \frac{c_0}{2}\int_\Omega \psi_\delta d(x)\mathrm{d}x = \varepsilon_\delta.$$

Hence

$$|(\mathrm{C})| \leq \frac{c_0}{2}\int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+u_n)^{\theta p}}\psi_\delta \mathrm{d}x + \varepsilon_\delta.$$

Clearly, (D) ≥ 0. Then

$$\frac{c_0}{2}\int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+u_n)^{\theta p}}\psi_\delta \mathrm{d}x \leq \varepsilon_{n,\delta}.$$

Due to

$$C\int_\Omega |\nabla T_k(u_n)|^p\psi_\delta \mathrm{d}x \leq \int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+k)^{\theta p}}\psi_\delta \mathrm{d}x \leq \int_\Omega \frac{|\nabla T_k(u_n)|^p}{(1+u_n)^{\theta p}}\psi_\delta \mathrm{d}x,$$

we get

$$\int_\Omega |\nabla T_k(u_n)|^p\psi_\delta \mathrm{d}x = \varepsilon_{n,\delta}. \tag{3.48}$$

Choose $T_k(u_n)(1-\psi_\delta)$ as a test function in (1.10), by the same way, we have

$$\int_\Omega |\nabla T_k(u_n)|^p(1-\psi_\delta) = \varepsilon_{n,\delta}. \tag{3.49}$$

By (3.48) and (3.49), we obtain

$$\int_\Omega |\nabla T_k(u_n)|^p \mathrm{d}x = \varepsilon_n,$$

that is $T_k(u_n) \to 0$ in $W_0^{1,p}(\Omega)$. Since the limit is independent of the choice of subsequence, the whole sequence $\{u_n\}$ is such that sequence $T_k(u_n) \to 0$ in $W_0^{1,p}(\Omega)$. Thus, $u = 0$ and so u_n converges weakly to 0 in $W_0^{1,p}(\Omega)$.

In order to prove the second part of this theorem, observe that the strong convergence to zero of $T_k(u_n)$ follows $\nabla u_n \to 0$ a.e. in Ω. Then (1.3) and $A(x,u_n,\nabla u_n) \rightharpoonup G$ in $(L^{p'}(\Omega))^N$ imply that $G = 0$. Choosing $\varphi \in C_0^1(\Omega)$ as test function in (1.10), we have

$$\int_\Omega A(x,u_n,\nabla u_n)\cdot\nabla\varphi \mathrm{d}x + \int_\Omega g(x,u_n,\nabla u_n)\varphi \mathrm{d}x = \int_\Omega f_n\varphi \mathrm{d}x. \tag{3.50}$$

Since $G = 0$, we have

$$\int_\Omega A(x,u_n,\nabla u_n)\cdot\nabla\varphi \mathrm{d}x = \varepsilon_n. \tag{3.51}$$

Combining (3.50) and (3.51) with (1.11), we obtain

$$\lim_{n\to+\infty}\int_\Omega g(x,u_n,\nabla u_n)\varphi \mathrm{d}x = \int_\Omega \varphi \mathrm{d}\lambda,$$

for every $\varphi \in C_0^1(\Omega)$. This concludes the proof of Theorem 1.7. □

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References

[1] P. BÉNILAN, L. BOCCARDO, T. GALLOUËT, et al. *An L^1 theory of existence and uniqueness of nonlinear elliptic equations*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 1995, **22**(2): 241–273.

[2] L. BOCCARDO, T. GALLOUËT, L. ORSINA. *Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data*. Ann. Inst. H. Poincaré Anal. Non Linéaire, 1996, **13**(5): 539–551.

[3] Shuibo HUANG, Qiaoyu TIAN, Jie WANG, et al. *Stability for noncoercive elliptic equations*. Electron. J. Differential Equations, 2016, **242**: 1–11.

[4] D. ARCOYA, L. BOCCARDO. *Regularizing effect of the interplay between coefficients in some elliptic equations*. J. Funct. Anal., 2015, **268**(5): 1153–1166.

[5] H. BAMDAD, N. NALIMA, M. S. JOHN. *On regularizations of the Dirac delta distribution*. J. Comput. Phys., 2016, **305**: 423–447.

[6] G. R. CIRMI, S. D'ASERO, S. LEONARDI. *Gradient estimate for solutions of nonlinear singular elliptic equations below the duality exponent*. Math. Methods Appl. Sci., 2018, **41**(1): 261–269.

