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New Sobolev-Weinstein Spaces and Applications

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Abstract In this paper, we consider the generalized Weinstein operator $\Delta_W^{d,\alpha,n}$, we introduce new Sobolev-Weinstein spaces denoted $\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$, $s \in \mathbb{R}$, associated with the generalized Weinstein operator and we investigate their properties. Next, as application, we study the extremal functions on the spaces $\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ using the theory of reproducing kernels.

Keywords generalized Weinstein operator; generalized Weinstein transform; sobolev spaces; extremal functions; reproducing kernels

MR(2020) Subject Classification 32A50; 32B10; 46E35; 46F12; 43A32

1. Introduction

In this paper, we consider the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ defined on $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times [0,+\infty]$, by

$$\Delta_W^{\alpha,d,n} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} - \frac{4n(\alpha+n)}{x_{d+1}^2} = \Delta_d + L_{\alpha,n} \tag{1.1}$$

where $n \in \mathbb{N}$, $\alpha > -\frac{1}{2}$, Δ_d is the Laplacian for the d first variables and $L_{\alpha,n}$ is the second-order singular differential operator on the half line given by

$$L_{\alpha,n} = \frac{\partial^2}{\partial x_{d+1}^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} - \frac{4n(\alpha+n)}{x_{d+1}^2}. \tag{1.2}$$

For $n = 0$, we regain the classical Weinstein operator $\Delta_W^{\alpha,d}$ given by

$$\Delta_W^{\alpha,d} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} = \Delta_d + L_{\alpha}, \tag{1.3}$$

$L_\alpha = L_{\alpha,0}$ is the Bessel operator [1–7].

The harmonic analysis associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ is studied by Aboulez, Achak, Daher and Loualid [8, 9].

For all $f \in L^1(\mathbb{R}^{d+1}_+, \mathrm{d}\mu_{\alpha,d}(x))$, we define the Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ by

$$\forall \lambda \in \mathbb{R}^{d+1}_+,\ \mathscr{F}_W^{\alpha,d,n}(f)(\lambda) = \int_{\mathbb{R}^{d+1}_+} f(x) \Lambda_{\alpha,d,n}(x,\lambda) \mathrm{d}\mu_{\alpha,d}(x)$$

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where $\mu_{\alpha,d}$ is the measure defined on $\mathbb{R}_+^{d+1}$ by

$$\mathrm{d}\mu_{\alpha,d}(x) = x_{d+1}^{2\alpha+1}\mathrm{d}x \tag{1.4}$$

and $\Lambda_{\alpha,d,n}$ is the generalized Weinstein kernel given by

$$\forall x, y\in \mathbb{C}^{d+1},\ \Lambda_{\alpha,d,n}(x,y) = x_{d+1}^{2n} e^{-i\langle x', y'\rangle} j_{\alpha+2n}(x_{d+1}y_{d+1}),$$

$x = (x', x_{d+1}),\ x' = (x_1, x_2, \dots, x_d)$ and j_α is the normalized Bessel function of index α defined by

$$\forall \xi\in \mathbb{C},\ j_\alpha(\xi) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\alpha+1)}(\frac{\xi}{2})^{2n}. \tag{1.5}$$

We design by $\mathscr{S}_*(\mathbb{R}^{d+1})$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable and $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ the subspace of $\mathscr{S}_*(\mathbb{R}^{d+1})$ consisting of functions f such that

$$\forall k \in \{1, \dots, 2n-1\},\ \frac{\partial^k f}{\partial x_{d+1}^k}(x', 0) = f(x', 0) = 0.$$

For all $s \in \mathbb{R}$, we define the generalized Sobolev-Weinstein space $\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}_+^{d+1})$ as the set of all $u \in \mathscr{S}'_{n,*}$ (the strong dual of the space $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$) such that $\mathscr{F}_W^{\alpha,d,n}(u)$ is a function and

$$\int_{\mathbb{R}_+^{d+1}} (1 + \|\xi\|^2)^s |\mathscr{F}_W^{\alpha,d,n}(u)(\xi)|^2 \mathrm{d}\mu_{\alpha+2n,d}(\xi) < \infty.$$

We investigate the properties of $\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}_+^{d+1})$. Using the theory of reproducing kernels, we study the extremal functions on the spaces $\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}_+^{d+1})$. The contents of the paper are as follows:

In the second section, we recapitulate some results related to the harmonic analysis associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ given by the relation (1.1).

The Section 3 is devoted to define and study the generalized Sobolev-Weinstein space $\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}_+^{d+1})$.

Finally, in the last section, as application, using the theory of reproducing kernels, we give good estimates of extremal functions on the spaces $\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}_+^{d+1})$.

2. Preliminaries

In this section, we shall collect some results and definitions from the theory of the harmonic analysis associated with the Generalized Weinstein operator $\Delta_W^{\alpha,d,n}$ defined on $\mathbb{R}_+^{d+1}$ by the relation (1.1).

Notations. In what follows, we need the following notations

- $\mathscr{C}_*(\mathbb{R}^{d+1})$, the space of continuous functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathscr{E}_*(\mathbb{R}^{d+1})$, the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathscr{S}_*(\mathbb{R}^{d+1})$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathscr{D}_*(\mathbb{R}^{d+1})$, the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^{d+1}$ which are of compact support, even with respect to the last variable.
- $\mathscr{H}_*(\mathbb{C}^{d+1})$, the space of entire functions on $\mathbb{C}^{d+1}$, even with respect to the last variable, rapidly decreasing and of exponential type.

• $\mathscr{M}_n$, the map defined by

$$\forall x \in \mathbb{R}_+^{d+1},\ \mathscr{M}_n(f)(x) = x_{d+1}^{2n} f(x), \tag{2.1}$$

where $x = (x', x_{d+1})$ and $x' = (x_1, x_2, \dots, x_d)$.

• $L^p_{\alpha,n}(\mathbb{R}_+^{d+1})$, $1 \le p \le +\infty$, the space of measurable functions on $\mathbb{R}_+^{d+1}$ such that

$$\|f\|_{\alpha,n,p} = \Big[\int_{\mathbb{R}_+^{d+1}} |\mathscr{M}_n^{-1} f(x)|^p \mathrm{d}\mu_{\alpha+2n,d}(x)\Big]^{\frac{1}{p}} < +\infty,\ \text{if } 1 \le p < +\infty;$$

$$\|f\|_{\alpha,n,\infty} = \operatorname{ess}\sup_{x\in\mathbb{R}_+^{d+1}} |\mathscr{M}_n^{-1} f(x)| < +\infty,$$

where $\mu_{\alpha,d}$ is the measure given by the relation (1.4).

• $L^p_\alpha(\mathbb{R}_+^{d+1}) := L^p_{\alpha,0}(\mathbb{R}_+^{d+1})$, $1 \le p \le +\infty$, and $\|f\|_{\alpha,p} := \|f\|_{\alpha,0,p}$.

• $\mathscr{E}_{n,*}(\mathbb{R}^{d+1})$, $\mathscr{D}_{n,*}(\mathbb{R}^{d+1})$ and $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$, repespectively, stand for the subspace of $\mathscr{E}_*(\mathbb{R}^{d+1})$, $\mathscr{D}_*(\mathbb{R}^{d+1})$ and $\mathscr{S}_*(\mathbb{R}^{d+1})$ consisting of functions f such that

$$\forall k \in \{1, \dots, 2n-1\},\ \frac{\partial^k f}{\partial x_{d+1}^k}(x', 0) = f(x', 0) = 0.$$

Let us begin by the following result.

