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Multiplicity of Solutions for a Class of Nearly Resonant Semilinear Elliptic Problems

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Abstract The existence of three nontrivial solutions for a class of superlinear Schrödinger equations is obtained by using variational theorems of mixed type due to Marino and Saccon combined a Linking Theorem.

Keywords Schrödinger equation; ∇-condition; linking

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1. Introduction

In this paper, we prove the existence of three nontrivial solutions for a class of semilinear Schrödinger equations when a parameter approaches one of the eigenvalues of the leading operator without assuming the Ambrosetti-Rabinowitz condition.

Going into details, we investigate the multiplicity of solutions for a class of semilinear equations of the form (P_λ) :

$$-\Delta u - V(x)u + \lambda u = f(x,u), \quad x \in \mathbb{R}^N,$$

where λ is a parameter, the nonlinearity $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $\lim_{|x|\to\infty} V(x) = v_\infty$, for some $v_\infty \in \mathbb{R}$. The Schrödinger operator S is defined as

$$Su(x) = -\Delta u(x) - V(x)u(x), \quad D(S) = H^2(\mathbb{R}^N). \tag{1.1}$$

Next, we recall a few basic facts in theory of Schrödinger operators which are relevant to our discussion [1].

(1) Since $\lim_{|x|\to\infty} V(x) = v_\infty$, one has $\sigma_{ess}(S) = [-v_\infty, \infty)$.

(2) The bottom of the spectrum $\sigma(S)$ of the operator S is given by

$$\Lambda = \lambda_0 = \inf_{0\neq u\in H^2(\mathbb{R}^N)} \frac{\int |\nabla u|^2 - V(x)u^2}{\int u^2}.$$

Therefore, we clearly have $\Lambda \le -v_\infty$. If $\Lambda < -v_\infty$, then by using the Concentration Compactness Principle of Lions, one shows that Λ is the principle eigenvalue of S with a positive eigenfunction Φ_0:

$$S\Phi_0 = \lambda_0\Phi_0,\ \Phi_0 \in H^2(\mathbb{R}^N),\ \Phi_0 > 0.$$

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(3) The spectrum of S in $(-\infty,-v_\infty)$, namely, $\sigma(S)\cap(-\infty,-v_\infty)$, is at most a countable set, which is denoted by

$$\Lambda=\lambda_0<\lambda_1<\lambda_2<\cdots,$$

where each λ_k is an isolated eigenvalue of S of the finite multiplicity. Let E_{λ_j} denote the eigenspace of S corresponding to the eigenvalue λ_j, and let $H_{i-1}=\oplus_{k\le i-1}E_{\lambda_k}$ and $H_i^0=E_{\lambda_i}$.

The main purpose of this paper is to use a critical point theorem of mixed type, one of the so-called ∇-theorems, posed by Marino and Saccon [2], which permit to find multiplicity results in a very beautiful way. These theorems have been successfully used in some different contexts [3–12]. In particular, in our previous work [13], we established a multiplicity result for problem (P_λ) by using this type theorem, assuming that f satisfies a growth condition of the Ambrosetti-Rabinowitz type. Here we want to improve our previous work from one main aspect, that is, we do not need the nonlinearity f satisfying the classical Ambrosetti-Rabinowitz type condition (see our condition (H_2)).

Now, we state our main result. In this paper, we often suppose that $\lim_{|x|\to\infty}V(x)=v_\infty$, $\Lambda<-v_\infty$, $v_\infty<0$ and C, C_1 and C_2 are defined as various different positive constants. Let $F(x,t)=\int_0^t f(x,s)\mathrm{d}s$. The conditions imposed on $f(x,t)$ are as follows.

(H_1) There exists $a(x)\in C(\mathbb{R}^N)$, $a(x)>0$ for all $x\in\mathbb{R}^N$ and $\lim_{|x|\to\infty}a(x)=0$ such that

$$\lim_{|t|\to\infty}\frac{f(x,t)}{a(x)|t|^\gamma}=0\ \text{ uniformly on } x\in\mathbb{R}^N,$$

for some $\gamma\in(1,(2^*-1))$ $(2^*=\frac{2N}{N-2})$;

(H_2) $f(x,t)=\circ(|t|)$ as $|t|\to0$ uniformly on $x\in\mathbb{R}^N$ and $\lim_{|t|\to\infty}\frac{F(x,t)}{|t|^2}=+\infty$ uniformly on any bounded subset of $\mathbb{R}^N$;

(H_3) There exists $\max\{\frac{2N\gamma}{N+2},\gamma\}<\beta<2^*$ and $a(x)$ in (H_1) such that

$$f(x,t)t-2F(x,t)\ge a(x)|t|^\beta,\quad \forall\, x\in\mathbb{R}^N,\ t\in\mathbb{R};$$

(H_4) For $a(x)$ in (H_1) and every $r>0$, there exists a positive constant M_r such that

$$\sup_{|t|\le r}\frac{|f(x,t)|}{|t|}\le M_r a(x)$$

for all $x\in\mathbb{R}^N$.

