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Bounded Weak Solutions to a Class of Parabolic Equations with Gradient Term and $L^r(0,T;L^q(\Omega))$ Sources

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Abstract We consider a class of nonlinear parabolic equations whose prototype is

$$\begin{cases} u_t-\Delta u=\overrightarrow{b}(x,t)\cdot\nabla u+\gamma|\nabla u|^2-\operatorname{div}\overrightarrow{F}(x,t)+f(x,t), & (x,t)\in\Omega_T,\\ u(x,t)=0, & (x,t)\in\Gamma_T,\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases}$$

where the functions $|\overrightarrow{b}(x,t)|^2, |\overrightarrow{F}(x,t)|^2, f(x,t)$ lie in the space $L^r(0,T;L^q(\Omega))$, γ is a positive constant. The purpose of this paper is to prove, under suitable assumptions on the integrability of the space $L^r(0,T;L^q(\Omega))$ for the source terms and the coefficient of the gradient term, a priori L^∞ estimate and the existence of bounded solutions.

The methods consist of constructing a family of perturbation problems by regularization, Stampacchia's iterative technique fulfilled by an appropriate nonlinear test function and compactness argument for the limit process.

Keywords parabolic equations; lower order gradient term; L^∞ estimate; bounded solutions

MR(2020) Subject Classification 35K20; 35K55

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ with $N>2$, and $\partial\Omega$ be the smooth boundary. $T>0$ is a finite number, $\Omega_T=\Omega\times(0,T)$ is the cylinder, and $\Gamma_T=\partial\Omega\times(0,T)$ is the lateral boundary. Consider the following parabolic equation:

$$\begin{cases} u_t-\operatorname{div}(a(x,t,u,\nabla u))+H(x,t,u,\nabla u)=-\operatorname{div}\overrightarrow{F}(x,t)+f(x,t), & (x,t)\in\Omega_T,\\ u(x,t)=0, & (x,t)\in\Gamma_T,\\ u(x,0)=u_0(x), & x\in\Omega. \end{cases}\tag{1.1}$$

We assume that

(H1): The Carathéodory function $a(x,t,s,\xi):\Omega_T\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ satisfies: for almost every $(x,t)\in\Omega_T$, for every s in $\mathbb{R}$, ξ, ξ' in $\mathbb{R}^N$ with $\xi\neq\xi'$,

$$a(x,t,s,\xi)\cdot\xi\geq\alpha|\xi|^2,\tag{1.2}$$

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$$|a(x,t,s,\xi)| \le b(|s|)|\xi|, \tag{1.3}$$

$$[a(x,t,s,\xi) - a(x,t,s,\xi')] \cdot (\xi - \xi') > 0, \tag{1.4}$$

where $\alpha > 0$, $b : [0,+\infty) \to (0,+\infty)$ is a continuous function.

(H2): The Carathéodory function $H : \Omega_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the growth condition

$$|H(x,t,s,\xi)| \le |\overrightarrow{b}(x,t)||\xi| + \gamma|\xi|^2, \tag{1.5}$$

for almost every $(x,t) \in \Omega_T$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, where $\gamma > 0$ is a constant.

(H3): The initial value function $u_0(x) \in L^\infty(\Omega)$. $|\overrightarrow{b}(x,t)|^2, |\overrightarrow{F}(x,t)|^2, f(x,t) \in L^r(0,T;L^q(\Omega))$, with

$$\frac{1}{r} + \frac{N}{2q} = 1 - \varrho_1 \quad 0 < \varrho_1 < 1, \tag{1.6}$$

and

$$q \in [\frac{N}{2(1-\varrho_1)}, \infty], \quad r \in [\frac{1}{1-\varrho_1}, \infty]. \tag{1.7}$$

Problem (1.1) with $\gamma = 0$ has been investigated in [1]. Without natural growth condition with respect to the gradient, the authors established the estimate of $\max |u|$ and the maximum principle. When $H = 0$, [2] studied a class of non-coercive parabolic equations with a divergence term $-\mathrm{div}(\overrightarrow{E}u)$. Under the assumption $|\overrightarrow{E}|^2 \in L^r(0,T;L^q(\Omega))$ with $\frac{1}{r} + \frac{N}{2q} < 1$, the authors introduced a new test function and proved the existence of bounded solutions.

The main feature of Problem (1.1) is that the nonlinear first order term has natural growth condition (the appearance of $\gamma|\nabla u|^2$), meanwhile the square of the coefficient $|\overrightarrow{b}(x,t)|^2$, the free terms $|\overrightarrow{F}(x,t)|^2$ and $f(x,t)$ lie in $L^r(0,T;L^q(\Omega))$. Now we explain that all these characteristics prevent us from directly observing the existence result. Let us look at the prototype of Problem (1.1) for the sake of clarity. On one hand, if we define $A(u) = -\Delta u - \overrightarrow{b}(x,t) \cdot \nabla u$, then, as pointed in [3], the operator A is lack of coercivity. On the other hand, it is obvious that the $L^r(0,T;L^q(\Omega))$ source is more complicated than the case of $r = q$. With the natural growth condition, this kind of integrability has a great influence on the existence of bounded solutions.