Lemma 2.1 ([8,9]) *(i) The map $\mathscr{M}_n$ is an isomorphism from $\mathscr{E}_*(\mathbb{R}^{d+1})$ (resp., $\mathscr{S}_*(\mathbb{R}^{d+1})$) onto $\mathscr{E}_{n,*}(\mathbb{R}^{d+1})$ (resp., $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$).*

(ii) For all $f \in \mathscr{E}_*(\mathbb{R}^{d+1})$, we have

$$L_{\alpha,n} \circ \mathscr{M}_n(f) = \mathscr{M}_n \circ L_{\alpha+2n}(f). \tag{2.2}$$

(iii) For all $f \in \mathscr{E}_*(\mathbb{R}^{d+1})$, we have

$$\Delta_W^{\alpha,d,n} \circ \mathscr{M}_n(f) = \mathscr{M}_n \circ \Delta_W^{\alpha+2n,d}(f). \tag{2.3}$$

(iv) For all $f \in \mathscr{E}_*(\mathbb{R}^{d+1})$ and $g \in \mathscr{D}_{n,*}(\mathbb{R}^{d+1})$, we have

$$\int_{\mathbb{R}_+^{d+1}} \Delta_W^{\alpha,d,n}(f)(x) g(x) \mathrm{d}\mu_{\alpha,d}(x) = \int_{\mathbb{R}_+^{d+1}} f(x) \Delta_W^{\alpha,d,n} g(x) \mathrm{d}\mu_{\alpha,d}(x). \tag{2.4}$$

Definition 2.2 *The generalized Weinstein kernel $\Lambda_{\alpha,d,n}$ is the function given by*

$$\forall x, y \in \mathbb{C}^{d+1},\ \Lambda_{\alpha,d,n}(x,y) = x_{d+1}^{2n} e^{-i\langle x', y'\rangle} j_{\alpha+2n}(x_{d+1} y_{d+1}), \tag{2.5}$$

where $x = (x', x_{d+1})$, $x' = (x_1, x_2, \dots, x_d)$ and j_α is the normalized Bessel function of index α defined by the relation (1.5).

It is easy to see that the generalized Weinstein kernel $\Lambda_{\alpha,d,n}$ satisfies the following properties.

Proposition 2.3 *(i) We have*

$$\forall x, y \in \mathbb{R}^{d+1},\ \overline{\Lambda_{\alpha,d,n}(x,y)} = \Lambda_{\alpha,d,n}(x,-y) = \Lambda_{\alpha,d,n}(-x,y). \tag{2.6}$$

(ii) We have

$$\forall x, y \in \mathbb{R}_+^{d+1},\ |\Lambda_{\alpha,d,n}(x,y)| \le x_{d+1}^{2n}. \tag{2.7}$$

(iii) The function $x \mapsto \Lambda_{\alpha,d,n}(x,y)$ *satisfies the differential equation*

$$\triangle_W^{\alpha,d,n}(\Lambda_{\alpha,d,n}(.,y))(x) = -\|y\|^2\Lambda_{\alpha,d,n}(x,y). \tag{2.8}$$

(iv) For all $x,\ y \in \mathbb{C}^{d+1}$*, we have*

$$\Lambda_{\alpha,d,n}(x,y) = a_{\alpha+2n}e^{-i\langle x',y'\rangle}x_{d+1}^{2n}\int_0^1 (1-t^2)^{\alpha+2n-\frac{1}{2}}\cos(tx_{d+1}y_{d+1})\mathrm{d}t, \tag{2.9}$$

where a_α *is the constant given by*

$$a_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}. \tag{2.10}$$

Definition 2.4 *The generalized Weinstein transform* $\mathscr{F}_W^{\alpha,d,n}$ *is given for* $f \in L^1_{\alpha,n}(\mathbb{R}^{d+1}_+)$ *by*

$$\forall\lambda \in \mathbb{R}^{d+1}_+,\ \mathscr{F}_W^{\alpha,d,n}(f)(\lambda) = \int_{\mathbb{R}^{d+1}_+} f(x)\Lambda_{\alpha,d,n}(x,\lambda)\mathrm{d}\mu_{\alpha,d}(x), \tag{2.11}$$

where $\mu_{\alpha,d}$ *is the measure on* $\mathbb{R}^{d+1}_+$ *given by the relation (1.4).*

Example 2.5 Let $E_{t,n}$, $t > 0$, $n \in \mathbb{N}$, be the function defined by

$$\forall x \in \mathbb{R}^{d+1},\ E_{t,n}(x) = C_{\alpha+2n,d}x_{d+1}^{2n}e^{-t\|x\|^2},$$

where $C_{\alpha,d}$ is the constant given by

$$C_{\alpha,d} = \frac{1}{(2\pi)^{\frac{d}{2}}2^\alpha\Gamma(\alpha+1)}. \tag{2.12}$$

Then the Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ of $E_{t,n}$ is given by

$$\forall\lambda \in \mathbb{R}^{d+1}_+,\ \mathscr{F}_W^{\alpha,d,n}(E_{t,n})(\lambda) = \frac{1}{(2t)^{\alpha+2n+\frac{d}{2}+1}}e^{-\frac{\|\lambda\|^2}{4t}}.$$

Remark 2.6 The generalized Weinstein transform $\mathscr{F}_W^{\alpha,d,n}$ can be written in the form:

$$\mathscr{F}_W^{\alpha,d,n} = \mathscr{F}_W^{\alpha+2n,d} \circ \mathscr{M}_n^{-1}, \tag{2.13}$$

where $\mathscr{F}_W^{\alpha,d} = \mathscr{F}_W^{\alpha,d,0}$ is the classical Weinstein transform.

Some basic properties of the transform $\mathscr{F}_W^{\alpha,d,n}$ are summarized in the following results.

Proposition 2.7 ([9]) *(i) For all* $f \in L^1_{\alpha,n}(\mathbb{R}^{d+1}_+)$*, we have*

$$\|\mathscr{F}_W^{\alpha,d,n}(f)\|_{\alpha,n,\infty} \le \|f\|_{\alpha,n,1}. \tag{2.14}$$

(ii) Let $m \in \mathbb{N}$ *and* $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$*. We have*

$$\forall\lambda \in \mathbb{R}^{d+1}_+,\ \mathscr{F}_W^{\alpha,d,n}[(\triangle_W^{\alpha,d,n})^m f](\lambda) = (-1)^m\|\lambda\|^{2m}\mathscr{F}_W^{\alpha,d,n}(f)(\lambda). \tag{2.15}$$

(iii) Let $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ *and* $m \in \mathbb{N}$*. For all* $\lambda \in \mathbb{R}^{d+1}_+$*,we have*

$$(\triangle_W^{\alpha,d,n})^m[\mathscr{M}_n\mathscr{F}_W^{\alpha,d,n}(f)](\lambda) = \mathscr{M}_n\mathscr{F}_W^{\alpha,d,n}(P_m f)(\lambda), \tag{2.16}$$

where $P_m(\lambda) = (-1)^m\|\lambda\|^{2m}$.