[7] M. DINDOŠ, J. PIPHER. *Regularity theory for solutions to second order elliptic operators with complex coefficients and the L^p Dirichlet problem*. Adv. Math., 2019, **341**: 255–298.

[8] A. JUNICHI. *Regularity of weak solutions for degenerate quasilinear elliptic equations involving operator curl*. J. Math. Anal. Appl., 2015, **426**(2): 872–892.

[9] D. KASTNER. *Existence and regularizing effect of degenerate lower order terms in elliptic equations beyond the Hardy constant*. Adv. Nonlinear Stud., 2018, **18**(4): 775–783.

[10] F. OLIVA. *Regularizing effect of absorption terms in singular problems*. J. Math. Anal. Appl., 2019, **472**(1): 1136–1166.

[11] L. ORSINA, A. PRIGNET. *Non-existence of solutions for some nonlinear elliptic equations involving measures*. Proc. Roy. Soc. Edinburgh Sect. A, 2000, **130**(1): 167–187.

[12] Maoji RI, Shuibo HUANG, Canyun HUANG. *Non-existence of solutions to some degenerate coercivity elliptic equations involving measures data*. Electron. Res. Arch., 2020, **28**(1): 165–182.

[13] L. BOCCARDO, T. GALLOUËT, L. ORSINA. *Existence and nonexistence of solutions for some nonlinear elliptic equations*. J. Anal. Math., 1997, **73**: 203–223.

[14] L. BOCCARDO. *Dirichlet problems with singular and gradient quadratic lower order terms*. ESAIM Control Optim. Calc. Var., 2008, **14**(3): 411–426.

[15] M. FRANCESCA BETTA, A. MERCALDO, F. MURAT, et al. *Existence of renormalized solutions to nonlinear elliptic equations with a lower-order term and right-hand side a measure*. J. Math. Pures Appl. (9), 2003, **82**(1): 90–124.

[16] O. GUIBÉ, A. MERCALDO. *Existence and stability results for renormalized solutions to noncoercive nonlinear elliptic equations with measure data*. Potential Anal., 2006, **25**(3): 223–258.

[17] O. GUUIBÉ, A. MERCALDO. *Existence of renormalized solutions to nonlinear elliptic equations with two lower order terms and measure data*. Trans. Amer. Math. Soc., 2008, **360**(2): 643–669.

[18] Shuibo HUANG, Tong SU, Xinsheng DU, et al. *Entropy solutions to noncoercive nonlinear elliptic equations with measure data*. Electron. J. Differential Equations, 2019, **97**: 1–22.

[19] A. ALVION, L. BOCCARDO, V. FERONE, et al. *Existence results for nonlinear elliptic equations with degenerate coercivity*. Ann. Mat. Pura Appl. (4), 2003, **182**(1): 53–79.

[20] V. DE CICCO, D. GIACHETTI, F. OLIVA, et al. *The Dirichlet problem for singular elliptic equations with general nonlinearities*. Calc. Var. Partial Differential Equations, 2019, **58**(4): Paper No. 129, 40 pp.

[21] S. FLAVIA. *On a class of quasilinear elliptic equations with degenerate coerciveness and measure data*. Adv. Nonlinear Stud., 2018, **18**(2): 361–392.

[22] T. KLIMAIAK, A. ROZKOSZ. *On semilinear elliptic equations with diffuse measures*. NoDEA Nonlinear Differential Equations Appl., 2018, **25**(4): Paper No. 35, 23 pp.

[23] Maoji RI, Shuibo HUANG, Qiaoyu TIAN, et al. *Existence of $W_0^{1,1}(\Omega)$ solutions to nonlinear elliptic equation with singular natural growth term*. AIMS Math., 2020, **5**(6): 5791–5800.

[24] J. LERAY, J. L. LIONS. *Quelques résultats de Višik sur les problèmes elliptiques semi-linéaires par les méthodes de Minty et Browder*. Bull. Soc. Math. France, 1965, **9**(3): 97–107. (in French)

[25] L. BOCCARDO, F. MURAT, J. P. PUEL. *L^∞-estimates for some nonlinear partial differential equations and application to an existence result*. SIAM J. Math. Anal., 1992, **23**(2): 326–333.

[26] G. STAMPACCHIA. *Le problème de Dirichlet pour les équations elliptiques du second ordre àcoefficients discontinus*. Ann. Inst. Fourier (Grenoble), 1965, **15**(1): 189–258. (in French)