In the stationary case, in order to have the L^∞ estimate, [4] added an extra sign condition on the gradient term. Boccardo, Murat and Puel introduced a nonlinear test function with exponential form in [5], which makes the Stampacchia's method adapt to the natural growth problem. For more detailed and systematical analysis on the bounded solutions of the elliptic equations, the readers may refer to monograph [6] and the references therein.

We absorb some ideas from elliptic equations to solve the parabolic case. Nevertheless, compared with the elliptic equations, the parabolic framework forces us to deal with some new technical issues, most of which are completely different from the stationary case. For instance, the presence of $L^r(0,T;L^q(\Omega))$ sources and the trick to handle the lower order term in the present paper. In the process of obtaining the necessary estimates and various results on the convergence, it should be remarked that some elementary functions will be employed as test functions in the evolution setting, such as exponential function $e^{\lambda|x|}(e^{\lambda|x|}-1)\mathrm{sign}(x)$ and hyperbolic sine function $\sinh(\lambda x)$, which simplify the calculations and the estimates.

The definition of a weak solution to Problem (1.1) is given in the following way [7,8].

Definition 1.1 *A measurable function $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$ is a weak solution to Problem (1.1), provided that $u_t \in L^2(0,T;H^{-1}(\Omega)) + L^1(\Omega_T)$; $u(x,0) = u_0(x)$ a.e., in Ω; $H(x,t,u,\nabla u) \in L^1(\Omega_T)$; and the equality*

$$\int_0^T \langle u_t, \phi\rangle \mathrm{d}t + \int_0^T\int_\Omega a(x,t,u,\nabla u)\cdot\nabla\phi \mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega H(x,t,u,\nabla u)\phi \mathrm{d}x\mathrm{d}t$$
$$= \int_0^T\int_\Omega \overrightarrow{F}(x,t)\cdot\nabla\phi \mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega f(x,t)\phi \mathrm{d}x\mathrm{d}t \tag{1.8}$$

holds for every $\phi(x,t) \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$. Here, $u_t = \alpha^{(1)} + \alpha^{(2)} \in L^2(0,T;H^{-1}(\Omega)) + L^1(\Omega_T)$, is understood as

$$\int_0^T \langle u_t, \phi\rangle \mathrm{d}t := \langle u_t, \phi\rangle_{L^2(0,T;H^{-1}(\Omega))+L^1(\Omega_T),\,L^2(0,T;H_0^1(\Omega))\cap L^\infty(\Omega_T)}$$
$$= \int_0^T \langle \alpha^{(1)}, \phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \mathrm{d}t + \int_0^T\int_\Omega \alpha^{(2)}\phi \mathrm{d}x\mathrm{d}t.$$

The bracket $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $H^{-1}(\Omega) + L^1(\Omega)$ and $H_0^1(\Omega) \cap L^\infty(\Omega)$.

2. Existence result

Now we give the existence result.

Theorem 2.1 *Suppose that (H1)–(H3) hold, then there exists at least one bounded weak solution $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$ to Problem (1.1).*

This section is devoted to proving the existence of bounded solutions.

2.1. Regularization

For Eq. (1.1), let us consider the following approximate problem:

$$\begin{cases} \frac{\partial u_n}{\partial t} - \mathrm{div}(a_n(x,t,u_n,\nabla u_n)) + H_n(x,t,u_n,\nabla u_n) = -\mathrm{div}\overrightarrow{F}(x,t) + f(x,t), & (x,t)\in\Omega_T, \\ u_n(x,t) = 0, & (x,t)\in\Gamma_T, \\ u_n(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{2.1}$$

with $a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi)$, $T_k(s) = \min(|s|,k)\mathrm{sign}(s)$, and

$$H_n(x,t,s,\xi) = \frac{H(x,t,s,\xi)}{1+\frac{1}{n}|H(x,t,s,\xi)|}. \tag{2.2}$$

It follows from the parabolic theory in [1,9,10] that, for every $n \in \mathbb{N}$, Problem (2.1) has a weak solution $u_n \in C(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$.

2.2. Uniform L^∞ estimate

In this subsection, our goal is to prove that $\{u_n\}_{n=1}^\infty$ is uniformly bounded in $L^\infty(\Omega_T)$. We modify the exponential test function in [5] in order to 'cancel' the natural growth with respect to the gradient. The method combines the technique to handle with the first order term and the trick to deal with the $L^r(0,T;L^q(\Omega))$ sources.

Define $\psi(s) = e^{\lambda|s|}(e^{\lambda|s|}-1)\mathrm{sign}(s)$ with $\lambda = \frac{\gamma}{\alpha}$. Let $k \geq \|u_0\|_{L^\infty(\Omega)}$, $G_k(s) = (|s|-k)_+\mathrm{sign}(s)$; and denote by χ_A the characteristic function of a set A. If we take $\psi[G_k(u_n)]\chi_{[0,\tau]}$ as a test function in Problem (2.1), then by (1.2), (1.5), (2.2) we have