Theorem 2.8 ([9]) *(i) Let $f \in L^1_{\alpha,n}(\mathbb{R}^{d+1}_+)$. If $\mathscr{F}^{\alpha,d,n}_W(f) \in L^1_{\alpha+2n}(\mathbb{R}^{d+1}_+)$, then we have*

$$f(x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(y)\Lambda_{\alpha,d,n}(-x,y)\mathrm{d}\mu_{\alpha+2n,d}(y), \text{ a.e., } x \in \mathbb{R}^{d+1}_+, \tag{2.17}$$

where $C_{\alpha,d}$ is the constant given by the relation (2.12).

(ii) The Weinstein transform $\mathscr{F}^{\alpha,d,n}_W$ is a topological isomorphism from $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ onto $\mathscr{S}_*(\mathbb{R}^{d+1})$ and from $\mathscr{D}_{n,*}(\mathbb{R}^{d+1})$ onto $\mathscr{H}_*(\mathbb{C}^{d+1})$.

The following Theorem is as an immediate consequence of the relation (2.13) and the properties of the transform $\mathscr{F}^{\alpha,d}_W$ (see [1–4]).

Theorem 2.9 *(i) For all $f, g \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$, we have the following Parseval formula*

$$\int_{\mathbb{R}^{d+1}_+} f(x)\overline{g(x)}\mathrm{d}\mu_{\alpha,d}(x) = C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(f)(\lambda)\overline{\mathscr{F}^{\alpha,d,n}_W(g)(\lambda)}\mathrm{d}\mu_{\alpha+2n,d}(\lambda). \tag{2.18}$$

(ii) (Plancherel formula) For all $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$, we have

$$\int_{\mathbb{R}^{d+1}_+} |f(x)|^2\mathrm{d}\mu_{\alpha,d}(x) = C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+} |\mathscr{F}^{\alpha,d,n}_W(f)(\lambda)|^2\mathrm{d}\mu_{\alpha+2n,d}(\lambda). \tag{2.19}$$

(iii) (Plancherel Theorem) The transform $\mathscr{F}^{\alpha,d,n}_W$ extends uniquely to an isometric isomorphism from $L^2(\mathbb{R}^{d+1}_+,\ d\mu_{\alpha,d}(x))$ onto $L^2(\mathbb{R}^{d+1}_+,\ C^2_{\alpha+2n,d}d\mu_{\alpha+2n,d}(x))$.

Definition 2.10 *The translation operator $T^{\alpha,d,n}_x$, $x \in \mathbb{R}^{d+1}_+$, associated with the operator $\Delta^{\alpha,d,n}_W$ is defined on $\mathscr{E}_{n,*}(\mathbb{R}^{d+1}_+)$ by*

$$\forall y \in \mathbb{R}^{d+1}_+,\ T^{\alpha,d,n}_x f(y) = x^{2n}_{d+1}y^{2n}_{d+1}T^{\alpha+2n,d}_x \mathscr{M}^{-1}_n f(y), \tag{2.20}$$

where

$$T^{\alpha,d}_x f(y) = \frac{a_\alpha}{2}\int_0^\pi f(x'+y',\ \sqrt{x^2_{d+1}+y^2_{d+1}+2x_{d+1}y_{d+1}\cos\theta})(\sin\theta)^{2\alpha}\mathrm{d}\theta, \tag{2.21}$$

$x'+y' = (x_1+y_1,\ldots,x_d+y_d)$ and a_α is the constant given by (2.10).

We need the following Lemmas.

Example 2.11 Let $\phi_{t,n}$, $t > 0$, be the function defined by

$$\forall x \in \mathbb{R}^{d+1}_+,\ \phi_{t,n}(x) = \frac{x^{2n}_{d+1}}{(2t)^{\alpha+2n+\frac{d}{2}+1}}e^{-\frac{\|x\|^2}{4t}}.$$

For all $x, y \in \mathbb{R}^{d+1}_+$, we have

$$T^{\alpha,d,n}_x(\phi_{t,n})(y) = \frac{x^{2n}_{d+1}y^{2n}_{d+1}}{(2t)^{\alpha+2n+\frac{d}{2}+.1}}e^{-\frac{\|x\|^2+\|y\|^2}{4t}}\Lambda_{\alpha+2n,d}(x,-i\frac{y}{2t}).$$

The following proposition summarizes some properties of the generalized Weinstein translation operator.

Proposition 2.12 *(i) For $f \in \mathscr{E}_{n,*}(\mathbb{R}^{d+1})$, we have*

$$\forall x,\ y \in \mathbb{R}^{d+1}_+,\ T^{\alpha,d,n}_x f(y) = T^{\alpha,d,n}_y f(x).$$

(ii) For all $f \in \mathscr{E}_{n,*}(\mathbb{R}^{d+1})$ *and* $y \in \mathbb{R}_+^{d+1}$*, the function* $x \mapsto T_x^{\alpha,d,n} f(y)$ *belongs to* $\mathscr{E}_{n,*}(\mathbb{R}^{d+1})$*.*

(iii) Let $f \in L^p_{\alpha,n}(\mathbb{R}_+^{d+1})$*,* $1 \le p \le +\infty$ *and* $x \in \mathbb{R}_+^{d+1}$*. Then* $T_x^{\alpha,d,n} f$ *belongs to* $L^p_{\alpha,n}(\mathbb{R}_+^{d+1})$ *and we have*

$$\|T_x^{\alpha,d,n} f\|_{\alpha,n,p} \le x_{d+1}^{2n} \|f\|_{\alpha,n,p}. \tag{2.22}$$

(iv) The function $t \mapsto \Lambda_{\alpha,d,n}(t,\lambda)$*,* $\lambda \in \mathbb{C}^{d+1}$*, satisfies on* $\mathbb{R}_+^{d+1}$ *the following product formula*

$$\forall x, y \in \mathbb{R}_+^{d+1},\ \Lambda_{\alpha,d,n}(x,\lambda)\Lambda_{\alpha,d,n}(y,\lambda) = T_x^{\alpha,d,n}[\Lambda_{\alpha,d,n}(.,\lambda)](y). \tag{2.23}$$

(v) Let $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$ *and* $x \in \mathbb{R}_+^{d+1}$*. We have*

$$\forall \lambda \in \mathbb{R}_+^{d+1},\ \mathscr{F}_W^{\alpha,d,n}(T_x^{\alpha,d,n} f)(\lambda) = \Lambda_{\alpha,d,n}(-x,\lambda)\mathscr{F}_W^{\alpha,d,n}(f)(\lambda). \tag{2.24}$$

(vi) Let $f \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1})$*. For all* $x, y \in \mathbb{R}_+^{d+1}$*, we have*

$$T_x^{\alpha,d,n} f(y) = C^2_{\alpha+2n,d} \int_{\mathbb{R}_+^{d+1}} \Lambda_{\alpha,d,n}(-x,\lambda)\Lambda_{\alpha,d,n}(-y,\lambda)\mathscr{F}_W^{\alpha,d,n}(f)(\lambda)\mathrm{d}\mu_{\alpha+2n,d}(\lambda). \tag{2.25}$$

Proof The results can be obtained by a simple calculation by using the relation (2.20). □

Definition 2.13 *Let* $f, g \in L^1_{\alpha,n}(\mathbb{R}_+^{d+1})$*. The generalized Weinstein convolution product of* f *and* g *is given by*

$$\forall x \in \mathbb{R}_+^{d+1},\ f *_{\alpha,n} g(x) = \int_{\mathbb{R}_+^{d+1}} T_x^{\alpha,d,n} f(-y)g(y)\mathrm{d}\mu_{\alpha,d}(y). \tag{2.26}$$