Theorem 1.1 *Assume that conditions (H_1)–(H_4) hold. Then, for any $i\ge2$, there exists $\delta_i>0$ such that for $\forall-\lambda\in(\lambda_i-\delta_i,\lambda_i)$, problem (P_λ) possesses at least three nontrivial solutions.*

Example 1.2 In case of $N=3$, let

$$f(x,t)=\frac{1}{1+|x|^2}|t|^{\frac{7}{2}}t,\quad \forall(x,t)\in\mathbb{R}^3\times\mathbb{R}.$$

It is easy to see that it satisfies our conditions (H_1)–(H_4). Here, $\gamma=\frac{109}{24}$ and $\beta=\frac{11}{2}$.

The paper is organized as follows: In Section 2 we introduce some preliminary knowledge and technical lemmas. In Section 3 we give the proof of our main result. Finally, in Appendix we introduce the important ∇ theorem used to prove our main result.

2. Preliminaries and technical Lemmas

The proof of Theorem 1.1 will be finished in several steps. On $H^1(\mathbb{R}^N)$ we define the inner product

$$\langle u,v\rangle_\lambda := \int \nabla u \nabla v + (\lambda - v_\infty)\int uv, \quad u,v\in H^1(\mathbb{R}^N)$$

and the induced norm

$$\|u\|_\lambda^2 = \langle u,u\rangle_\lambda \text{ for all } u\in H^1(\mathbb{R}^N).$$

It is easy to see that $\|\cdot\|_\lambda$ is equivalent to the standard norm of $H^1(\mathbb{R}^N)$. Now regard the functional $I_\lambda : H^1(\mathbb{R}^N)\to\mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2}\int (V(x)-v_\infty)u^2 - \int F(x,u).$$

From $f\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$ and (H$_1$) we know that I_λ is a C^1 functional on $H^1(\mathbb{R}^N)$. We want to show that if there exists i in $\mathbb{N}$ such that $\lambda_{i-1} < -\lambda < \lambda_i < \lambda_{i+1}$, and $-\lambda$ is sufficiently close to λ_i, then the topological situation described in Theorem 4.1 (see Appendix) holds (with $X_1 = H_{i-1}$, $X_2 = H_i^0$ and $X_3 = H_i^\perp$) and in particular that $(\bigtriangledown)$ $(I_\lambda, H_{i-1}\oplus H_i^\perp, a, b)$ (see Appendix) holds for suitable a and b.

To show that, we begin from the following notations. If $i<k$ and $i\in\mathbb{N}$, then we denote:

$$B_i(R) = \{u\in H_i : \|u\|_\lambda \le R\}$$

and

$$\Gamma_{i-1,i}(R) = \{u\in H_{i-1} : \|u\|_\lambda \le R\}\cup\{u\in H_i : \|u\|_\lambda = R\},$$
$$S_k^+(\rho) = \{u\in H_k^\perp : \|u\|_\lambda = \rho\},\ B_k^+(\rho) = \{u\in H_k^\perp : \|u\|_\lambda \le \rho\}.$$

Lemma 2.1 *If $\lambda_{i-1}<\lambda_i$ and $-\lambda\in(\lambda_{i-1},\lambda_i)$, then there exist R and ρ, $R>\rho>0$, such that*

$$\sup I_\lambda(\Gamma_{i-1,i}(R)) < \inf I_\lambda(S_{i-1}^+(\rho)).$$

Proof From (H$_3$), we know that $\frac{F(x,t)}{t^2}$ is increasing for $t>0$ and decreasing for $t<0$. This together with the first condition of (H$_2$), we have $F(x,t)\ge 0$ for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$. So, for any $u\in H_{i-1}$ and $-\lambda\in(\lambda_{i-1},\lambda_i)$, we get

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2}\int (V(x)-v_\infty)u^2 - \int F(x,u) \\ &\le -C\|u\|_\lambda^2 \le 0, \end{aligned} \tag{2.1}$$

where $C>0$.

