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A Note on Some Inequalities of GG-E-Convexity

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Abstract In this paper, we define GG-E-convexity and GG-E-concavity. We also establish some inequalities in relation to GG-E-convex function using various integral inequalities with enough examples of various categories, which verifies our results.

Keywords convex function; E-convex set; E-convex function; GG-E-convex function

MR(2020) Subject Classification 26A51; 26D10 ; 26D15

1. Introduction

The study convex sets is a branch of geometry and linear algebra which has many links to other areas of mathematics and is useful in uniting many different mathematical phenomena. It is also relevant to many areas of science and technology.

Although the convex set is defined in different settings [1], the most useful definitions are based on the notion of middle. When E is an A space in which such a notion is defined, the subset C of E is said to be convex provided that for each two points x and y of C, C covers all points between x and y. The most supreme setting, and the only one to be discussed here, is that in which E is a vector space over the real number field R or, in particular, is the N-dimensional Euclidean space E^n, and the points between x and y are those of the line segment. Thus Victor Klee in 1971 (see [2]) defined a subset C of a real vector space to be convex provided that C contains every segment whose endpoints both belong to C. (For example, a cube in E^3 is convex, but its boundary is not, for the boundary does not contain the segment unless x and y lie together in some 2-dimensional face of the cube.) The importance of the convexity stems from the fact that convex sets arise frequently in many areas of mathematics and are often attributed to rather elementary logic.

The systematic study of convexity was soon conducted by Minkowski (1864–1909), whose works [3] contain, at least in germinal form, most of the important considerations of the subject. The early events of convexity theory were: finite-dimensional and mainly directed towards the solution of quantitative problems; a classic survey of them was made by Bonnesen and Fenchel [4] in 1934. Since 1940, however, the mixed, qualitative, and dimension-free parts of the theory have tended to predominate, perhaps because of their many applications in other areas of mathematics.

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In 2020, Chen et al. elaborated n-polynomial P-convex function with several types of convexities [5], also in 2021 Chu et al. introduced two new classes of convex functions known as GEH convex functions and GEH s-convex functions on the fractal domain [6]. A class of functions which is based on the effect of an operator E on the sets and domain of definition of the functions is called an E-convex function. Youness [7] introduced the concepts of an E-convex set and an E-convex function. E-convexity is a basic notion in geometry, but also is widely used in multiobjective programming problems [8], optimality criteria in E-convex programming [9], epi-graph [10], duality [11] and generalized convexity [12]. It is often hidden in other areas of mathematics: functional analysis, complex analysis, calculus of variations, graph theory, partial differential equations, discrete mathematics, algebraic geometry, probability theory, coding theory, crystallography and many other fields. E-convexity can play an important role also in areas outside mathematics, such as physics, chemistry, biology and other sciences, but this is beyond the scope of this note for consideration of these applications.

2. Definitions and preliminaries

Some definitions and examples are given:

Definition 2.1 *The function $g : I \subset R \to R$ is a convex function on I, if the inequality*

$$g(tx + (1-t)y) \leq tg(x) + (1-t)g(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. We say that g is concave if $-g$ is convex [13].

Let $g : I \subset R \to R$ be a convex function where $a, b \in I$ with $a < b$. Then the following double inequality holds:

$$g(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x \leq \frac{g(a)+g(b)}{2}.$$

This inequality is well-known in the literature as Hermite-Hadamard inequality [14].

Definition 2.2 *A set $N \subset R^n$ is said to be E-convex iff there is a map $E : R^n \to R^n$ such that*

$$(1-t)E(x) + tE(y) \in N$$

for each $x, y \in N$ and $0 \leq t \leq 1$ (see [7]).

Definition 2.3 *A function $g : R^n \to R^n$ is said to be E-convex on a set $N \subset R^n$, iff there is a map $E : R^n \to R^n$ such that N is a convex set and*

$$g(tE(x) + (1-t)E(y)) \leq tg(E(x)) + (1-t)g(E(y))$$

for each $x, y \in N$ and $0 \leq t \leq 1$ on the other hand, if

$$g(tE(x) + (1-t)E(y)) \geq tg(E(x)) + (1-t)g(E(y))$$

then g is called E-concave on N. If the inequality signs in the previous two inequalities are strict, then g is called strictly E-convex and strictly E-concave, respectively [7].