Lemma 2.14 *Let* $f, g \in L^1_{\alpha,n}(\mathbb{R}_+^{d+1})$*. We have*

$$f *_{\alpha,n} g = \mathscr{M}_n(\mathscr{M}_n^{-1} f *_{\alpha} \mathscr{M}_n^{-1} g),$$

where for all $\varphi, \psi \in L^1_\alpha(\mathbb{R}_+^{d+1})$*, we have*

$$\forall x \in \mathbb{R}_+^{d+1},\ \varphi *_\alpha \psi(x) := \varphi *_{\alpha,0} \psi(x) = \int_{\mathbb{R}_+^{d+1}} T_x^{\alpha,d}\varphi(-y)\psi(y)\mathrm{d}\mu_{\alpha,d}(y).$$

Proposition 2.15 ([9]) *(i) Let* $p, q, r \in [1,\ +\infty]$ *such that* $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$*. Then for all* $f \in L^p_{\alpha,n}(\mathbb{R}_+^{d+1})$ *and* $g \in L^q_{\alpha,n}(\mathbb{R}_+^{d+1})$*, the function* $f *_{\alpha,n} g \in L^r_{\alpha,n}(\mathbb{R}_+^{d+1})$ *and we have*

$$\|f *_{\alpha,n} g\|_{\alpha,n,r} \le \|f\|_{\alpha,n,p}\|g\|_{\alpha,n,q}. \tag{2.27}$$

(ii) For all $f, g \in L^1_{\alpha,n}(\mathbb{R}_+^{d+1})$*,* $f *_{\alpha,n} g \in L^1_{\alpha,n}(\mathbb{R}_+^{d+1})$ *and we have*

$$\mathscr{F}_W^{\alpha,d,n}(f *_{\alpha,n} g) = \mathscr{F}_W^{\alpha,d,n}(f)\mathscr{F}_W^{\alpha,d,n}(g). \tag{2.28}$$

(iii) Let $f, g \in L^2_{\alpha,n}(\mathbb{R}_+^{d+1})$*. Then, we have*

$$\int_{\mathbb{R}_+^{d+1}} |f *_{\alpha,n} g(x)|^2 \mathrm{d}\mu_{\alpha,d}(x) = C^2_{\alpha+2n,d} \int_{\mathbb{R}_+^{d+1}} |\mathscr{F}_W^{\alpha,d,n}(f)(\lambda)|^2 |\mathscr{F}_W^{\alpha,d,n}(g)(\lambda)|^2 \mathrm{d}\mu_{\alpha+2n,d}(\lambda), \tag{2.29}$$

where both sides are finite or infinite.

Notation. We denote by $\mathscr{S}'_*$, (resp., $\mathscr{S}'_{n,*}$) the strong dual of the space $\mathscr{S}_*(\mathbb{R}^{d+1})$, (resp., $\mathscr{S}_{n,*}(\mathbb{R}^{d+1})$).

Definition 2.16 *The generalized Fourier-Weinstein transform of a distribution* $u \in \mathscr{S}'_{n,*}$ *is defined by*

$$\forall \phi \in \mathscr{S}_*(\mathbb{R}^{d+1}),\ \langle \mathscr{F}_W^{\alpha,d,n}(u),\ \phi \rangle = \langle u, (\mathscr{F}_W^{\alpha,d,n})^{-1}(\phi) \rangle. \tag{2.30}$$

The following proposition is as an immediate consequence of Theorem 2.8.

Proposition 2.17 *The transform* $\mathscr{F}_W^{\alpha,d,n}$ *is a topological isomorphism from* $\mathscr{S}'_{n,*}$ *onto* $\mathscr{S}'_*$.

Lemma 2.18 ([9]) *Let* $m \in \mathbb{N}$ *and* $u \in \mathscr{S}'_{n,*}$. *We have*

$$(\mathscr{F}_W^{\alpha,d,n})[(\Delta_W^{\alpha,d,n})^m u] = (-1)^m \|x\|^{2m} (\mathscr{F}_W^{\alpha,d,n})(u), \tag{2.31}$$

where

$$\forall \phi \in \mathscr{S}_{n,*}(\mathbb{R}^{d+1}),\ \langle \Delta_W^{\alpha,d,n} u, \phi \rangle = \langle u, \Delta_W^{\alpha,d,n} \phi \rangle. \tag{2.32}$$

3. Sobolev spaces associated with the generalized Weinstein operator

The goal of this section is to introduce and study the Sobolev spaces associated with the generalized Weinstein operator $\Delta_W^{\alpha,d,n}$.

Definition 3.1 *Let* $s \in \mathbb{R}$ *and* $p \in [1, +\infty]$. *We define the generalized Sobolev-Weinstein space of order* s, *that will be denoted* $\mathscr{W}^{s,p}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$, *as the set of all* $u \in \mathscr{S}'_{n,*}$ *such that* $\mathscr{F}_W^{\alpha,d,n}(u)$ *is a function and*

$$\int_{\mathbb{R}^{d+1}_+} (1 + \|\xi\|^2)^{\frac{sp}{2}} |\mathscr{F}_W^{\alpha,d,n}(u)(\xi)|^p \mathrm{d}\mu_{\alpha+2n,d}(\xi) < \infty. \tag{3.1}$$

We provide the space $\mathscr{W}^{s,p}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ *with the norm*

$$\|u\|_{\mathscr{W}^{s,p}_{\alpha,d,n}} = \Big[C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} (1 + \|\xi\|^2)^{\frac{sp}{2}} |\mathscr{F}_W^{\alpha,d,n}(u)(\xi)^p \mathrm{d}\mu_{\alpha+2n,d}(\xi) \Big]^{\frac{1}{p}}. \tag{3.2}$$

For $p = 2$, *we provide the* $\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+) := \mathscr{W}^{s,2}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ *with the inner product*

$$\langle u, v \rangle_{(1),\mathscr{H}^s_{\alpha,d,n}} = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} (1 + \|\xi\|^2)^s \mathscr{F}_W^{\alpha,d,n}(u)(\xi) \overline{\mathscr{F}_W^{\alpha,d,n}(v)(\xi)} \mathrm{d}\mu_{\alpha+2n,d}(\xi) \tag{3.3}$$

and the norm

$$\|u\|_{(1),\mathscr{H}^s_{\alpha,d,n}} = \Big[C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} (1 + \|\xi\|^2)^s |\mathscr{F}_W^{\alpha,d}(u)(\xi)|^2 \mathrm{d}\mu_{\alpha+2n,d}(\xi) \Big]^{\frac{1}{2}}. \tag{3.4}$$

We give the following properties of the spaces $\mathscr{W}^{s,p}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$.

Proposition 3.2 *(i) Let* $1 \le p < +\infty$ *and* $s, t \in \mathbb{R}$ *such that* $t > s$. *Then the space* $\mathscr{W}^{t,p}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ *is continuously contained in* $\mathscr{W}^{s,p}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$.

(ii) Let $s \in \mathbb{R}$ *and* $1 \le p < +\infty$. *The space* $\mathscr{W}^{s,p}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ *provided with the norm* $\|.\|_{\mathscr{W}^{s,p}_{\alpha,d,n}}$ *is a Banach space.*

(iii) For all $s \in \mathbb{R}$ *and* $1 \le p < +\infty$, *the space* $\mathscr{D}_*(\mathbb{R}^{d+1})$ *is dense in* $\mathscr{W}^{s,p}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$.

Proof (i) The result is immediately from the Definition 3.1.

