Journal of Mathematical Research with Applications
Nov., 2022, Vol. 42, No. 6, pp. 580–586
DOI:10.3770/j.issn:2095-2651.2022.06.003
Http://jmre.dlut.edu.cn

On a Problem of Q. H. YANG and Y. G. CHEN

Xiaohui YAN
School of Mathematics and Statistics, Anhui Normal University, Anhui 241002*, P. R. China*

Abstract For any positive integers k_1, k_2 and any set $A \subseteq \mathbb{N}$, let $R_{k_1,k_2}(A,n)$ be the number of solutions of the equation $n = k_1a_1 + k_2a_2$ with $a_1, a_2 \in A$. Let $\bar{A} = \mathbb{N}\backslash A$. Yang and Chen proved that if k_1 and k_2 are two integers with $k_2 > k_1 \geq 2$ and $(k_1, k_2) = 1$, then there does not exist any set $A \subseteq \mathbb{N}$ such that $R_{k_1,k_2}(A,n) = R_{k_1,k_2}(\bar{A},n)$ for all sufficiently large integers n. For two integers $k > 1$ and $t \geq 1$, define $f_k(t)$ to be the number of sets $A \subseteq \mathbb{N}$ such that $R_{1,k}(A,n) = R_{1,k}(\bar{A},n)$ holds for all integers $n \geq t$. Yang and Chen proved that $f_k(t)$ is finite and $\lim_{t\to\infty} \frac{\log f_k(t)}{t} = \log 2$. They also asked if it is true that for any integers $k, l > 1$ there exists $t_0(k,l)$ such that $f_k(t) = f_l(t)$ for all integers $t \geq t_0$. In this paper, we give the exact formula of $f_k(t)$ when $t \leq k$, which implies that $f_k(t) = f_l(t)$ for all integers $t \leq \min\{k,l\}$.

Keywords representation function; partition; Sárközy problem

MR(2020) Subject Classification 11B34

1. Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers. For a set $A \subseteq \mathbb{N}$, let $R_1(A,n)$, $R_2(A,n)$ and $R_3(A,n)$ denote the number of solutions of $a_1+a_2 = n, a_1, a_2 \in A$; $a_1+a_2 = n, a_1, a_2 \in A, a_1 < a_2$ and $a_1 + a_2 = n, a_1, a_2 \in A, a_1 \leq a_2$, respectively. For $i = 1, 2, 3$, Sárközy asked whether there exist two sets A and B with infinite symmetric difference such that $R_i(A,n) = R_i(B,n)$ for all sufficiently large integers n. Dombi [1] proved that the answer is negative for $i = 1$ and positive for $i = 2$. For $i = 3$, Chen and Wang [2] proved that the answer is also positive. Later, Lev [3], Sándor [4] and Tang [5] provided new and nice proofs, respectively.

In [6], Yang and Chen considered the Sárközy problem with weighted representation functions. For any positive integers $k_1, \ldots, k_t$ and any set $A \subseteq \mathbb{N}$, let $R_{k_1,\ldots,k_t}(A,n)$ be the number of solutions of the equation $n = k_1a_1 + \cdots + k_ta_t$ with $a_1, \ldots, a_t \in A$. Let $\bar{A} = \mathbb{N}\backslash A$. They posed the following question:

Question 1.1 ([6]) Is there a set $A \subseteq \mathbb{N}$ such that $R_{k_1,\ldots,k_t}(A,n) = R_{k_1,\ldots,k_t}(\bar{A},n)$ for all $n \geq n_0$?

They answered this question for $t = 2$ and proved the following results.

Theorem 1.2 ([6]) *If k_1 and k_2 are two integers with $k_2 > k_1 \geq 2$ and $(k_1, k_2) = 1$, then*

Received September 22, 2021; Accepted May 7, 2022
Supported by the National Natural Science Foundation of China (Grant No. 12101009) and the Natural Science Foundation of Anhui Province (Grant No. 2108085QA02).
E-mail address: yanxiaohui_1992@163.com

there does not exist any set $A \subseteq \mathbb{N}$ such that $R_{k_1,k_2}(A,n) = R_{k_1,k_2}(\bar{A},n)$ for all sufficiently large integers n.

Theorem 1.3 ([6]) *If k is an integer with $k > 1$, then there exists a set $A \subseteq \mathbb{N}$ such that*

$$R_{1,k}(A,n) = R_{1,k}(\bar{A},n) \tag{1.1}$$

for all integers $n \ge 1$.