Next, we claim that $I_\lambda(u)\to-\infty$, as $\|u\|_\lambda\to\infty$ for all $u\in H_i$. Argue by the contradiction that there exists a sequence $\{u_n\}$ such that

$$I_\lambda(u_n) = \frac{1}{2}\|u_n\|_\lambda^2 - \frac{1}{2}\int (V(x)-v_\infty)u_n^2 - \int F(x,u_n) \ge C. \tag{2.2}$$

Set $v_n = \frac{u_n}{\|u_n\|_\lambda}$. Then $\{v_n\}$ is a bounded sequence in H_i. Because $\dim H_i$ is finite, there exists $v\in H_i\backslash\{0\}$ such that

$$\|v_n - v\|_\lambda \to 0, \quad v_n\to v \text{ a.e., } x\in\mathbb{R}^N.$$

Thus, by the assumptions of $V(x)$ and condition (H$_2$), for $R > 0$ large enough and n large enough, from (2.2), we get

$$\begin{aligned}\frac{I_\lambda(u_n)}{\|u_n\|_\lambda^2} =&\frac{1}{2}-\frac{1}{2}\int (V(x)-v_\infty)v_n^2-\int \frac{F(x,u_n)}{\|u_n\|_\lambda^2}\\ \le&\frac{1}{2}-\frac{1}{2}\int_{B_R(0)}(V(x)-v_\infty)v^2\mathrm{d}x+\circ(1)-\int_{B_R(0)}\frac{F(x,u_n)}{u_n^2}v_n^2\mathrm{d}x\\ \le& C_1+\circ(1)-\int_{B_R(0)}\liminf_{n\to\infty}\frac{F(x,u_n)}{u_n^2}v_n^2\mathrm{d}x\to-\infty.\end{aligned}$$

This is a contradiction and our claim holds.

On the other hand, from (H$_1$), (H$_2$) and (H$_4$), for any $\epsilon > 0$ small enough, there exists $C > 0$ such that

$$F(x,t)\le \frac{\epsilon}{2}t^2+C|t|^{2^*},\quad \forall (x,t)\in\mathbb{R}^N\times\mathbb{R}.$$

Hence, we have

$$\Big|\int F(x,u)\Big|\le \frac{\epsilon}{2}\|u\|_{L^2}^2+C\|u\|_{L^{2^*}}^{2^*},\quad \forall u\in H^1(\mathbb{R}^N). \tag{2.3}$$

For any $u \in H_{i-1}^{\perp}$, by (2.3), we get

$$\begin{aligned}I_\lambda(u) =&\frac{1}{2}\|u\|_\lambda^2-\frac{1}{2}\int (V(x)-v_\infty)u^2-\int F(x,u)\\ \ge& C_1\|u\|_\lambda^2-C_2\|u\|_\lambda^{2^*}. \end{aligned}\tag{2.4}$$

In fact, we define the linear operator

$$L : H^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N) : u\to -\Delta u-V(x)u+\lambda u.$$

Then it is invertible since $-\lambda \in (\lambda_{i-1}, \lambda_i)$. Then, we have

$$a(u,v):=\int |L|^{\frac{1}{2}}u|L|^{\frac{1}{2}}v$$

and the corresponding norm

$$\|u\|_a=\sqrt{a(u,v)}.$$

Hence, from the equivalent property of the norm, there exists $C_1 > 0$ such that

$$\|u\|_a^2\ge C_1\|u\|_\lambda^2.$$

Therefore, the last inequality of (2.4) holds. Thus, from (2.1), (2.2) and (2.4), there exist two constants $R > \rho > 0$ such that

$$\sup I_\lambda(\Gamma_{i-1,i}(R))<\inf I_\lambda(S_{i-1}^{+}(\rho)).\quad\square$$

Lemma 2.2 *Assume that (H$_1$)–(H$_4$) hold. Then for any $\delta \in (0, \min\{\lambda_{i+1} - \lambda_i, \lambda_i - \lambda_{i-1}\})$, there exists $\epsilon_0 > 0$ such that for any $-\lambda \in [\lambda_i - \delta, \lambda_i + \delta]$, the unique critical point u of I_λ is constrained on $H_{i-1} \oplus H_i^{\perp}$ such that $I_\lambda(u) \in [-\epsilon_0, \epsilon_0]$, is the trivial.*

Proof Argue by the contradiction that there exist $\delta > 0$, $-\lambda_n \in [\lambda_i - \delta, \lambda_i + \delta]$, and $\{u_n\} \subseteq$

$H_{i-1} \oplus H_i^{\perp} \setminus \{0\}$ such that

$$I_{\lambda_n}(u_n) = \frac{1}{2}\|u_n\|^2_{\lambda_n} - \frac{1}{2}\int (V(x) - v_\infty)u_n^2 - \int F(x, u_n) \to 0, \tag{2.5}$$

$$\langle I'_{\lambda_n}(u_n), v\rangle = \langle u_n, v\rangle_{\lambda_n} - \int (V(x) - v_\infty)u_n v - \int f(x, u_n)v = 0, \tag{2.6}$$

for any v in $H_{i-1} \oplus H_i^{\perp}$. Of course, up to a subsequence, we can suppose $-\lambda_n \to -\lambda$ in $[\lambda_i - \delta, \lambda_i + \delta]$, as $n \to \infty$. Taking $v = u_n$ in (2.6), by (H$_3$) we get

$$2I_{\lambda_n}(u_n) - \langle I'_{\lambda_n}(u_n), u_n\rangle = \int f(x, u_n) - 2F(x, u_n) \geq \int a(x)|u_n|^{\beta}.$$