(ii) Let $(f_m)_{m\in\mathbb{N}}$ be a Cauchy sequence of $\mathscr{W}^{s,p}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$. From the definition of the norm $\|.\|_{\mathscr{W}^{s,p}_{\alpha,d,n}}$, it is clear that $(\mathscr{F}^{\alpha,d,n}_W(f_m))_{m\in\mathbb{N}}$ is a Cauchy sequence of $L^p_{s,n,\alpha}(\mathbb{R}^{d+1}_+) := L^p(\mathbb{R}^{d+1}_+,\ (1+\|\xi\|^2)^{\frac{sp}{2}} d\mu_{\alpha+2n,d}(x))$.

Since $L^p_{s,n,\alpha}(\mathbb{R}^{d+1}_+)$ is complete, there exists a function $g \in L^p_{s,n,\alpha}(\mathbb{R}^{d+1}_+)$ such that

$$\lim_{m\to+\infty} \|\mathscr{F}^{\alpha,d,n}_W(f_m) - g\|_{L^p_{s,n,\alpha}(\mathbb{R}^{d+1}_+)} = 0. \tag{3.5}$$

Then $g \in \mathscr{S}'_*$ and $f = (\mathscr{F}^{\alpha,d,n}_W)^{-1}(g) \in \mathscr{S}'_{n,*}$. So, $\mathscr{F}^{\alpha,d,n}_W(f) = g \in L^p_{s,n,\alpha}(\mathbb{R}^{d+1}_+)$ which proves that $f \in \mathscr{W}^{s,p}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ and we have

$$\|f_m - f\|_{\mathscr{W}^{s,p}_{\alpha,d,n}} = C^{\frac{2}{p}}_{\alpha+2n,d} \|\mathscr{F}^{\alpha,d,n}_W(f_m) - g\|_{L^p_{s,n,\alpha}(\mathbb{R}^{d+1}_+)} \underset{m\to+\infty}{\to} 0.$$

Hence, $\mathscr{W}^{s,p}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ is complete.

(iii) We proceed as [10] to prove the result. □

The following theorem gives a relation between the dual of $\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ and $\mathscr{H}^{-s}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$.

Theorem 3.3 *The dual of $\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ can be identified with $\mathscr{H}^{-s}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$. The relation of the identification is as follows*

$$\langle u, v\rangle_{(0)} = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \mathscr{F}^{\alpha,d,n}_W(u)(\xi)\overline{\mathscr{F}^{\alpha,d,n}_W(v)(\xi)} \mathrm{d}\mu_{\alpha+2n,d}(\xi) \tag{3.6}$$

with $u \in \mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ and $v \in \mathscr{H}^{-s}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$.

Proof For all $u \in \mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ and $v \in \mathscr{H}^{-s}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$, we have

$$|\langle u,\ v\rangle_{(0)}| \le \|u\|_{(1),\mathscr{H}^s_{\alpha,d,n}} \|v\|_{(1),\mathscr{H}^{-s}_{\alpha,d,n}}. \tag{3.7}$$

Then, $(u,\ v) \mapsto \langle u,\ v\rangle_{(0)}$ is a continuous bilinear form on

$$\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+) \times \mathscr{H}^{-s}_{\alpha,d,n}(\mathbb{R}^{d+1}_+).$$

Let $v \in \mathscr{H}^{-s}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$. We consider the function $\phi_v : u \mapsto \langle u,\ v\rangle_{(0)}$.

From the relation (3.7), we see that ϕ_v is a continuous linear form on $\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ and we have

$$\|\phi_v\| \le \|v\|_{(1),\mathscr{H}^{-s}_{\alpha,d,n}}.$$

On the other hand for $u_0(\lambda) = [\mathscr{F}^{\alpha,d,n}_W]^{-1}((1+\|\lambda\|^2)^{-s}\mathscr{F}^{\alpha,d,n}_W(v))(\lambda)$, we obtain

$$u_0 \in \mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+) \text{ and } \langle u_0,\ v\rangle_{(0)} = \|v\|^2_{(1),\mathscr{H}^{-s}_{\alpha,d,n}}.$$

Then $\|\phi_v\| = \|v\|_{(1),\mathscr{H}^{-s}_{\alpha,d,n}}$.

Let now $v^* \in (\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+))'$. By the Riesz representation theorem and the relation (3.3), one can see that there exists $w \in \mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$, such that for all $u \in \mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$, we have

$$\begin{aligned} v^*(u) &= \langle u,\ w\rangle_{(1),\mathscr{H}^s_{\alpha,d,n}} \\ &= \int_{\mathbb{R}^{d+1}_+} (1+\|\lambda\|^2)^s \mathscr{F}^{\alpha,d,n}_W(w)(\lambda)\overline{\mathscr{F}^{\alpha,d,n}_W(u)(\lambda)} \mathrm{d}\mu_{\alpha,d}(\lambda). \end{aligned}$$

We put

$$v(\lambda) = [\mathscr{F}_W^{\alpha,d,n}]^{-1}((1+\|\lambda\|^2)^s \mathscr{F}_W^{\alpha,d,n}(w)(\lambda)).$$

Then, $v \in \mathscr{H}^{-s}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ and we have

$$\forall u \in \mathscr{H}^{s}_{\alpha,d,n}(\mathbb{R}^{d+1}_+),\ v^*(u) = \langle u,\ v\rangle_{(0)}.$$

Hence the map $v \mapsto \langle .,\ v\rangle_{(0)}$ is an isometry from $\mathscr{H}^{-s}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ into $(\mathscr{H}^{s}_{\alpha,d,n}(\mathbb{R}^{d+1}_+))'$.

Thus the proof is completed. □

Proposition 3.4 *For $s > \frac{d}{2}+\alpha+2n+1$, the Hilbert space $\mathscr{H}^{s}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ admits the reproducing kernel*

$$\mathscr{R}_s^{\alpha,d,n}(x,y) = C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}(1+\|\xi\|^2)^{-s}\Lambda_{\alpha,d,n}(-x,\xi)\Lambda_{\alpha,d,n}(y,\xi)\mathrm{d}\mu_{\alpha+2n,d}(\xi), \tag{3.8}$$

where $C_{\alpha,d}$ is the constant given by the relation (2.12). That is

(i) For every $y \in \mathbb{R}^{d+1}_+$, the distribution given by the function $x \mapsto \mathscr{R}_s^{\alpha,d,n}(x,y)$ belongs to $\mathscr{H}^{s}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$.

(ii) For every $f \in \mathscr{H}^{s}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$, we have

$$\forall y \in \mathbb{R}^{d+1}_+,\ \langle f,\ \mathscr{R}_s^{\alpha,d,n}(.,y)\rangle_{(1),\mathscr{H}^s_{\alpha,d,n}} = f(y).$$

Proof (i) It is easy to see that

$$\int_{\mathbb{R}^{d+1}_+}(1+\|\xi\|^2)^{-s}\mathrm{d}\mu_{\alpha+2n,d}(\xi) < +\infty \text{ if and only if } s > \frac{d}{2}+\alpha+2n+1. \tag{3.9}$$

Then using the relation (2.7), we deduce that the function $(x,y) \mapsto \mathscr{R}_s^{\alpha,d,n}(x,y)$ is well-defined.