Furthermore, if $0 \in A$, then (1.1) holds for all integers $n \ge 1$ if and only if

$$A = \{0\} \bigcup \Big(\bigcup_{i=0}^{\infty} [(k+1)k^{2i}, (k+1)k^{2i+1} - 1] \Big),$$

where $[x,y] = \{n : n \in \mathbb{Z}, x \le n \le y\}$.

Later, Li and Ma [7] proved the same results by using generating function. For two integers $k > 1$ and $t \ge 1$, define $f_k(t)$ to be the number of sets $A \subseteq \mathbb{N}$ such that (1.1) holds for all integers $n \ge t$. By Theorem 1.3, we have $f_k(1) = 2$. Yang and Chen [6] proved the following result and posed a question.

Theorem 1.4 ([6]) *Let k be an integer with $k > 1$. Then, for each integer $t \ge 1$, $f_k(t)$ is finite and*

$$\lim_{t\to\infty} \frac{\log f_k(t)}{t} = \log 2.$$

Question 1.5 ([6]) Is it true for any integers $k, l > 1$ there exists $t_0 = t_0(k,l)$ such that $f_k(t) = f_l(t)$ for all integers $t \ge t_0$?

In this paper, we give the exact formula of $f_k(t)$ when $t \le k$.

Theorem 1.6 *Let k be an integer with $k > 1$. Then*

$$f_k(t) = \begin{cases} 2, & t = 1, \\ 2^{t-1}, & 2 \le t \le k. \end{cases}$$

Furthermore, for any integers $k, l > 1$, $f_k(t) = f_l(t)$ holds for all integers $t \le \min\{k, l\}$.

2. Proof of Theorem 1.6

To prove Theorem 1.6, we need some Lemmas.

Lemma 2.1 ([7]) *There exists a set $A \subseteq \mathbb{N}$ such that $R_{k_1,\dots,k_t}(A,n) = R_{k_1,\dots,k_t}(\bar{A},n)$ for all integers $n \ge n_0$ if and only if there is a polynomial $p(x)$ (including zero polynomial) of degree at most $n_0 - 1$ such that for $|x| < 1$,*

$$\prod_{i=1}^{t} \Big(\sum_{a\in A} x^{k_i a} \Big) - \prod_{i=1}^{t} \Big(\frac{1}{1 - x^{k_i}} - \sum_{a\in A} x^{k_i a} \Big) = p(x).$$

Lemma 2.2 *Let k and t be two positive integers with $k \ge t > 1$ and let $A \subseteq \mathbb{N}$. The equality $R_{1,k}(A,n) = R_{1,k}(\bar{A},n)$ holds for all integers $n \ge t$ if and only if*

$$\chi_A(0) + \chi_A(1) = 1, \tag{2.1}$$

$$\chi_A(i)+\chi_A(i+k)=1 \quad \text{for } 0\le i\le t-1, \tag{2.2}$$

$$\chi_A(i)+\chi_A([\frac{i}{k}])=1 \quad \text{for} \quad i\ge k+t \quad \text{and} \quad t-1<i<k, \tag{2.3}$$

where $\chi_A(m)$ is the characteristic function of the set A, that is, $\chi_A(m)=1$ if $m\in A$ and $\chi_A(m)=0$ if $m\notin A$.

Proof Let $f(x)$ be the generating function associated with A, that is,

$$f(x)=\sum_{a\in A}x^a=\sum_{i=0}^{\infty}\chi_A(i)x^i.$$

Suppose that $R_{1,k}(A,n)=R_{1,k}(\bar{A},n)$ holds for $n\ge t$. It follows from Lemma 2.1 that there is a polynomial $p(x)$ (including zero polynomial) of degree at most $t-1$ such that for $|x|<1$,

$$\begin{aligned}p(x)=&f(x)f(x^k)-(\frac{1}{1-x}-f(x))(\frac{1}{1-x^k}-f(x^k))\\=&\frac{f(x^k)}{1-x}+\frac{f(x)}{1-x^k}-\frac{1}{(1-x)(1-x^k)},\end{aligned}$$

that is,

$$f(x)=\frac{1}{1-x}-f(x^k)(1+x+\cdots+x^{k-1})+p(x)(1-x^k). \tag{2.4}$$

Let

$$p(x)=\sum_{i=0}^{t-1}\alpha_i x^i.$$

Then

$$p(x)(1-x^k)=\sum_{i=0}^{t-1}\alpha_i x^i-\sum_{i=0}^{t-1}\alpha_i x^{k+i}.$$

Since

$$f(x^k)(1+x+\cdots+x^{k-1})=\sum_{i=0}^{\infty}\chi_A([\frac{i}{k}])x^i$$

and

$$\frac{1}{1-x}=\sum_{i=0}^{\infty}x^i,$$

it follows that

$$\frac{1}{1-x}-f(x^k)(1+x+\cdots+x^{k-1})=\sum_{i=0}^{\infty}(1-\chi_A([\frac{i}{k}]))x^i.$$