$$\overbrace{\int_0^\tau \langle \frac{\partial u_n}{\partial t}, \psi[G_k(u_n)]\rangle \mathrm{d}t}^{A_0} + \lambda\alpha \int_0^\tau \int_{A_k(t)} |\nabla G_k(u_n)|^2 e^{2\lambda|G_k(u_n)|}\mathrm{d}x\mathrm{d}t$$
$$\leq \overbrace{\int_0^\tau \int_{A_k(t)} |\overrightarrow{b}(x,t)||\nabla G_k(u_n)|e^{\lambda|G_k(u_n)|}(e^{\lambda|G_k(u_n)|}-1)\mathrm{d}x\mathrm{d}t}^{A_1} +$$
$$\overbrace{\int_0^\tau \int_{A_k(t)} 2\lambda|\overrightarrow{F}(x,t)||\nabla G_k(u_n)|e^{2\lambda|G_k(u_n)|}\mathrm{d}x\mathrm{d}t}^{A_2} + \overbrace{\int_0^\tau \int_{A_k(t)} |f(x,t)|e^{2\lambda|G_k(u_n)|}\mathrm{d}x\mathrm{d}t}^{A_3}, \qquad (2.3)$$

where $A_k(t) = \{x \in \Omega : |u_n(x,t)| > k\}$.

Denote $\widetilde{\psi}(s) = \int_0^s \psi(\tau)\mathrm{d}\tau$. First, we consider the time derivative term A_0. Since $k \geq \|u_0\|_{L^\infty(\Omega)}$ and $\widetilde{\psi}(s) = \frac{1}{2\lambda}[e^{2\lambda|s|}-1-2e^{\lambda|s|}] = \frac{1}{2\lambda}(e^{\lambda|s|}-1)^2$, the integration by parts in [11] helps us get that

$$A_0 = \int_\Omega \widetilde{\psi}[G_k(u_n(\tau))]\mathrm{d}x - \int_\Omega \widetilde{\psi}[G_k(u_0)]\mathrm{d}x$$
$$= \int_\Omega \widetilde{\psi}[G_k(u_n(\tau))]\mathrm{d}x = \frac{1}{2\lambda}\int_\Omega (e^{\lambda|G_k(u_n(\tau))|}-1)^2\mathrm{d}x.$$

We estimate A_i $(i=1,2,3)$, by Cauchy's inequality with ϵ, as follows:

$$A_1 \leq \int_0^\tau \int_{A_k(t)} |\overrightarrow{b}||\nabla G_k(u_n)|e^{2\lambda|G_k(u_n)|}\mathrm{d}x\mathrm{d}t$$
$$\leq \frac{\epsilon}{2}\int_0^\tau \int_{A_k(t)} |\nabla G_k(u_n)|^2 e^{2\lambda|G_k(u_n)|}\mathrm{d}x\mathrm{d}t + \frac{1}{2\epsilon}\int_0^\tau \int_{A_k(t)} |\overrightarrow{b}|^2 e^{2\lambda|G_k(u_n)|}\mathrm{d}x\mathrm{d}t,$$
$$A_2 \leq \frac{\epsilon}{2}\int_0^\tau \int_{A_k(t)} |\nabla G_k(u_n)|^2 e^{2\lambda|G_k(u_n)|}\mathrm{d}x\mathrm{d}t + \frac{1}{2\epsilon}\int_0^\tau \int_{A_k(t)} 4\lambda^2|\overrightarrow{F}|^2 e^{2\lambda|G_k(u_n)|}\mathrm{d}x\mathrm{d}t.$$

Furthermore, applying the inequality $A^2 \leq 2(A-1)^2+2$, we have

$$A_1 + A_2 \leq \epsilon\int_0^\tau \int_{A_k(t)} |\nabla G_k(u_n)|^2 e^{2\lambda|G_k(u_n)|}\mathrm{d}x\mathrm{d}t + \frac{1}{2\epsilon}\int_0^\tau \int_{A_k(t)} (|\overrightarrow{b}|^2 + 4\lambda^2|\overrightarrow{F}|^2)e^{2\lambda|G_k(u_n)|}\mathrm{d}x\mathrm{d}t$$

$$\leq \epsilon \int_0^{\tau} \int_{A_k(t)} |\nabla G_k(u_n)|^2 e^{2\lambda|G_k(u_n)|} \mathrm{d}x\mathrm{d}t + \frac{1}{\epsilon} \int_0^{\tau} \int_{A_k(t)} (|\overrightarrow{b}|^2 + 4\lambda^2|\overrightarrow{F}|^2)(e^{\lambda|G_k(u_n)|} - 1)^2 \mathrm{d}x\mathrm{d}t +$$
$$\frac{1}{\epsilon} \int_0^{\tau} \int_{A_k(t)} (|\overrightarrow{b}|^2 + 4\lambda^2|\overrightarrow{F}|^2) \mathrm{d}x\mathrm{d}t;$$

and

$$A_3 \leq 2 \int_0^{\tau} \int_{A_k(t)} |f|(e^{\lambda|G_k(u_n)|} - 1)^2 \mathrm{d}x\mathrm{d}t + 2 \int_0^{\tau} \int_{A_k(t)} |f| \mathrm{d}x\mathrm{d}t.$$