Moreover for all $y \in \mathbb{R}^{d+1}_+$ and $s > \frac{d}{2}+\alpha+2n+1$, the function $\xi \mapsto (1+\|\xi\|^2)^{-s}\Lambda_{\alpha,d,n}(y,\xi)$ belongs to $L^1_{\alpha,n}(\mathbb{R}^{d+1}_+)\cap L^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$. Then, from the relation (2.17), the function $\mathscr{R}_{s,y}^{\alpha,d,n} : x \mapsto \mathscr{R}_s^{\alpha,d,n}(x,y)$ belongs to $L^2_\alpha(\mathbb{R}^{d+1}_+)$ and we have

$$\forall \xi \in \mathbb{R}^{d+1}_+,\ \mathscr{F}_W^{\alpha,d,n}[\mathscr{R}_{s,y}^{\alpha,d,n}](\xi) = (1+\|\xi\|^2)^{-s}\Lambda_{\alpha,d,n}(y,\xi). \tag{3.10}$$

Then

$$\|\mathscr{R}_{s,y}^{\alpha,d,n}\|_{(1),\mathscr{H}^s_{\alpha,d,n}} \le k_s y_{d+1}^{2n}, \tag{3.11}$$

where for all $s > \frac{d}{2}+\alpha+2n+1$, we have

$$k_s = k_s(\alpha,d,n) = C_{\alpha+2n,d}\Big(\int_{\mathbb{R}^{d+1}_+}(1+\|\xi\|^2)^{-s}\mathrm{d}\mu_{\alpha+2n,d}(\xi)\Big)^{\frac{1}{2}}. \tag{3.12}$$

Hence for all $y \in \mathbb{R}^{d+1}_+$ and $s > \frac{d}{2}+\alpha+2n+1$, $\mathscr{R}_{s,y}^{\alpha,d,n} \in \mathscr{H}^{s}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$.

(ii) Let $f \in \mathscr{H}^{s}_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ and $y \in \mathbb{R}^{d+1}_+$. Using the relations (3.3), (3.10) and (2.17), we obtain

$$\begin{aligned}\langle f,\mathscr{R}_s^{\alpha,d,n}(.,y)\rangle_{(1),\mathscr{H}^s_{\alpha,d,n}} &= C^2_{\alpha+2n,d}\int_{\mathbb{R}^{d+1}_+}\mathscr{F}_W^{\alpha,d,n}(f)(\xi)\Lambda_{\alpha,d,n}(-y,\xi)\mathrm{d}\mu_{\alpha+2n,d}(\xi)\\ &= f(y).\ \square\end{aligned}$$

4. Extremal functions on $\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$

The theory of reproducing kernels started with two papers of 1921 (see [11]) and 1922 (see [12]) which dealt with typical reproducing kernels of Szegö and Bergman and since then the theory has been developed into a large and deep theory in complex analysis by many mathematicians. In this section, using the theory of reproducing kernels, we study the extremal functions on the Hilbert space $\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$.

Definition 4.1 *Let $r > 0$, $\mathcal{H}$ be a Hilbert space and $\mathscr{L} : \mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+) \to \mathcal{H}$ be a bounded linear operator. For all $f, h \in \mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$, we define the inner product in $\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ by*

$$\langle f, h\rangle_{(2),\mathscr{H}^s_{\alpha,d,n}} = r\langle f, h\rangle_{(1),\mathscr{H}^s_{\alpha,d,n}} + \langle \mathscr{L}f, \mathscr{L}h\rangle_{\mathcal{H}}. \tag{4.1}$$

The norm associated with this inner product is given by

$$\|f\|^2_{(2),\mathscr{H}^s_{\alpha,d,n}} = r\|f\|^2_{(1),\mathscr{H}^s_{\alpha,d,n}} + \|\mathscr{L}f\|^2_{\mathcal{H}}. \tag{4.2}$$

Lemma 4.2 *The norms $\|.\|_{(1),\mathscr{H}^s_{\alpha,d,n}}$ and $\|.\|_{(2),\mathscr{H}^s_{\alpha,d,n}}$ are equivalent.*

Proof Let $u \in \mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$. We have

$$\sqrt{r}\|u\|_{(1),\mathscr{H}^s_{\alpha,d,n}} \le \|u\|_{(2),\mathscr{H}^s_{\alpha,d,n}} \le \sqrt{r + \|\mathscr{L}\|^2}\|u\|_{(1),\mathscr{H}^s_{\alpha,d,n}}.$$

This clearly yields the result. □

Proposition 4.3 *Let $r > 0$ and $s > \frac{d}{2} + \alpha + 2n + 1$. The space $(\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+), \langle ., .\rangle_{(2),\mathscr{H}^s_{\alpha,d,n}})$ possesses a reproducing $\mathscr{R}^{s,r}_{\mathscr{L}}$ satisfying the identity*

$$\mathscr{R}^{s,r}_{\mathscr{L}}(., y) = (rI + \mathscr{L}^*\mathscr{L})^{-1}\mathscr{R}^{\alpha,d,n}_s(., y) \tag{4.3}$$

where $I = Id$ and $\mathscr{L}^* : \mathcal{H} \to \mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ is the adjoint of $\mathscr{L}$ given by

$$\forall f \in \mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+), \forall h \in \mathcal{H},\ \langle \mathscr{L}f, h\rangle_{\mathcal{H}} = \langle f, \mathscr{L}^*h\rangle_{(1),\mathscr{H}^s_{\alpha,d,n}}.$$

Proof From [13], the space $(\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+), \langle ., .\rangle_{(2),\mathscr{H}^s_{\alpha,d,n}})$ has a reproducing kernel denoted by $\mathscr{R}^{s,r}_{\mathscr{L}}$ and we have

$$\begin{aligned} f(y) &= \langle f, \mathscr{R}^{s,r}_{\mathscr{L}}(., y)\rangle_{(2),\mathscr{H}^s_{\alpha,d,n}} \\ &= r\langle f, \mathscr{R}^{s,r}_{\mathscr{L}}(., y)\rangle_{(1),\mathscr{H}^s_{\alpha,d,n}} + \langle \mathscr{L}f, \mathscr{L}\mathscr{R}^{s,r}_{\mathscr{L}}(., y)\rangle_{\mathcal{H}} \\ &= \langle f, (rI + \mathscr{L}^*\mathscr{L})\mathscr{R}^{s,r}_{\mathscr{L}}(., y)\rangle_{(1),\mathscr{H}^s_{\alpha,d,n}}. \end{aligned}$$

Then for all $y \in \mathbb{R}^{d+1}_+$, we have

$$(rI + \mathscr{L}^*\mathscr{L})\mathscr{R}^{s,r}_{\mathscr{L}}(., y) = \mathscr{R}^{\alpha,d,n}_s(., y). \tag{4.4}$$

Thus the proof is completed. □

The following proposition summarizes some properties of the kernel $\mathscr{R}^{s,r}_{\mathscr{L}}$.

Proposition 4.4 *The kernel $\mathscr{R}^{s,r}_{\mathscr{L}}$ satisfies the following properties*

(i) We have

$$\|\mathscr{R}^{s,r}_{\mathscr{L}}(., y)\|_{(1),\mathscr{H}^s_{\alpha,d,n}} \le \frac{k_s}{r} y^{2n}_{d+1}, \tag{4.5}$$

where k_s is the constant given by the relation (3.12).