Definition 2.4 *Let $g : S \to (0,\infty)$ be continuous, where I is subinterval of $(0,\infty)$. Let N and P be any two mean functions, $T \subset R$ and there is a map $E : R \to R$. Then we say g is NP-E-convex (concave) on T if*

$$g(N(E(x),E(y)) \leq (\geq) P(g(E(x),g(E(y))$$

for all $x, y \in S$.

Definition 2.5 *The GG-E-convex functions are those functions $g : S \subseteq R_+ \to R$ and there is a map $E : R \to R$ such that $x, y \in S$ and*

$$t \in [0,1] \Rightarrow g((E(x))^{1-t}(E(y))^t) \leq (g(E(x))^{1-t}(g(E(y))^t.$$

Definition 2.6 *The GG-E-concave functions are those functions $g : S \subseteq R_+ \to R$ and there is a map $E : R \to R$ such that $x, y \in S$ and*

$$t \in [0,1] \Rightarrow g((E(x))^{1-t}(E(y))^t) \geq (g(E(x))^{1-t}(g(E(y))^t.$$

Lemma 2.7 *Let $g : S \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on S^o and $x, y \in S^o$ with $a < b$, and there is a map $E :\to R$. If $g' \in L([E(a)],[E(b)])$, then the following identity holds:*

$$\begin{aligned}
&E(b)g(E(b)) - E(a)g(E(a)) - \int_{E(a)}^{E(b)} g(E(u))\mathrm{d}u \\
&= (\ln E(x) - \ln E(y)) \int_0^1 ((E(x))^{2t}(E(a))^{2(1-t)})g'((E(x))^t(E(a))^{(1-t)})\mathrm{d}t - \\
&(\ln E(x) - \ln E(b)) \int_0^1 ((E(x))^{2t}(E(b))^{2(1-t)})g'((E(x))^t(E(b))^{(1-t)})\mathrm{d}t
\end{aligned}$$

for all $E(x) \in [E(a),E(b)]$.

Lemma 2.8 *Let $g : S \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on S^o and $\alpha, \beta \in S^o$ with $\alpha < \beta$ and $E : R \to R$ is a non decreasing function so $E(\alpha) < E(\beta)$. If $g' \in L([E(\alpha)],[E(\beta)])$. If $|g'|$ is GG-E-convex on $[E(\alpha),E(\beta)]$, for all $E(\gamma) \in [E(\alpha),E(\beta)]$, then the following inequality holds:*

$$\begin{aligned}
&(E(\beta))^2 g(E(\beta) - (E(\alpha))^2 g(E(\alpha) - 2\int_{E(\alpha)}^{E(\beta)} E(\gamma)g(E(\gamma))\mathrm{d}(E(\gamma)) \\
&= (\ln E(\beta) - \ln E(\gamma)) \int_0^1 (E(\beta)^{3\tau'}E(\gamma)^{3(1-\tau')})g'(E(\beta)^{\tau'}E(\gamma)^{(1-\tau')}\mathrm{d}\tau' + \\
&(\ln E(\gamma) - \ln E(\alpha)) \int_0^1 (E(\gamma)^{3\tau'}E(\alpha)^{3(1-\tau')})g'(E(\gamma)^{\tau'}E(\alpha)^{(1-\tau')}\mathrm{d}\tau'.
\end{aligned}$$

Proof Let

$$J_1 = \int_0^1 (E(\beta)^{3\tau'}E(\gamma)^{3(1-\tau')})g'(E(\beta)^{\tau'}E(\gamma)^{(1-\tau')})\mathrm{d}\tau'$$

and

$$J_2 = \int_0^1 (E(\gamma)^{3\tau'}E(\alpha)^{3(1-\tau')})g'(E(\gamma)^{\tau'}E(\alpha)^{(1-\tau')})\mathrm{d}\tau'$$

then we notice that

$$J_1 = \int_0^1 (E(\beta)^{3\tau'} E(\gamma)^{3(1-\tau')}) g'(E(\beta)^{\tau'} E(\gamma)^{(1-\tau')}) \mathrm{d}\tau'$$
$$= (\frac{1}{\ln E(\beta) - \ln E(\gamma)}) \int_0^1 (E(\beta)^{2\tau'} E(\gamma)^{2(1-\tau')}) g'(E(\beta)^{\tau'} E(\gamma)^{(1-\tau')}) \mathrm{d}(E(\beta)^{\tau'} E(\gamma)^{(1-\tau')}).$$