Choosing $\epsilon = \frac{\gamma}{2}$, substituting the above estimates for A_i into (2.3), and taking the supremum for $\tau \in [0, t_1]$ in it, we deduce that

$$\frac{\alpha}{2\gamma} \min\{1, \alpha\} \overbrace{\left[\operatorname*{ess\,sup}_{\tau \in [0,t_1]} \int_{\Omega} (e^{\lambda|G_k(u_n(\tau))|} - 1)^2 \mathrm{d}x + \iint_{\Omega_{t_1}} |\nabla(e^{\lambda|G_k(u_n)|} - 1)|^2 \mathrm{d}x\mathrm{d}t \right]}^{L}$$
$$\leq \overbrace{\int_0^{t_1} \int_{A_k(t)} \mathcal{D}(e^{\lambda|G_k(u_n)|} - 1)^2 \mathrm{d}x\mathrm{d}t}^{I_1} + \overbrace{\int_0^{t_1} \int_{A_k(t)} \mathcal{D} \mathrm{d}x\mathrm{d}t}^{I_2}, \tag{2.4}$$

where $\mathcal{D} = \frac{2}{\gamma}(|\overrightarrow{b}|^2 + \frac{4\gamma^2}{\alpha^2}|\overrightarrow{F}|^2) + 2|f|$, t_1 will be chosen later.

Define $|v|_{\Omega_t} = \|v\|_{L^\infty(0,t;L^2(\Omega))} + \|\nabla v\|_{L^2(\Omega_t)}$, $\Omega_\tau = \Omega \times [0, \tau]$, then

$$L \geq \frac{1}{2} |e^{\lambda|G_k(u_n)|} - 1|^2_{\Omega_{t_1}}.$$

Denote $\varrho = \frac{2\varrho_1}{N}$, $\hat{q} = 2q'(1 + \varrho)$, $\hat{r} = 2r'(1 + \varrho)$, q' is the Hölder conjugate exponent of q. It follows from (1.6) that $\frac{2}{\hat{r}} + \frac{N}{\hat{q}} = \frac{N}{2}$. Therefore, by means of Hölder's inequality, based on parabolic embedding $L^\infty(0, t_1; L^2(\Omega)) \cap L^2(0, t_1; H_0^1(\Omega)) \hookrightarrow L^{\hat{r}}(0, t_1; L^{\hat{q}}(\Omega))$, we have an estimate for I_1:

$$\begin{aligned} I_1 \leq & \|\mathcal{D}\|_{q,r,Q_{t_1}(k)} \|e^{\lambda|G_k(u_n)|} - 1\|^2_{2q',2r',Q_{t_1}(k)} \\ \leq & \|\mathcal{D}\|_{q,r,Q_{t_1}(k)} \left[\|e^{\lambda|G_k(u_n)|} - 1\|_{2q'(1+\varrho),2r'(1+\varrho),\Omega_{t_1}} \left(\int_0^{t_1} |A_k(t)|^{\frac{r'}{q'}} \mathrm{d}t \right)^{\frac{1}{2r'} - \frac{1}{2r'(1+\varrho)}} \right]^2 \\ = & \|\mathcal{D}\|_{q,r,Q_{t_1}(k)} \|e^{\lambda|G_k(u_n)|} - 1\|^2_{\hat{q},\hat{r},\Omega_{t_1}} [\mu(k)]^{\frac{2\varrho}{\hat{r}}} \\ \leq & \|\mathcal{D}\|_{q,r,Q_{t_1}(k)} \beta^2 |e^{\lambda|G_k(u_n)|} - 1|^2_{\Omega_{t_1}} (|\Omega|^{\frac{\hat{r}}{\hat{q}}} t_1)^{\frac{2\varrho}{\hat{r}}}, \end{aligned}$$

where $Q_{t_1}(k) = \{(x,t) \in \Omega \times (0, t_1) : |u(x,t)| > k\}$, $\beta = \beta(N)$ is the embedding constant, $\mu(k) = \int_0^{t_1} |A_k(t)|^{\frac{\hat{r}}{\hat{q}}} \mathrm{d}t$, and $\frac{\hat{r}}{\hat{q}} = \frac{r'}{q'}$ is used.

By Hölder's inequality,

$$I_2 \leq \|\mathcal{D}\|_{q,r,\Omega_{t_1}} \|1\|_{q',r',Q_{t_1}(k)} = \|\mathcal{D}\|_{q,r,\Omega_{t_1}} \left(\int_0^{t_1} |A_k(t)|^{\frac{r'}{q'}} \mathrm{d}t \right)^{\frac{1}{r'}} \leq \|\mathcal{D}\|_{q,r,\Omega_T} [\mu(k)]^{\frac{2(1+\varrho)}{\hat{r}}}.$$

Taking into account the estimates L, I_1, I_2 in (2.4) together, we obtain

$$\frac{\alpha}{4\gamma}\min\{1,\alpha\}|e^{\lambda|G_k(u_n)|}-1|^2_{\Omega_{t_1}} \leq \|\mathcal{D}\|_{q,r,Q_{t_1}(k)}\beta^2|\Omega|^{\frac{2\varrho}{\hat{q}}}t_1^{\frac{2\varrho}{\hat{r}}}|e^{\lambda|G_k(u_n)|}-1|^2_{\Omega_{t_1}}+ \|\mathcal{D}\|_{q,r,\Omega_T}[\mu(k)]^{\frac{2(1+\varrho)}{\hat{r}}}.$$