So

$$\sum_{i=0}^{\infty}\chi_A(i)x^i=\sum_{i=0}^{\infty}(1-\chi_A([\frac{i}{k}]))x^i+\sum_{i=0}^{t-1}\alpha_i x^i-\sum_{i=0}^{t-1}\alpha_i x^{k+i}.$$

Noting that $t\le k$, we have

$$\chi_A(i)=1-\chi_A([\frac{i}{k}]) \quad \text{for} \quad i\ge k+t \quad \text{and} \quad t-1<i<k$$

and

$$\chi_A(i)=1-\chi_A([\frac{i}{k}])+\alpha_i, \quad \chi_A(i+k)=1-\chi_A([\frac{i+k}{k}])-\alpha_i \quad \text{for} \quad 0\le i\le t-1.$$

Then

$$\chi_A(i) + \chi_A([\frac{i}{k}]) = 1 \quad \text{for} \quad i \ge k+t \quad \text{and} \quad t-1 < i < k$$

and

$$\chi_A(i) + \chi_A(i+k) + \chi_A(0) + \chi_A(1) = 2 \quad \text{for} \quad 0 \le i \le t-1. \tag{2.5}$$

Take $i = 0$ in (2.5), and we have

$$2\chi_A(0) + \chi_A(k) + \chi_A(1) = 2.$$

If $\chi_A(0) = 1$, then $\chi_A(1) = 0$. If $\chi_A(0) = 0$, then $\chi_A(1) = 1$. Therefore,

$$\chi_A(0) + \chi_A(1) = 1.$$

It follows from (2.5) that

$$\chi_A(i) + \chi_A(i+k) = 1 \quad \text{for} \quad 0 \le i \le t-1,$$

which proves the necessary part of Lemma 2.2.

Next, we prove the sufficient part of Lemma 2.2. It follows from Lemma 2.1 and (2.4) that we only need to prove there exists a polynomial $p(x)$ of degree at most $t-1$ such that

$$f(x) = \frac{1}{1-x} - f(x^k)(1 + x + \cdots + x^{k-1}) + p(x)(1 - x^k).$$

It follows from (2.3) that

$$\begin{aligned}
f(x) &= \sum_{i=0}^{\infty} \chi_A(i)x^i = \sum_{i=0}^{k+t-1} \chi_A(i)x^i + \sum_{i=k+t}^{\infty} \chi_A(i)x^i \\
&= \sum_{i=0}^{k+t-1} \chi_A(i)x^i + \sum_{i=k+t}^{\infty} (1 - \chi_A([\frac{i}{k}]))x^i \\
&= \sum_{i=0}^{k+t-1} (\chi_A(i) + \chi_A([\frac{i}{k}]) - 1)x^i + \sum_{i=0}^{\infty}(1 - \chi_A([\frac{i}{k}]))x^i \\
&= \frac{1}{1-x} - f(x^k)(1 + x + \cdots + x^{k-1}) + \sum_{i=0}^{k+t-1} (\chi_A(i) + \chi_A([\frac{i}{k}]) - 1)x^i.
\end{aligned}$$

By (2.1)–(2.3) and $t \le k$ we have

$$\begin{aligned}
&\sum_{i=0}^{k+t-1} (\chi_A(i) + \chi_A([\frac{i}{k}]) - 1)x^i \\
&= \sum_{i=0}^{t-1}(\chi_A(i) + \chi_A([\frac{i}{k}]) - 1)x^i + \sum_{i=k}^{k+t-1} (\chi_A(i) + \chi_A([\frac{i}{k}]) - 1)x^i \\
&= \sum_{i=0}^{t-1}(\chi_A(i) + \chi_A([\frac{i}{k}]) - 1)x^i + \sum_{i=0}^{t-1}(\chi_A(i+k) + \chi_A([\frac{i+k}{k}]) - 1)x^{k+i} \\
&= \sum_{i=0}^{t-1}(\chi_A(i) + \chi_A(0) - 1)x^i + \sum_{i=0}^{t-1}(1 - \chi_A(i) + \chi_A(1) - 1)x^{k+i}
\end{aligned}$$