By (2.5) and (2.6), the above expression implies that

$$\int a(x)|u_n|^{\beta} < C. \tag{2.7}$$

Take v_n in H_{i-1} and w_n in $H_i^{\perp}$ such that $u_n = v_n + w_n$ and choose $v = v_n - w_n$ in (2.6). Then we have

$$\begin{aligned}&\int (|\nabla w_n|^2 - V(x)w_n^2 + \lambda_n|w_n|^2) - \int (|\nabla v_n|^2 - V(x)w_n^2 + \lambda_n|v_n|^2)\\ &= \int f(x, u_n)(w_n - v_n).\end{aligned}$$

From the above equality, similar to the proof of (2.4), we have

$$C\|u_n\|^2_{\lambda} \leq \int f(x, u_n)(w_n - v_n). \tag{2.8}$$

Moreover, from (H$_1$), we have

$$\Big|\int f(x, u_n)(w_n - v_n)\Big| \leq \Big(\int |f(x, u_n)|^{\frac{1+\gamma}{\gamma}}\Big)^{\frac{\gamma}{\gamma+1}}\Big(\int |w_n - v_n|^{1+\gamma}\Big)^{\frac{1}{1+\gamma}}.$$

Using Sobolev's embedding theorem there exists a constant $C > 0$ such that

$$\|u_n\|_{\lambda} \leq C\Big(\int |f(x, u_n)|^{\frac{1+\gamma}{\gamma}}\Big)^{\frac{\gamma}{\gamma+1}}. \tag{2.9}$$

On the other hand, from (H$_1$), (H$_2$), (H$_4$), (2.7) and Hölder's inequality, we get

$$\begin{aligned}\Big|\int f(x, u_n)(v_n - w_n)\Big| &\leq \int |f(x, u_n)(v_n - w_n)|\\ &\leq (\epsilon\int |v_n - w_n|^2 + C\int a(x)|u_n|^{\gamma}|v_n - w_n|)\\ &\leq \epsilon\|v_n - w_n\|^2_{L^2} + C\Big(\int a(x)|u_n|^{\beta}\Big)^{\frac{\gamma}{\beta}}\Big(\int a(x)|v_n - w_n|^{\frac{\beta}{\beta-\gamma}}\Big)^{\frac{\beta-\gamma}{\beta}}\\ &\leq C_1\epsilon\|v_n - w_n\|^2_{\lambda} + C_2\|v_n - w_n\|_{\lambda}\\ &= C_1\epsilon\|u_n\|^2_{\lambda} + C_2\|u_n\|_{\lambda},\end{aligned}$$

where see (H$_3$) for the restriction of β. Hence, from (2.8) and the above inequality, we know that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. There is a subsequence of $\{u_n\}$, without any loss of generality, also denoted by $\{u_n\}$, and $u \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, $u_n \rightharpoonup u$ strongly in

$L^p_{\rm Loc}(\mathbb{R}^N)$, and $u_n(x)\to u(x)$ a.e., $x\in\mathbb{R}^N$, where $2\le p<2^*$. By (2.5), (2.6) with $v=u_n$ and Fatou's lemma, we obtain

$$\begin{aligned}0=&\lim_{n\to\infty}(2I_{\lambda_n}(u_n)-\langle I'_{\lambda_n}(u_n),u_n\rangle)=\lim_{n\to\infty}\int(f(x,u_n)u_n-2F(x,u_n))\\ \ge&\int\liminf_{n\to\infty}(f(x,u_n)u_n-2F(x,u_n))=\int(f(x,u)u-2F(x,u)).\end{aligned}$$

From (H_2) and (H_3), the above inequality means that $u=0$.

If $u_n\to0$ as $n\to\infty$ in $H^1(\mathbb{R}^N)$, by (H_1), (H_2), (H_4), for any $\epsilon>0$, we get

$$\Big(\int|f(x,u_n)|^{\frac{\gamma+1}{\gamma}}\Big)^{\frac{\gamma}{\gamma+1}}\le\epsilon\|u_n\|_\lambda+C\|u_n\|_\lambda^\gamma.$$

Thus, from (2.9) we have

$$1\le\lim_{n\to\infty}C\frac{(\int|f(x,u_n)|^{\frac{\gamma+1}{\gamma}})^{\frac{\gamma}{\gamma+1}}}{\|u_n\|_\lambda}=0.$$

This is a contradiction.