Choosing t_1 so small that

$$\|\mathcal{D}\|_{q,r,Q_{t_1}(k)}\beta^2|\Omega|^{\frac{2\varrho}{\hat{q}}}t_1^{\frac{2\varrho}{\hat{r}}} \leq \frac{\alpha}{8\gamma}\min\{1,\alpha\}, \tag{2.5}$$

and utilizing the inequality $e^x - 1 \geq x$, $\forall\, x \geq 0$, as a consequence, one has

$$\lambda|G_k(u_n)|_{\Omega_{t_1}} \leq |e^{\lambda|G_k(u_n)|}-1|_{\Omega_{t_1}} \leq C[\mu(k)]^{\frac{(1+\varrho)}{\hat{r}}}, \tag{2.6}$$

where $\varrho = \frac{2\varrho_1}{N} > 0$; and the constant C depends on $\alpha, \gamma, \||\overrightarrow{b}|^2\|_{q,r,\Omega_T}, \||\overrightarrow{F}|^2\|_{q,r,\Omega_T}, \|f\|_{q,r,\Omega_T}$.

Now we are in a position to apply the Iteration Lemma in [1] to (2.6), and we obtain

$$\operatorname*{ess\,sup}_n \|u_n(x,t)\|_{L^\infty(\Omega_{t_1})} \leq C. \tag{2.7}$$

If the time interval $[0,T]$ is partitioned into a finite number of subintervals $[0,t_1], [t_1,t_2], \ldots, [t_{s-1}, t_s = T]$, meanwhile for each subinterval, a condition of the form (2.5) is fulfilled, then analogous arguments are valid for the cylinder $\Omega_{t_i} = \Omega \times (t_{i-1}, t_i)$, $i = 1, 2, \ldots$, and finally we arrive at

$$|u_n(x,t)| \leq C, \tag{2.8}$$

for all n and almost all $(x,t) \in \Omega_T$.

2.3. Almost everywhere convergence of u_n

Now we focus on the energy estimate. The test function is related to $\sinh(x)$, which is used in [7,12].[1] Taking $\sinh(\hat{\lambda}u_n)\chi_{[0,\tau]}$ with $\hat{\lambda} = \frac{2\gamma+3}{\alpha}$ as a test function in (2.1) gives

$$\begin{aligned}&\frac{1}{\hat{\lambda}}\int_\Omega[\cosh(\hat{\lambda}u_n(\tau))-1]\mathrm{d}x - \frac{1}{\hat{\lambda}}\int_\Omega[\cosh(\hat{\lambda}u_0)-1]\mathrm{d}x + \hat{\lambda}\alpha\iint_{\Omega_\tau}\cosh(\hat{\lambda}u_n)|\nabla u_n|^2\mathrm{d}x\mathrm{d}t\\ &\leq \overbrace{\iint_{\Omega_\tau}|\overrightarrow{b}(x,t)||\nabla u_n||\sinh(\hat{\lambda}u_n)|\mathrm{d}x\mathrm{d}t}^{J_1} + \overbrace{\gamma\iint_{\Omega_\tau}|\nabla u_n|^2|\sinh(\hat{\lambda}u_n)|\mathrm{d}x\mathrm{d}t}^{J_2} + \\ &\overbrace{\hat{\lambda}\iint_{\Omega_\tau}|\overrightarrow{F}(x,t)|\cosh(\hat{\lambda}u_n)|\nabla u_n|\mathrm{d}x\mathrm{d}t}^{J_3} + \iint_{\Omega_\tau}|f(x,t)||\sinh(\hat{\lambda}u_n)|\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{2.9}$$

where $\tau \in [0,T]$.

Note the properties of $\sinh(x)$, $\cosh(x)$ stated in the footnote, by virtue of Cauchy's inequality and the uniform L^∞ boundness of u_n, J_1, J_2, J_3 are estimated as follows:

$$J_1 \leq \frac{1}{2}\iint_{\Omega_\tau}|\nabla u_n|^2|\sinh(\hat{\lambda}u_n)|\mathrm{d}x\mathrm{d}t + \frac{1}{2}\iint_{\Omega_\tau}|\overrightarrow{b}|^2|\sinh(\hat{\lambda}u_n)|\mathrm{d}x\mathrm{d}t$$

1. $\sinh(x) := \frac{e^x - e^{-x}}{2}$, $\cosh(x) := \frac{e^x+e^{-x}}{2}$. It is clear that $\sinh(0) = 0$, $\cosh(0) = 1$; $\sinh'(x) = \cosh(x)$, $\cosh'(x) = \sinh(x)$; $|\sinh(x)| \leq \cosh(x)$; and $\cosh(x) \geq 1$, $\forall\, x \in \mathbb{R}$.