(ii) We have

$$\|\mathscr{L}\mathscr{R}^{s,r}_{\mathscr{L}}(.,y)\|_{\mathcal{H}} \leq \frac{k_s}{\sqrt{2r}} y^{2n}_{d+1}. \tag{4.6}$$

(iii) For all $y \in \mathbb{R}^{d+1}_+$, we have

$$\|\mathscr{L}^*\mathscr{L}\mathscr{R}^{s,r}_{\mathscr{L}}(.,y)\|_{(1),\mathscr{H}^s_{\alpha,d,n}} \leq k_s y^{2n}_{d+1}. \tag{4.7}$$

Proof Using the relation (3.11) and (4.4), for all $y \in \mathbb{R}^{d+1}_+$, we get

$$\begin{aligned} & r^2\|\mathscr{R}^{s,r}_{\mathscr{L}}(.,y)\|^2_{(1),\mathscr{H}^s_{\alpha,d,n}} + 2r\|\mathscr{L}\mathscr{R}^{s,r}_{\mathscr{L}}(.,y)\|^2_{\mathcal{H}} + \|\mathscr{L}^*\mathscr{L}\mathscr{R}^{s,r}_{\mathscr{L}}(.,y)\|^2_{(1),\mathscr{H}^s_{\alpha,d,n}} \\ &= \|\mathscr{R}^{\alpha,d,n}_s(.,y)\|^2_{(1),\mathscr{H}^s_{\alpha,d,n}} \leq k_s^2 y^{4n}_{d+1}. \end{aligned}$$

Then the assertions (i)–(iii) are an immediate consequence of the above result. □

The main result of this section can be stated as follows.

Theorem 4.5 *Let $s > \frac{d}{2} + \alpha + 2n + 1$. For all $h \in \mathcal{H}$ and for all $r > 0$, the infimum*

$$\inf_{f \in \mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)} [r\|f\|^2_{(1),\mathscr{H}^s_{\alpha,d,n}} + \|h - \mathscr{L}f\|^2_{\mathcal{H}}] \tag{4.8}$$

is attained by a unique function $f^*_{r,h}$ given by

$$\forall y \in \mathbb{R}^{d+1}_+, \ f^*_{r,h}(y) = \langle h, \mathscr{L}\mathscr{R}^{s,r}_{\mathscr{L}}(.,y)\rangle_{\mathcal{H}}. \tag{4.9}$$

Moreover, the extremal function $f^*_{r,h}$ satisfies the following inequality

$$\forall y \in \mathbb{R}^{d+1}_+, |f^*_{r,h}(y)| \leq \frac{k_s}{\sqrt{2r}} \|h\|_{\mathcal{H}} y^{2n}_{d+1}. \tag{4.10}$$

Proof The existence and unicity of extremal function $f^*_{r,h}$ represented by the relation (4.8) is given by [13]. On the other hand from the relation (4.6), we get

$$\forall y \in \mathbb{R}^{d+1}_+, |\ f^*_{r,h}(y)| \leq \|\mathscr{L}\mathscr{R}^{s,r}_{\mathscr{L}}(.,y)\|_{\mathcal{H}} \|h\|_{\mathcal{H}} \leq \frac{k_s}{\sqrt{2r}} \|h\|_{\mathcal{H}} y^{2n}_{d+1}. \ \square$$

Corollary 4.6 *Let $s > \dfrac{d}{2} + \alpha + 2n + 1$ and $r > 0$. If $\mathscr{L}$ is isometry ($\mathscr{L}^*\mathscr{L} = \mathrm{Id}$), then*

(i) $\langle .,. \rangle_{(2),\mathscr{H}^s_{\alpha,d,n}} = (r+1)\langle .,. \rangle_{(1),\mathscr{H}^s_{\alpha,d,n}}$.

(ii) For all $x, y \in \mathbb{R}^{d+1}_+$, we have $\mathscr{R}^{s,r}_{\mathscr{L}}(x,y) = \frac{1}{r+1}\mathscr{R}^{\alpha,d,n}_s(.,y)$.

(iii) For all $h \in \mathcal{H}$, we have $\forall y \in \mathbb{R}^{d+1}_+, \ f^*_{r,h}(y) = \frac{1}{r+1}\mathscr{L}^* h(y)$.

(iv) For all $f \in \mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$, we have $\forall y \in \mathbb{R}^{d+1}_+, \ f^*_{r,\mathscr{L}f}(y) = \frac{1}{r+1} f(y)$.

Corollary 4.7 *Let $s > \frac{d}{2} + \alpha + 2n + 1$ and $r > 0$. Let $f \in \mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ and $h = \mathscr{L}f$.*

(i) For all $y \in \mathbb{R}^{d+1}_+$, we have $f(y) = \lim\limits_{r \to 0^+} f^*_{r,h}(y)$.

(ii) We have $\forall y \in \mathbb{R}^{d+1}_+, |f(y) - f^*_{r,h}(y)| \leq k_s \|f\|_{(1),\mathscr{H}^s_{\alpha,d,n}} y^{2n}_{d+1}$.

(iii) We have $\forall y \in \mathbb{R}^{d+1}_+, |f^*_{r,h}(y)| \leq k_s \|f\|_{(1),\mathscr{H}^s_{\alpha,d,n}} y^{2n}_{d+1}$.

Proof Let $f \in \mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ and $h = \mathscr{L}f$.

(i) From the relations (4.4) and (4.9), we get

$$\forall y \in \mathbb{R}^{d+1}_+, \ f^*_{r,h}(y) = \langle f, \mathscr{L}^*\mathscr{L}\mathscr{R}^{s,r}_{\mathscr{L}}(.,y)\rangle_{(1),\mathscr{H}^s_{\alpha,d,n}}. \tag{4.11}$$

Then, for all $y \in \mathbb{R}_+^{d+1}$, we obtain

$$f_{r,h}^*(y) = \langle f, \mathscr{R}_s^{\alpha,d,n}(.,y) - r\mathscr{R}_{\mathscr{L}}^{s,r}(.,y)\rangle_{(1),\mathscr{H}_{\alpha,d,n}^s}.$$

Hence

$$\forall y \in \mathbb{R}_+^{d+1},\ f_{r,h}^*(y) = f(y) - r\langle f, \mathscr{R}_{\mathscr{L}}^{s,r}(.,y)\rangle_{(1),\mathscr{H}_{\alpha,d,n}^s} \tag{4.12}$$

and we have

$$\lim_{r\to 0^+} f_{r,h}^*(y) = \lim_{r\to 0^+}[f(y) - r\langle f, \mathscr{R}_{\mathscr{L}}^{s,r}(.,y)\rangle_{(1),\mathscr{H}_{\alpha,d,n}^s}] = f(y).$$

(ii) By invoking (4.5) and (4.12), for all $y \in \mathbb{R}_+^{d+1}$, we can write

$$\begin{aligned}|f(y) - f_{r,h}^*(y)| &= r|\langle f, \mathscr{R}_{\mathscr{L}}^{s,r}(.,y)\rangle_{(1),\mathscr{H}_{\alpha,d,n}^s}| \leq r\|f\|_{(1),\mathscr{H}_{\alpha,d,n}^s}\|\mathscr{R}_{\mathscr{L}}^{s,r}(.,y)\|_{(1),\mathscr{H}_{\alpha,d,n}^s}\\ &\leq k_s\|f\|_{(1),\mathscr{H}_{\alpha,d,n}^s} y_{d+1}^{2n}.\end{aligned}$$