$$= (1-x^k)\sum_{i=0}^{t-1}(\chi_A(i)+\chi_A(0)-1)x^i.$$

Let

$$p(x)=\sum_{i=0}^{t-1}(\chi(i)+\chi(0)-1)x^i.$$

Then the degree of $p(x)$ at most $t-1$ and

$$f(x)=\frac{1}{1-x}-f(x^k)(1+x+\cdots+x^{k-1})+p(x)(1-x^k).$$

This completes the proof of Lemma 2.2. □

Now, we will prove Theorem 1.6.

Proof of Theorem 1.6. It follows from Theorem 1.3 that $f_k(1)=2$. We will use induction on $2\le t\le k$ to prove $f_k(t)=2^{t-1}$.

If $t=2$, by Lemma 2.2, then $R_{1,k}(A,n)=R_{1,k}(\bar{A},n)$ holds for all integers $n\ge 2$ if and only if

$$\chi_A(0)+\chi_A(1)=1,\quad \chi_A(0)+\chi_A(k)=1,\quad \chi_A(1)+\chi_A(k+1)=1$$

and

$$\chi_A(i)+\chi_A([\frac{i}{k}])=1 \ \text{ for } \ i\ge k+2 \ \text{ and } \ 1<i<k.$$

Then

$$A=\{0\}\bigcup\Big(\bigcup_{i=0}^{\infty}[(k+1)k^{2i},(k+1)k^{2i+1}-1]\Big)$$

or

$$A=[1,k]\bigcup\Big(\bigcup_{i=0}^{\infty}[(k+1)k^{2i+1},(k+1)k^{2i+2}-1]\Big).$$

Therefore, $f_k(2)=2$.

Assume that $f_k(t)=2^{t-1}$ with $2\le t\le k-1$. Then there are 2^{t-1} different sets $A_1,A_2,\dots,A_{2^{t-1}}$ such that

$$R_{1,k}(A_j,n)=R_{1,k}(\bar{A}_j,n)\ (1\le j\le 2^{t-1})$$

holds for all integers $n\ge t$. It is clear that every set A_j also satisfies

$$R_{1,k}(A_j,n)=R_{1,k}(\bar{A}_j,n)$$

for all integers $n\ge t+1$. By Lemma 2.2 we have

$$\chi_{A_j}(t)+\chi_{A_j}(t+k)=1$$

and

$$\chi_{A_j}(m)+\chi_{A_j}([\frac{m}{k}])=1,$$
$$m\in\bigcup_{s=1}^{\infty}([k^st,k^s(t+1)-1]\cup[k^s(t+k),k^s(t+k+1)-1]).$$

For every set A_j, let A'_j be a set satisfying

$$\chi_{A'_j}(m) = \chi_{A_j}(m), \quad m \notin \bigcup_{s=0}^{\infty}([k^s t, k^s(t+1)-1] \cup [k^s(t+k), k^s(t+k+1)-1])$$

and

$$\chi_{A'_j}(m) = 1 - \chi_{A_j}(m), \quad m \in \bigcup_{s=0}^{\infty}([k^s t, k^s(t+1)-1] \cup [k^s(t+k), k^s(t+k+1)-1]).$$

Then

$$\chi_{A'_j}(t) + \chi_{A'_j}(t+k) = 1$$

and

$$\chi_{A'_j}(m) + \chi_{A'_j}([\frac{m}{k}]) = 1, \quad m \in \bigcup_{s=1}^{\infty}([k^s t, k^s(t+1)-1] \cup [k^s(t+k), k^s(t+k+1)-1]).$$

Moreover,

$$\chi_{A'_j}(0) + \chi_{A'_j}(1) = 1,$$

$$\chi_{A'_j}(i) + \chi_{A'_j}(i+k) = 1 \ \text{ for } \ 0 \le i \le t,$$

$$\chi_{A'_j}(i) + \chi_{A'_j}([\frac{i}{k}]) = 1 \ \text{ for } \ i \ge k+t+1 \ \text{ and } \ t < i < k.$$

It follows from Lemma 2.2 that $R_{1,k}(A'_j, n) = R_{1,k}(\bar{A}'_j, n)$ holds for all integers $n \ge t+1$.