$$
\begin{aligned}
&\leq \frac{1}{2}\iint\limits_{\Omega_\tau} |\nabla u_n|^2 \cosh(\hat{\lambda} u_n) \mathrm{d}x\mathrm{d}t + \frac{1}{2}\|\sinh(\hat{\lambda} u_n)\|_{L^\infty(\Omega_T)} \| |\overrightarrow{b}|^2\|_{L^1(\Omega_T)},\\
J_2 &\leq \gamma \iint\limits_{\Omega_\tau} |\nabla u_n|^2 \cosh(\hat{\lambda} u_n) \mathrm{d}x\mathrm{d}t,\\
J_3 &\leq \frac{\hat{\lambda}\alpha}{2}\iint\limits_{\Omega_\tau} \cosh(\hat{\lambda} u_n)|\nabla u_n|^2 \mathrm{d}x\mathrm{d}t + \frac{\hat{\lambda}}{2\alpha}\iint\limits_{\Omega_\tau} \cosh(\hat{\lambda} u_n)|\overrightarrow{F}|^2 \mathrm{d}x\mathrm{d}t\\
&\leq \frac{\hat{\lambda}\alpha}{2}\iint\limits_{\Omega_\tau} \cosh(\hat{\lambda} u_n)|\nabla u_n|^2 \mathrm{d}x\mathrm{d}t + \frac{\hat{\lambda}}{2\alpha}\| |\overrightarrow{F}|^2\|_{L^1(\Omega_T)}\|\cosh(\hat{\lambda} u_n)\|_{L^\infty(\Omega_T)}.
\end{aligned}
$$

These calculations help us estimate (2.9). Taking the supremum for $\tau \in [0,T]$, recalling that $\cosh(x) \geq 1, \forall x \in \mathbb{R}$ and the definition of $\hat{\lambda}$, we have

$$
\begin{aligned}
&\frac{1}{\hat{\lambda}} \operatorname*{ess\,sup}_{\tau\in[0,T]} \int\limits_{\Omega} [\cosh(\hat{\lambda} u_n(\tau)) - 1]\mathrm{d}x + \iint\limits_{\Omega_T} |\nabla u_n|^2 \mathrm{d}x\mathrm{d}t\\
&\leq \|\sinh(\hat{\lambda} u_n)\|_{L^\infty(\Omega_T)}\|f\|_{L^1(\Omega_T)} + \frac{1}{2}\|\sinh(\hat{\lambda} u_n)\|_{L^\infty(\Omega_T)}\| |\overrightarrow{b}|^2\|_{L^1(\Omega_T)} +\\
&\quad \frac{\hat{\lambda}}{2\alpha}\| |\overrightarrow{F}|^2\|_{L^1(\Omega_T)}\|\cosh(\hat{\lambda} u_n)\|_{L^\infty(\Omega_T)} + \frac{1}{\hat{\lambda}}\int\limits_{\Omega} \cosh(\hat{\lambda} u_0)\mathrm{d}x,
\end{aligned}
$$

which shows that

$$
\{u_n\}_{n=1}^\infty \ \text{ is bounded in } \ L^2(0,T;H_0^1(\Omega)). \tag{2.10}
$$

From (2.10) and (2.8), there exist a subsequence of $\{u_n\}_{n=1}^\infty$, not relabeled, and a function $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$, such that

$$
u_n \rightharpoonup u \ \text{ weakly in } \ L^2(0,T;H_0^1(\Omega)); \tag{2.11}
$$

$$
\nabla u_n \rightharpoonup \nabla u \ \text{ weakly in } \ (L^2(\Omega_T))^N; \tag{2.12}
$$

$$
u_n \rightharpoonup u \ \text{ weakly* in } \ L^\infty(\Omega_T). \tag{2.13}
$$

We infer from (2.10), (2.8), (1.5) and (1.3) that the term $\operatorname{div}(a_n(x,t,u_n,\nabla u_n))$ is bounded in $L^2(0,T;H^{-1}(\Omega))$, while $H_n(x,t,u_n,\nabla u_n)$ is bounded in $L^1(\Omega_T)$, thus the equality

$$
\frac{\partial u_n}{\partial t} = \operatorname{div}(a_n(x,t,u_n,\nabla u_n)) - H_n(x,t,u_n,\nabla u_n) - \operatorname{div}\overrightarrow{F}(x,t) + f(x,t)
$$

implies that

$$
\{(u_n)_t\}_{n=1}^\infty \ \text{ is bounded in } \ L^2(0,T;H^{-1}(\Omega)) + L^1(\Omega_T). \tag{2.14}
$$

For $r < \frac{N}{N-1}$, there hold

$$
\{(u_n)_t\}_{n=1}^\infty \ \text{ is bounded in } \ L^1(0,T;W^{-1,r}(\Omega)) \tag{2.15}
$$

and

$$
H_0^1(\Omega) \overset{\text{compact}}{\hookrightarrow} L^2(\Omega) \hookrightarrow W^{-1,r}(\Omega). \tag{2.16}
$$

Combining (2.10), (2.15), (2.16) with Simon's Compactness Theorem in [13], we deduce that

$$
u_n \to u \ \text{ strongly in } \ L^2(0,T;L^2(\Omega)).
$$

Extracting a subsequence of $\{u_n\}_{n=1}^{\infty}$, still denoted by $\{u_n\}_{n=1}^{\infty}$ for simplicity, we have the almost everywhere convergence of u_n:

$$u_n \to u \ \text{ a.e., in } \ \Omega_T. \tag{2.17}$$

2.4. The strong convergence of ∇u_n

Subtracting $(2.1)_m$ from $(2.1)_n$, we have

$$\begin{cases} (u_n - u_m)_t - \operatorname{div}[a_n(u_n, \nabla u_n) - a_m(u_m, \nabla u_m)] + & \\ \quad H_n(u_n, \nabla u_n) - H_m(u_m, \nabla u_m) = 0, & (x,t) \in \Omega_T, \\ (u_n - u_m)(x,t) = 0, & (x,t) \in \Gamma_T, \\ (u_n - u_m)(x,0) = 0, & x \in \Omega, \end{cases} \tag{2.18}$$

where $a_n(s,\xi) := a_n(x,t,s,\xi)$, $H_n(s,\xi) := H_n(x,t,s,\xi)$ for the sake of brevity.