If there exists $\alpha>0$ such that $\|u_n\|_\lambda\ge\alpha$, from (H_1) and H_4), for any $\epsilon>0$, there exist $R,M_T>0$ such that

$$a(x)<\epsilon\text{ for }|x|>R$$

and

$$|f(x,t)|\le a(x)(M_T|t|+|t|^\gamma),\quad\forall(x,t)\in\mathbb{R}^N\times\mathbb{R}.$$

Thus, by $u_n\rightharpoonup0$, for n large enough, we get

$$\begin{aligned}\Big(\int|f(x,u_n)|^{\frac{\gamma+1}{\gamma}}\Big)^{\frac{\gamma}{\gamma+1}}=&\Big(\int_{\mathbb{R}^N\setminus B_R(0)}|f(x,u_n)|^{\frac{\gamma+1}{\gamma}}\Big)^{\frac{\gamma}{\gamma+1}}+\Big(\int_{B_R(0)}|f(x,u_n)|^{\frac{\gamma+1}{\gamma}}\Big)^{\frac{\gamma}{\gamma+1}}\\ \le&C\epsilon(\|u_n\|_\lambda+\|u_n\|_\lambda^\gamma)+\epsilon.\end{aligned}$$

Hence, we have

$$\alpha\le\lim_{n\to\infty}C\Big(\int|f(x,u_n)|^{\frac{1+\gamma}{\gamma}}\Big)^{\frac{\gamma}{1+\gamma}}=0,$$

which also leads to a contradiction. The lemma is thus completely proved. □

For the following lemma, for given $i\in\mathbb{N}$, we denote by $P:H^1(\mathbb{R}^N)\to H_i^0$ and $Q:H^1(\mathbb{R}^N)\to H_{i-1}\oplus H_i^\perp$ the orthogonal projections.

Lemma 2.3 *Assume that (H_1)–(H_4) hold, $-\lambda\in(\lambda_{i-1},\lambda_{i+1})$ and $\{u_n\}$ in $H^1(\mathbb{R}^N)$ are such that $I_\lambda(u_n)$ is bounded, $Pu_n\to0$ and $QI'_\lambda(u_n)\to0$ as $n\to\infty$. Then $\{u_n\}$ is bounded.*

Proof Arguing by the contradiction that $\{u_n\}$ is unbounded in $H^1(\mathbb{R}^N)$, we can suppose that

$$\|u_n\|_\lambda\to\infty,\tag{2.10}$$

as $n\to\infty$ and that there exists $u\in H^1(\mathbb{R}^N)$ such that

$$\frac{u_n}{\|u_n\|_\lambda}\rightharpoonup u.\tag{2.11}$$

Let $u_n = Pu_n + Qu_n$. By (H_1), (H_2), (H_4), Hölder's inequality and dim $H_i^0 < +\infty$, we get

$$\begin{aligned}\Big|\int f(x,u_n)Pu_n\Big| \le & \int |f(x,u_n)Pu_n| \\ \le & \epsilon\|Pu_n\|_{L^2}^2 + C\Big(\int a(x)|u_n|^\beta\Big)^{\frac{\gamma}{\beta}}\Big(\int |Pu_n|^{\frac{\beta}{\beta-\gamma}}\Big)^{\frac{\beta-\gamma}{\beta}} \\ \le & C_1\epsilon\|Pu_n\|_\lambda^2 + C_2\|Pu_n\|_\lambda\Big(\int a(x)|u_n|^\beta\Big)^{\frac{\gamma}{\beta}},\end{aligned}$$

where $\epsilon > 0$ small enough. Combining the above inequality and (H_3), we obtain

$$\begin{aligned}&2I_\lambda(u_n) - \langle QI'_\lambda(u_n), u_n\rangle \\ &= \int (f(x,u_n)u_n - 2F(x,u_n)) + \|Pu_n\|_\lambda^2 - \int (V(x)-v_\infty)|Pu_n|^2 - \int f(x,u_n)Pu_n \\ &\ge C\Big(\int a(x)|u_n|^\beta\Big) + \|Pu_n\|_\lambda^2 - \int (V(x)-v_\infty)|Pu_n|^2 - C_1\epsilon\|Pu_n\|_\lambda^2 - \\ &\quad C_2\|Pu_n\|_\lambda\Big(\int a(x)|u_n|^\beta\Big)^{\frac{\gamma}{\beta}}.\end{aligned}$$

Thanks to $\beta > \gamma$, dim $H_i^0 < +\infty$ and $\|Pu_n\|_\lambda \to 0$ as $n \to \infty$, dividing by $\|u_n\|_\lambda$ in the two sides of inequality above, we get

$$\frac{(\int a(x)|u_n|^\beta)^{\frac{\gamma}{\beta}}}{\|u_n\|_\lambda} \to 0 \text{ as } n \to \infty. \tag{2.12}$$