Now by the change of variable $E(\gamma) = (E(\beta)^{\tau'} E(\gamma)^{(1-\tau')})$ and integrating by parts, we have

$$J_1 = (\frac{1}{\ln E(\beta) - \ln E(\gamma)}) \Big[(E(\beta)^2 g(E(\beta)) - (E(\gamma)^2 g(E(\gamma)) - 2 \int_{E(\gamma)}^{E(\beta)} E(\gamma) g(E(\gamma)) \mathrm{d}E((\gamma)) \Big].$$

Correspondingly, we have

$$J_2 = (\frac{1}{\ln E(\gamma) - \ln E(\alpha)}) \Big[(E(\gamma)^2 g(E(\gamma)) - (E(\alpha)^2 g(E(\alpha)) - 2 \int_{E(\alpha)}^{E(\gamma)} E(\gamma) g(E(\gamma)) \mathrm{d}E((\gamma)) \Big].$$

Multiplying J_1 by $(\ln E(\beta) - \ln E(\gamma))$ and J_2 by $(\ln E(\gamma) - \ln E(\alpha))$ and adding the results we get the appealed identity.

The following examples illustrate our results.

Example 2.9 Let $f : [0, \frac{\pi}{2}] \to [0, \infty)$ such that

$$f(x) = -\int_0^{\sqrt{x}} \ln(\cos(t)) \mathrm{d}t$$

is not *GG*-convex on $(0, \frac{\pi}{2})$ and there is a map $E : [0, \sqrt{\frac{\pi}{2}}] \to [0, \frac{\pi}{2}]$ such that

$$E(x) = x^2.$$

Then the function

$$f(E(x)) = -\int_0^{x} \ln(\cos(t)) \mathrm{d}t$$

is *GG*-*E*-convex function on $(0, \frac{\pi}{2})$.

Example 2.10 Let $f : [0, \frac{\pi}{2}] \to [0, \infty)$ such that

$$f(x) = \ln(\sin x)$$

is not *GG*-convex on $(0, \frac{\pi}{2})$ and there is a map $E : [0, \frac{\pi}{2}] \to [0, \frac{\pi}{2}]$ such that

$$E(x) = \frac{\pi}{2} - x.$$

Then the function

$$f(E(x)) = \ln(\sin(\frac{\pi}{2} - x)), \quad f(E(x)) = \ln(\cos x)$$

is *GG*-*E*-convex function on $(0, \frac{\pi}{2})$.

Example 2.11 Let $f : [1, \infty) \to [1, \infty)$ such that

$$f(x) = \Gamma(x)$$

and there is a map $E : [0, \infty] \to [1, \infty]$ such that $E(x) = x + 1$. Then the function

$$f(E(x)) = \Gamma(x + 1)$$

is GG-E-convex on $[0,\infty)$.

Example 2.12 Let $f:[0,\frac{\pi}{2})\to[0,\infty)$ such that

$$f(x)=\cot(x)$$

and there is a map $E:[0,\frac{\pi}{2}]\to[0,\frac{\pi}{2}]$ such that $E(x)=\frac{\pi}{2}-x$. Then the function

$$f(E(x))=\tan(x)$$

is GG-E-convex on $[0,\frac{\pi}{2})$.

3. Main results

We have the following results based on the previous lemmas:

Theorem 3.1 *Let $g:S\subseteq R_{+}=(0,\infty)\to R$ be a differentiable function on S^{o} and $\alpha,\beta\in S^{o}$ with $\alpha<\beta$ and $E:R\to R$ is a non decreasing function so $E(\alpha)<E(\beta)$. Suppose $g'\in L([E(\alpha)],[E(\beta)])$. If $|g'|$ is GG-E-convex on $[E(\alpha),E(\beta)]$, for all $E(\gamma)\in[E(\alpha),E(\beta)]$, then the following inequality holds:*

$$\begin{aligned}&\Big|(E(\beta))^2g(E(\beta)-(E(\alpha))^2g(E(\alpha)-2\int_{E(\alpha)}^{E(\beta)}E(\gamma)g(E(\gamma))\mathrm{d}(E(\gamma))\Big|\\&\le(\ln E(\beta)-\ln E(\gamma))L((E(\beta))^3|g'(E(\beta)|,(E(\gamma))^3|g'(E(\gamma)|)+\\&\quad(\ln E(\gamma)-\ln E(\alpha))L(E(\gamma))^3|g'(E(\gamma)|,E(\alpha))^3|g'(E(\alpha)|).\end{aligned}$$