(iii) Using the relations (4.7) and (4.11), for all $y \in \mathbb{R}_+^{d+1}$, we obtain

$$\begin{aligned}|f_{r,h}^*(y)| &\leq \|f\|_{(1),\mathscr{H}_{\alpha,d,n}^s}\|\mathscr{L}^*\mathscr{L}\mathscr{R}_{\mathscr{L}}^{s,r}(.,y)\|_{(1),\mathscr{H}_{\alpha,d,n}^s}\\ &\leq k_s\|f\|_{(1),\mathscr{H}_{\alpha,d,n}^s} y_{d+1}^{2n}. \quad \square\end{aligned}$$

Example 4.8 For all $s \geq 0$, the identity operator $\mathrm{id} : \mathscr{H}_{\alpha,d,n}^s \to L_{\alpha,n}^2(\mathbb{R}_+^{d+1})$ is bounded and we have

$$\forall u \in \mathscr{H}_{\alpha,d,n}^s(\mathbb{R}_+^{d+1}),\ \|\mathrm{id}(u)\|_{\alpha,n,2} \leq \|u\|_{(1),\mathscr{H}_{\alpha,d,n}^s}.$$

Its adjoint operator $\mathrm{id}^* : L_{\alpha,n}^2(\mathbb{R}_+^{d+1}) \to \mathscr{H}_{\alpha,d,n}^s(\mathbb{R}_+^{d+1})$ is given by

$$\forall v \in L_{\alpha,n}^2(\mathbb{R}_+^{d+1}),\ \mathrm{id}^*(v) = (\mathscr{F}_W^{\alpha,d,n})^{-1}[(1+\|\xi\|^2)^{-s}\mathscr{F}_W^{\alpha,d,n}(v)].$$

On the other hand, the inner product associated with the operator id can be written

$$\langle u, v\rangle_{(2),\mathscr{H}_{\alpha,d,n}^s} = C_{\alpha+2n,d}^2\int_{\mathbb{R}_+^{d+1}}[1 + r(1+\|\xi\|^2)^s]\mathscr{F}_W^{\alpha,d,n}(u)(\xi)\overline{\mathscr{F}_W^{\alpha,d,n}(v)}(\xi)\mathrm{d}\mu_{\alpha+2n,d}(\xi).$$

In this case, the Hilbert space $\mathscr{H}_{\alpha,d,n}^s(\mathbb{R}_+^{d+1})$ admits the following reproducing kernel

$$\mathscr{R}_{id}^{s,r}(x,y) = C_{\alpha+2n,d}^2\int_{\mathbb{R}_+^{d+1}}\frac{\Lambda_{\alpha,d,n}(-x,\xi)\Lambda_{\alpha,d,n}(y,\xi)}{1 + r(1+\|\xi\|^2)^s}\mathrm{d}\mu_{\alpha+2n,d}(\xi).$$

For all $h \in L_{\alpha,n}^2(\mathbb{R}_+^{d+1})$ and for all $r > 0$, the infimum

$$\inf_{f\in\mathscr{H}_{\alpha,d,n}^s(\mathbb{R}_+^{d+1})}[r\|f\|_{(1),\mathscr{H}_{\alpha,d,n}^s}^2 + \|h - f\|_{\alpha,n,2}^2]$$

exists and it is attained by a unique function $f_{r,h}^*$ given by

$$\begin{aligned}f_{r,h}^*(y) &= \langle h, \mathscr{R}_{\mathrm{id}}^{s,r}(.,y)\rangle_{L_{\alpha,n}^2(\mathbb{R}_+^{d+1})}\\ &= C_{\alpha+2n,d}^2\int_{\mathbb{R}_+^{d+1}}\mathscr{F}_W^{\alpha,d,n}(h)(\xi)\overline{\mathscr{F}_W^{\alpha,d,n}(\mathscr{R}_{id}^{s,r}(.,y))(\xi)}\mathrm{d}\mu_{\alpha+2n,d}(\xi)\\ &= C_{\alpha+2n,d}^2\int_{\mathbb{R}_+^{d+1}}\frac{\Lambda_{\alpha,d,n}(-y,\xi)\mathscr{F}_W^{\alpha,d,n}(h)(\xi)}{1 + r(1+\|\xi\|^2)^s}\mathrm{d}\mu_{\alpha+2n,d}(\xi).\end{aligned}$$

Moreover, the extremal function $f^*_{r,h}$ satisfies the following inequality

$$\forall y \in \mathbb{R}^{d+1}_+, |f^*_{r,h}(y)| \leq \frac{k_s}{\sqrt{2r}} \|h\|_{\alpha,n,2} y^{2n}_{d+1},$$

where k_s is the constant given by the relation (3.12).

Example 4.9 For $m \in L^\infty_{\alpha,n}(\mathbb{R}^{d+1}_+)$, we define the multiplier operator $\mathscr{L}_m$ by:

$$\forall u \in \mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+),\ \mathscr{L}_m u := (\mathscr{F}^{\alpha,d,n}_W)^{-1}[m\mathscr{F}^{\alpha,d,n}_W(u)].$$

For all $s \geq 0$, the operator $\mathscr{L}_m$ is bounded from $\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ into $L^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$. The inner product associated with the operator $\mathscr{L}_m$ is given by

$$\langle u, v\rangle_{(2),\mathscr{H}^s_{\alpha,d,n}} = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} [|m(\xi)|^2 + r(1 + \|\xi\|^2)^s]\mathscr{F}^{\alpha,d,n}_W(u)(\xi)\overline{\mathscr{F}^{\alpha,d,n}_W(v)}(\xi)\mathrm{d}\mu_{\alpha+2n,d}(\xi).$$

The Hilbert space $\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)$ admits the following reproducing kernel

$$\mathscr{R}^{s,r}_{\mathscr{L}_m}(x, y) = C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \frac{\Lambda_{\alpha,d,n}(-x, \xi)\Lambda_{\alpha,d,n}(y, \xi)}{r(1 + \|\xi\|^2)^s + |m(\xi)|^2} \mathrm{d}\mu_{\alpha+2n,d}(\xi).$$

For all $h \in L^2_{\alpha,n}(\mathbb{R}^{d+1}_+)$ and for all $r > 0$, the infimum

$$\inf_{f\in\mathscr{H}^s_{\alpha,d,n}(\mathbb{R}^{d+1}_+)} [r\|f\|^2_{(1),\mathscr{H}^s_{\alpha,d,n}} + \|h - \mathscr{L}_m f\|^2_{\alpha,n,2}]$$

exists and it is attained by a unique function $f^*_{r,h}$ given by

$$\begin{aligned} f^*_{r,h}(y) &= \langle h, \mathscr{L}_m\mathscr{R}^{s,r}_{\mathscr{L}_m}(., y)\rangle_{L^2_{\alpha,n}(\mathbb{R}^{d+1}_+)} \\ &= C^2_{\alpha+2n,d} \int_{\mathbb{R}^{d+1}_+} \frac{\overline{m(\xi)}\Lambda_{\alpha,d,n}(-y, \xi)\mathscr{F}^{\alpha,d,n}_W(h)(\xi)}{r(1 + \|\xi\|^2)^s + |m(\xi)|^2} \mathrm{d}\mu_{\alpha+2n,d}(\xi). \end{aligned}$$

Moreover, the extremal function $f^*_{r,h}$ satisfies the following inequality

$$\forall y \in \mathbb{R}^{d+1}_+, |f^*_{r,h}(y)| \leq \frac{k_s}{\sqrt{2r}} \|h\|_{\alpha,n,2} y^{2n}_{d+1},$$

where k_s is the constant given by the relation (3.12).

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