Denote $\sigma(s) = \frac{1}{\bar\lambda}(e^{\bar\lambda|s|} - 1)\operatorname{sign}(s)$, with $\bar\lambda = \frac{\gamma+1}{\alpha}$, then $\widetilde\sigma(s) = \int_0^s \sigma(\tau)\mathrm{d}\tau = \frac{1}{\bar\lambda^2}(e^{\bar\lambda|s|} - 1 - \bar\lambda|s|)$. It is obvious that $\widetilde\sigma(s) \ge 0$, $\widetilde\sigma(0) = 0$.

The use of $\sigma(u_n - u_m)$ as a test function in (2.18) yields

$$\int_\Omega \widetilde\sigma(u_n - u_m)(T)\mathrm{d}x + \overbrace{\iint_{\Omega_T} [a_n(u_n, \nabla u_n) - a_m(u_m, \nabla u_m)] \cdot (\nabla u_n - \nabla u_m)\sigma'(u_n - u_m)\mathrm{d}x\mathrm{d}t}^{I1}$$
$$\le \overbrace{\iint_{\Omega_T} |H_n(u_n, \nabla u_n) - H_m(u_m, \nabla u_m)||\sigma(u_n - u_m)|\mathrm{d}x\mathrm{d}t}^{I2}. \tag{2.19}$$

For $I1$, let $n, m > \operatorname{ess\,sup}_n \|u_n\|_{L^\infty(\Omega_T)}$, then

$$\begin{aligned} I1 &= \iint_{\Omega_T} [a(u_n, \nabla u_n) - a(u_m, \nabla u_m)] \cdot (\nabla u_n - \nabla u_m)\sigma'(u_n - u_m)\mathrm{d}x\mathrm{d}t \\ &= \iint_{\Omega_T} \mathcal{A}(u_n, u_m)\sigma'(u_n - u_m)\mathrm{d}x\mathrm{d}t, \end{aligned}$$

where $\mathcal{A}(u,v) := [a(u,\nabla u) - a(v, \nabla v)] \cdot (\nabla u - \nabla v)$.

For $I2$, by the hypotheses (1.5), (1.2) and Cauchy's inequality, $I2$ is estimated in the following manner:

$$\begin{aligned} I2 \le& \iint_{\Omega_T} |\overrightarrow{b}|(|\nabla u_n| + |\nabla u_m|)|\sigma(u_n - u_m)|\mathrm{d}x\mathrm{d}t + \gamma \iint_{\Omega_T} (|\nabla u_n|^2 + |\nabla u_m|^2)|\sigma(u_n - u_m)|\mathrm{d}x\mathrm{d}t \\ \le& \frac{1}{2}\iint_{\Omega_T} |\overrightarrow{b}|^2|\sigma(u_n - u_m)|\mathrm{d}x\mathrm{d}t + (\gamma+1)\iint_{\Omega_T} (|\nabla u_n|^2 + |\nabla u_m|^2)|\sigma(u_n - u_m)|\mathrm{d}x\mathrm{d}t \\ \le& \frac{1}{2}\iint_{\Omega_T} |\overrightarrow{b}|^2|\sigma(u_n - u_m)|\mathrm{d}x\mathrm{d}t + \frac{\gamma+1}{\alpha}\iint_{\Omega_T} a(u_n, \nabla u_n) \cdot \nabla u_n|\sigma(u_n - u_m)|\mathrm{d}x\mathrm{d}t + \\ & \frac{\gamma+1}{\alpha}\iint_{\Omega_T} a(u_m, \nabla u_m) \cdot \nabla u_m|\sigma(u_n - u_m)|\mathrm{d}x\mathrm{d}t \\ \le& \frac{1}{2}\iint_{\Omega_T} |\overrightarrow{b}|^2|\sigma(u_n - u_m)|\mathrm{d}x\mathrm{d}t + \frac{\gamma+1}{\alpha}\iint_{\Omega_T} \mathcal{A}(u_n, u_m)|\sigma(u_n - u_m)|\mathrm{d}x\mathrm{d}t + \end{aligned}$$

$$\frac{\gamma+1}{\alpha}\iint\limits_{\Omega_T} a(u_n,\nabla u_n)\cdot\nabla u_m|\sigma(u_n-u_m)|\mathrm{d}x\mathrm{d}t+$$
$$\frac{\gamma+1}{\alpha}\iint\limits_{\Omega_T} a(u_m,\nabla u_m)\cdot\nabla u_n|\sigma(u_n-u_m)|\mathrm{d}x\mathrm{d}t.$$