As a consequence of (2.12), we also obtain $u = 0$. Now, from our assumptions and (2.10), we get

$$\frac{I_\lambda(u_n)}{\|u_n\|_\lambda^2} = \frac{1}{2} - \frac{1}{2}\frac{\int (V(x)-v_\infty)u_n^2}{\|u_n\|_\lambda^2} - \frac{F(x,u_n)}{\|u_n\|_\lambda^2} \to 0,$$

which implies that

$$\frac{\int F(x,u_n)}{\|u_n\|_\lambda^2} \to \frac{1}{2} \text{ as } n \to \infty. \tag{2.13}$$

On the other hand, again by (H_1), (H_2) and (H_4), for $\epsilon > 0$ small enough, there exists $C > 0$ such that

$$\Big|\int F(x,u_n)\Big| \le \epsilon\int u_n^2 + C\Big(\int a(x)|u_n|^\beta\Big)^{\frac{\gamma}{\beta}}\|u_n\|_\lambda,$$

and by (2.12) we obtain a contradiction with (2.13). □

Lemma 2.4 *Suppose that (H_1)–(H_4) hold. Then for any $\delta \in (0, \min\{\lambda_{i+1}-\lambda_i, \lambda_i - \lambda_{i-1}\})$, there exists $\epsilon_0 > 0$ such that for any $-\lambda \in [\lambda_i - \delta, \lambda_i + \delta]$ and for any $\epsilon_1, \epsilon_2 \in (0, \epsilon_0)$ with $\epsilon_1 < \epsilon_2$, the condition $(\nabla)(I_\lambda, H_{i-1} \oplus H_i^\perp, \epsilon_1, \epsilon_2)$ holds.*

Proof Argue by contradiction that there exists $\delta_0 > 0$ such that for all $\epsilon_0 > 0$, there exist $-\lambda \in [\lambda_i - \delta, \lambda_i + \delta]$ and $\epsilon_1, \epsilon_2 \in (0, \epsilon_0)$ with $\epsilon_1 < \epsilon_2$, the condition $(\nabla)(I_\lambda, H_{i-1} \oplus H_i^\perp, \epsilon_1, \epsilon_2)$ does not hold.

Let $\epsilon_0 > 0$ be as in Lemma 2.2. There exists a sequence of $\{u_n\}$ in $H^1(\mathbb{R}^N)$ such that $\text{dist}(u_n, H_{i-1} \oplus H_i^\perp) \to 0, I_\lambda(u_n) \in (\epsilon_1, \epsilon_2)$ and $QI'_\lambda(u_n) \to 0$. By Lemma 2.3, we know that $\|u_n\|_\lambda$ is bounded. Hence, there exists a subsequence of $\{u_n\}$, without loss of generality, also denoted by $\{u_n\}$, and $u \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$. From (H_1) and (H_4), similarly to the proof of Lemma 2.6 (see next context), we have $u_n \to u$ and $u = 0$ is a critical point of I_λ constrained on $H_{i-1} \oplus H_i^\perp$ by Lemma 2.2. But $0 < \epsilon_1 \le I_\lambda(u)$, which leads to a contradiction. □

Lemma 2.5 *Suppose that* (H_2) *holds. Then*

$$\limsup_{\lambda \to -\lambda_i} I_\lambda(H_i) = 0.$$

Proof Argue by contradiction that there exist $\lambda_n \to -\lambda_i$, $\{u_n\}$ in H_i and $\epsilon > 0$ such that

$$\sup I_{\lambda_n}(H_i) = I_{\lambda_n}(u_n) \ge \epsilon.$$

Note that I_λ attains a maximum in H_i by (H_2).

If $\{u_n\}$ is bounded, we can assume that $u_n \to u$ in H_i. Then we have

$$\epsilon \le I_{-\lambda_i}(u) \le 0.$$

Hence, we can suppose that $\|u_n\|_\lambda \to \infty$. In this case, the condition (H_2) easily implies a contradiction. □

Lemma 2.6 *Assume that* (H_1)*–*(H_4) *hold. Then* I_λ *satisfies* (PS) *condition for* $-\lambda \in (\lambda_{i-1}, \lambda_i)$.

Proof Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a sequence such that $\{I_\lambda(u_n)\}$ is bounded and $I'_\lambda(u_n) \to 0$. We need to verify that $\{u_n\}$ possesses a convergent subsequence. We will show that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Arguing by the contradiction, we can suppose that $\|u_n\|_\lambda \to +\infty$.