Next, we will prove these sets $A_1, A'_1, \ldots, A_{2^{t-1}}, A'_{2^{t-1}}$ are different. Then $f_k(t+1) \ge 2^t$. If there exist $A'_i = A'_j$ $(i \ne j)$, then

$$1 - \chi_{A_i}(m) = \chi_{A'_i}(m) = \chi_{A'_j}(m) = 1 - \chi_{A_j}(m)$$

for all $m \in \bigcup_{s=0}^{\infty}([k^s t, k^s(t+1)-1] \cup [k^s(t+k), k^s(t+k+1)-1])$, which implies

$$\chi_{A_i}(m) = \chi_{A_j}(m), m \in \bigcup_{s=0}^{\infty}([k^s t, k^s(t+1)-1] \cup [k^s(t+k), k^s(t+k+1)-1]).$$

Also, it is clear that

$$\chi_{A_i}(m) = \chi_{A'_i}(m) = \chi_{A'_j}(m) = \chi_{A_j}(m)$$

for all $m \notin \bigcup_{s=0}^{\infty}([k^s t, k^s(t+1)-1] \cup [k^s(t+k), k^s(t+k+1)-1])$. Hence, $A_i = A_j$, a contradiction. Therefore, $A'_1, \ldots, A'_{2^{t-1}}$ are different. It is clear that $A_i \ne A'_i$. If there exist $A'_i = A_j$ $(i \ne j)$, then

$$\chi_{A'_i}(t) = \chi_{A_j}(t), \quad \chi_{A'_i}(t) \ne \chi_{A_i}(t), \quad \chi_{A_i}(0) = \chi_{A'_i}(0) = \chi_{A_j}(0).$$

Noting that $t \le k-1$, we have

$$1 = \chi_{A_j}(t) + \chi_{A_j}([\frac{t}{k}]) = \chi_{A_j}(t) + \chi_{A_j}(0) = \chi_{A'_i}(t) + \chi_{A_i}(0).$$

Since

$$1 = \chi_{A_i}(t) + \chi_{A_i}([\frac{t}{k}]) = \chi_{A_i}(t) + \chi_{A_i}(0),$$

it follows that $\chi_{A_i}(t) = \chi_{A'_i}(t)$, a contradiction. Therefore, $A_1, A'_1, \ldots, A_{2^{t-1}}, A'_{2^{t-1}}$ are different. Then $f_k(t+1) \geq 2^t$. Let A be a set with $R_{1,k}(A,n) = R_{1,k}(\bar{A},n)$ for all integers $n \geq t+1$. It follows from Lemma 2.2 that

$$\chi_A(0) + \chi_A(1) = 1,$$

$$\chi_A(i) + \chi_A(i+k) = 1 \quad \text{for} \quad 0 \leq i \leq t,$$

$$\chi_A(i) + \chi_A([\frac{i}{k}]) = 1 \quad \text{for} \quad i \geq k+t+1 \quad \text{and} \quad t < i < k.$$

Since the set A is completely determined by $\chi_A(1), \chi_A(2), \ldots, \chi_A(t)$, it follows that $f_k(t+1) \leq 2^t$. Therefore,

$$f_k(t+1) = 2^t.$$

This completes the proof of Theorem 1.6. $\square$

Acknowledgements We thank the referees for their time and comments.

References

[1] G. DOMBI. *Additive properties of certain sets*. Acta Arith., 2002, **103**(2): 137–146.

[2] Yonggao CHEN, Bin WANG. *On additive properties of two special sequences*. Acta Arith., 2003, **110**(3): 299–303.

[3] V. F. LEV. *Reconstructing integer sets from their representation functions*. Electron. J. Combin., 2004, **11**(1): Research Paper 78, 6 pp.

[4] C. SÁNDOR. *Partitions of natural numbers and their representation functions*. Integers, 2004, **4**: A18, 5 pp.

[5] Min TANG. *Partitions of the set of natural numbers and their representation functions*. Discrete Math., 2008, **308**(12): 2614–2616.

[6] Quanhui YANG, Yonggao CHEN. *Partitions of natural numbers with the same weighted representation functions*. J. Number Theory, 2012, **132**(12): 3047–3055.

[7] Yali LI, Wuxia MA. *Partitions of natural numbers with the same weighted representation functions*. Colloq. Math., 2020, **159**(1): 1–5.