From the estimates $I1, I2$, and that $\sigma'(s)-\bar{\lambda}|\sigma(s)|=1$, $\forall\, s\in\mathbb{R}$, (2.19) is estimated as:

$$\begin{aligned}\iint\limits_{\Omega_T}\mathcal{A}(u_n,u_m)\mathrm{d}x\mathrm{d}t \le & \frac{1}{2}\iint\limits_{\Omega_T}|\overrightarrow{b}|^2|\sigma(u_n-u_m)|\mathrm{d}x\mathrm{d}t+\\ & \frac{\gamma+1}{\alpha}\iint\limits_{\Omega_T} a(u_n,\nabla u_n)\cdot\nabla u_m|\sigma(u_n-u_m)|\mathrm{d}x\mathrm{d}t+\\ & \frac{\gamma+1}{\alpha}\iint\limits_{\Omega_T} a(u_m,\nabla u_m)\cdot\nabla u_n|\sigma(u_n-u_m)|\mathrm{d}x\mathrm{d}t.\end{aligned}$$

By the uniform L^∞ bound of u_n, (1.3) and (2.12), we claim that $a(u_n,\nabla u_n)$ is bounded in $(L^2(\Omega_T))^N$. Thus one may assume that $a(u_n,\nabla u_n)\rightharpoonup\zeta$ weakly in $(L^2(\Omega_T))^N$. For fixed n, letting $m\to\infty$, we have

$$\begin{aligned}&\iint\limits_{\Omega_T} a(u_n,\nabla u_n)\cdot\nabla u_n\mathrm{d}x\mathrm{d}t-\iint\limits_{\Omega_T} a(u_n,\nabla u_n)\cdot\nabla u\mathrm{d}x\mathrm{d}t-\\ &\quad\iint\limits_{\Omega_T}\zeta\cdot\nabla u_n\mathrm{d}x\mathrm{d}t+\limsup_{m\to\infty}\iint\limits_{\Omega_T} a(u_m,\nabla u_m)\cdot\nabla u_m\mathrm{d}x\mathrm{d}t\\ &\le\frac{1}{2}\iint\limits_{\Omega_T}|\overrightarrow{b}|^2|\sigma(u_n-u)|\mathrm{d}x\mathrm{d}t+\frac{\gamma+1}{\alpha}\iint\limits_{\Omega_T} a(u_n,\nabla u_n)\cdot\nabla u|\sigma(u_n-u)|\mathrm{d}x\mathrm{d}t+\\ &\quad\frac{\gamma+1}{\alpha}\iint\limits_{\Omega_T}\zeta\cdot\nabla u_n|\sigma(u_n-u)|\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{2.20}$$

where we employ (2.8), (2.12), (2.17), the Vitali Theorem (see Theorem 3.2 in page 14 in [6]) and the Lebesgue Dominated Convergence Theorem to the limit.

Now letting $n\to\infty$ in (2.20), we obtain

$$\limsup_{n\to\infty}\iint\limits_{\Omega_T} a(u_n,\nabla u_n)\cdot\nabla u_n\mathrm{d}x\mathrm{d}t\le\iint\limits_{\Omega_T}\zeta\cdot\nabla u\mathrm{d}x\mathrm{d}t.$$

From (1.3), (2.8), (2.17) and the Lebesgue Dominated Convergence Theorem it follows

$$a(u_n,\nabla u)\to a(u,\nabla u)\ \text{ strongly in }\ (L^2(\Omega_T))^N.$$

Thus we get

$$\limsup_{n\to\infty}\iint\limits_{\Omega_T}[a(u_n,\nabla u_n)-a(u_n,\nabla u)]\cdot(\nabla u_n-\nabla u)\mathrm{d}x\mathrm{d}t\le 0.$$

It follows from (1.4) and the result in [14] that

$$\nabla u_n\to\nabla u\ \text{ strongly in }\ (L^2(\Omega_T))^N.\tag{2.21}$$

2.5. Passing to the limit

Let $\phi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$ be a test function in Problem (2.1). Then

$$\int_0^T \langle \frac{\partial u_n}{\partial t}, \phi\rangle \mathrm{d}t + \iint_{\Omega_T} a_n(x,t,u_n,\nabla u_n)\cdot\nabla\phi \mathrm{d}x\mathrm{d}t + \iint_{\Omega_T} H_n(x,t,u_n,\nabla u_n)\phi \mathrm{d}x\mathrm{d}t$$
$$= \iint_{\Omega_T} \overrightarrow{F}(x,t)\cdot\nabla\phi\mathrm{d}x\mathrm{d}t + \iint_{\Omega_T} f(x,t)\phi\mathrm{d}x\mathrm{d}t. \tag{2.22}$$

By (2.17), (2.21), (1.5), (2.2) and the Vitali Theorem, we have that

$$H_n(x,t,u_n,\nabla u_n) \to H(x,t,u,\nabla u) \ \text{ strongly in } \ L^1(\Omega_T). \tag{2.23}$$

In view of the Lebesgue Dominated Convergence Theorem and the boundedness of u_n, (2.12), (2.23) and (2.17) permit the limit process in (2.22).

By virtue of (2.10) and (2.14), we know that $\{u_n\}_{n=1}^\infty$ is bounded in $W^{1,1}(0,T;H^{-s}(\Omega))$ with $s > \frac{N}{2}+1$, thus

$$u_n \to u \ \text{ strongly in } \ C([0,T];H^{-s}(\Omega)).$$

Therefore, the initial value has meaning and $u(x,0)=u_0(x)$. $\square$

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