Now, set $u_n = z_n + h_n$, where $z_n \in H_{i-1}$ and $h_n \in H_{i-1}^\perp$ for every $i \in \mathbb{N}$. From Hölder's inequality, we have

$$\int a(x)|u_n|^\gamma |z_n| \le C\Big(\int a(x)|u_n|^\beta\Big)^{\frac{\gamma}{\beta}} \|z_n\|_\lambda, \tag{2.14}$$

$$\int a(x)|u_n|^\gamma |h_n| \le C\Big(\int a(x)|u_n|^\beta\Big)^{\frac{\gamma}{\beta}} \|h_n\|_\lambda \tag{2.15}$$

for some $C > 0$ and all $n \in \mathbb{N}$. Using (H_3), we again obtain

$$\lim_{n\to\infty} \frac{(\int a(x)|u_n|^\beta)^{\frac{\gamma}{\beta}}}{\|u_n\|_\lambda} = 0, \tag{2.16}$$

and so also (2.12) holds again. Now, from (H_1), (H_2), (H_4) and (2.14), it follows

$$\begin{aligned}\|z_n\|_\lambda o(1) =& \langle I'_\lambda(u_n), -z_n\rangle = -\|z_n\|_\lambda^2 + \int (V(x) - v_\infty) z_n^2 + \int f(x, u_n) z_n \\ \ge& C\|z_n\|_\lambda^2 - \epsilon\|z_n\|_{L^2}^2 - C_1 \int a(x)|u_n|^\gamma |z_n| \\ \ge& C\|z_n\|_\lambda^2 - C_2\Big(\int a(x)|u_n|^\beta\Big)^{\frac{\gamma}{\beta}} \|z_n\|_\lambda.\end{aligned}$$

From (2.12) and the above expression, we can imply that

$$\lim_{n\to\infty} \frac{\|z_n\|_\lambda}{\|u_n\|_\lambda} = 0. \tag{2.17}$$

Similarly, we can also obtain that

$$\lim_{n\to\infty}\frac{\|h_n\|_\lambda}{\|u_n\|_\lambda}=0. \tag{2.18}$$

Thus, from (2.17) and (2.18), we have

$$1=\frac{\|u_n\|_\lambda}{\|u_n\|_\lambda}\leq\frac{\|z_n\|_\lambda+\|h_n\|_\lambda}{\|u_n\|_\lambda}\to 0,\ \text{as } n\to\infty,$$

which is a contradiction. So $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Next, we prove that $\{u_n\}$ contains a convergent subsequence.

In fact, we know that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Passing to a subsequence, we may assume that $u_n\rightharpoonup u$ in $H^1(\mathbb{R}^N)$. In order to establish strong convergence, it suffices to show that

$$\|u_n\|_\lambda\to\|u\|_\lambda.$$

Since $\langle I'_\lambda(u_n),\, u_n-u\rangle\to 0$, we know that

$$0\leq\limsup_{n\to\infty}(||u_n||^2_\lambda-||u||^2_\lambda)=\lim_{n\to\infty}\sup\int f(x,u_n)(u_n-u). \tag{2.19}$$

By the condition (H$_1$), for $\epsilon>0$ small enough and $r>1$ large enough, we have

$$\int_{|u_n|\geq r}f(x,u_n)(u_n-u)<\frac{\epsilon}{3}. \tag{2.20}$$

Moreover, by (H$_4$) there exists $R>0$ such that

$$\int_{|u_n|\leq r;|x|\geq R}f(x,u_n)(u_n-u)\leq\frac{\epsilon}{3}. \tag{2.21}$$

Finally, by (H$_1$), using Lebesgue convergence theorem, we find that

$$\int_{|u_n|\leq r;|x|\leq R}f(x,u_n)(u_n-u)\leq\frac{\epsilon}{3}, \tag{2.22}$$

for n large enough. Thus, from (2.20)–(2.22), we obtain

$$\int f(x,u_n)(u_n-u)\leq\epsilon,$$

for n large enough. So from above the inequality and (2.19), our conclusion holds. □

3. The proof of main result

Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1 The argument will be divided into two steps.

(a) Two critical points are obtained.

Take $\delta'>0$ and find ϵ_0 as in Lemma 2.4. Denote $\epsilon_1<\epsilon_2<\epsilon_0$. By Lemma 2.5 there exists $\delta_1\leq\delta'$ such that, if $-\lambda\in(\lambda_i-\delta_1,\lambda_i)$, then $\sup I_\lambda(H_i)<\epsilon_2$ and by Lemma 2.4, $\bigtriangledown(I_\lambda,H_{i-1}\oplus H_i^\perp,\epsilon_1,\epsilon_2)$ holds. Moreover, since $-\lambda<\lambda_i$, the topological strcture of Lemma 2.1 is satisfied. By Theorem 4.1, there exist two critical points u_1,u_2 of I_λ such that $I_\lambda(u_i)\in[\epsilon_1,\epsilon_2], i=1,2$. In particular u_1 and u_2 are nontrivial solutions of problem (P_λ).

(b) The third critical point is obtained.

Since the classical Linking Theorem [14] and Lemma 2.6, it suffices to prove that there exist $\delta_i > 0$, $\rho_1 > 0$ and $R_1 > \rho_1$ such that for any $-\lambda$ in $(\lambda_i - \delta_i, \lambda_i)$

$$\sup I_\lambda(\Gamma_{i,i+1}(R_1)) < \inf I_\lambda(S_i^+(\rho_1)). \tag{3.1}$$

Hence, there is a critical point u of I_λ such that $I_\lambda(u) > \inf I_\lambda(S_i^+(\rho_1))$.

In fact, from (2.3), we conclude that for any $u \in H_i^\perp$

$$\begin{aligned} I_\lambda(u) =& \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2}\int (V(x) - v_\infty)u^2 - \int F(x,u) \\ \geq& C\|u\|_\lambda^2 - C_1\|u\|_\lambda^{2^*}. \end{aligned} \tag{3.2}$$

By above the inequality, there exist $\rho_1 > 0, \alpha > 0$ such that

$$\inf I_\lambda(S_i^+(\rho_1)) \geq \alpha.$$

On the other hand, using (H_2), we can obtain that

$$I_\lambda(u) \to -\infty \text{ as } \|u\|_\lambda \to \infty, u \in H_{i+1}.$$

By Lemma 2.5, there exists $\delta_i > 0$ such that for any $-\lambda$ in $(\lambda_i - \delta_i, \lambda_i)$, we get

$$\sup I_\lambda(H_i) < \alpha.$$

Thus (3.1) holds. Hence u is different from the critical point u_i $(i = 1, 2)$, since

$$I_\lambda(u_i) \leq \sup I_\lambda(H_i) < \alpha \leq I_\lambda(u). \quad \square$$

4. Appendix

In this section we recall one theorem belonging to a class of recent variational ones which provide the existence of several critical points under a "mixed type" assumption on the functional, in the sense that there are hypotheses both on the values of functional on some suitable sets and on the values of its gradient. Theorems of this kind were first introduced in [2] as follows.

Definition 4.1 ([2]) *Let X be a Hilbert space, let $I : X \to R$ be a C^1 function, let M be a closed subspace of X, and let $a, b \in R \cup \{-\infty, +\infty\}$. We say that condition $(\nabla)(I, M, a, b)$ holds if there exists $\gamma > 0$ such that*

$$\inf\{\|P_M \nabla I(u)\| \;|a \leq I(u) \leq b,\ \operatorname{dist}(u, M) \leq \gamma\} > 0,$$

where $P_M : X \to M$ is the orthogonal projection of X onto M.

Theorem 4.2 ([2, (∇)-Theorem]) *Let X be a Hilbert space and X_i, $i = 1, 2, 3$ three subspaces of X such that $X = X_1 \oplus X_2 \oplus X_3$ and $\dim X_i < \infty$ for $i = 1, 2$. Denote by P_i the orthogonal projection of X onto X_i. Let $I : X \to R$ be a $C^{1,1}$ function. Let ρ, ρ', ρ'', ρ_1 be such that $\rho_1 > 0$, $0 \leq \rho' < \rho < \rho''$ and define*

$$\Delta = \{u \in X_1 \oplus X_2 | \rho' \leq \|P_2 u\| \leq \rho'', \|P_1 u\| \leq \rho_1\} \text{ and } \Gamma = \partial_{X_1 \oplus X_2}\Delta,$$

$$S_{23}(\rho) = \{u \in X_2 \oplus X_3 | \ \|u\| = \rho\} \text{ and } B_{23}(\rho) = \{u \in X_2 \oplus X_3 | \ \|u\| \le \rho\}.$$

Assume that

$$a' = \sup I(\Gamma) < \inf I(S_{23}(\rho)) = a''.$$

Let a *and* b *be such that* $a' < a < a''$ *and* $b > \sup I(\Delta)$. *Assume* $(\nabla)(I, X_1 \oplus X_3, a, b)$ *holds and that* $(PS)_c$ *holds at any* c *in* $[a, b]$. *Then* I *has at least two critical points in* $I^{-1}([a, b])$. *Moreover, if*

$$\inf I(B_{23}(\rho)) > -\infty \text{ and } a_1 < \inf I(B_{23}(\rho))$$

and $(PS)_c$ *holds at any* c *in* $[a_1, b]$, *then* I *has another critical level in* $[a_1, a']$.

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