Proof From Lemma 2.8, using the property of the modulus and GG-E-convexity of $|g'|$, we can write

$$\begin{aligned}&\Big|(E(\beta))^2g(E(\beta)-(E(\alpha))^2g(E(\alpha)-2\int_{E(\alpha)}^{E(\beta)}E(\gamma)g(E(\gamma))\mathrm{d}(E(\gamma))\Big|\\&\le(\ln E(\beta)-\ln E(\gamma))\int_0^1(E(\beta)^{3\tau'}E(\gamma)^{3(1-\tau')})|g'(E(\beta))^{\tau'}(E(\gamma))^{(1-\tau')}|\mathrm{d}\tau'+\\&\quad(\ln E(\gamma)-\ln E(\alpha))\int_0^1(E(\gamma)^{3\tau'}E(\alpha)^{3(1-\tau')})|g'(E(\gamma))^{\tau'}(E(\alpha))^{(1-\tau')}|\mathrm{d}\tau'\\&\le(\ln E(\beta)-\ln E(\gamma))\int_0^1(E(\beta)^{3\tau'}E(\gamma)^{3(1-\tau')})|g'(E(\beta)|^{\tau'}|g'(E(\gamma)|^{(1-\tau')}|\mathrm{d}\tau'\\&\quad(\ln E(\gamma)-\ln E(\alpha))\int_0^1(E(\gamma)^{3\tau'}E(\alpha)^{3(1-\tau')})|g'(E(\alpha)|^{\tau'}|g'(E(\beta)|^{(1-\tau')}|\mathrm{d}\tau'\\&=(\ln E(\beta)-\ln E(\gamma))(E(\gamma))^3|g'(E(\gamma)|\int_0^1(\frac{(E(\beta))^3|g'(E(\beta)|}{(E(\gamma))^3|g'(E(\gamma)|})^{\tau'}\mathrm{d}\tau'+\\&\quad(\ln E(\gamma)-\ln E(\alpha))(E(\alpha))^3|g'(E(\alpha)|\int_0^1(\frac{(E(\gamma))^3|g'(E(\gamma)|}{(E(\alpha))^3|g'(E(\alpha)|})^{\tau'}\mathrm{d}\tau'.\end{aligned}$$

Then we get the desired result. This theorem is applicable to all differentiable functions whose first derivative is GG-E-convex function.

Theorem 3.2 *Let $g:S\subseteq R_{+}=(0,\infty)\to R$ be a differentiable function on S^{o} and $\alpha,\beta\in S^{o}$*

with $\alpha < \beta$ and $E : R \to R$ is a non decreasing function so $E(\alpha) < E(\beta)$. Suppose $g' \in L([E(\alpha)], [E(\beta)])$. If $|g'|^n$ is GG-E-convex on $[E(\alpha), E(\beta)]$, for all $E(\gamma) \in [E(\alpha), E(\beta)]$, then the following inequality holds:

$$\begin{aligned}
&\Big|(E(\beta))^2 g(E(\beta) - (E(\alpha))^2 g(E(\alpha) - 2\int_{E(\alpha)}^{E(\beta)} E(\gamma)g(E(\gamma))\mathrm{d}(E(\gamma))\Big| \\
&\quad \le (\ln E(\beta) - \ln E(\gamma))((L((E(\beta))^{3m}, (E(\gamma))^{3m}))^{\frac{1}{m}}(L(|g'(E(\beta))|^n, |g'(E(\gamma))|^n)^{\frac{1}{n}} + \\
&\qquad (\ln E(\gamma) - \ln E(\alpha))((L((E(\gamma))^{3m}, (E(\alpha))^{3m}))^{\frac{1}{m}}(L(|g'(E(\gamma))|^n, |g'(E(\alpha))|^n)^{\frac{1}{n}},
\end{aligned}$$

where $n > 1$ and $\frac{1}{m} + \frac{1}{n} = 1$.

Proof From Lemma 2.8, using the property of the modulus, *GG-E*-convexity of $|g'|^m$ and Holder integral inequality, we can write

$$\begin{aligned}
&\Big|(E(\beta))^2 g(E(\beta) - (E(\alpha))^2 g(E(\alpha) - 2\int_{E(\alpha)}^{E(\beta)} E(\gamma)g(E(\gamma))\mathrm{d}(E(\gamma))\Big| \\
&= (\ln E(\beta) - \ln E(\gamma))\int_0^1 (E(\beta)^{3\tau'}E(\gamma)^{3(1-\tau')})|g'(E(\beta))^{\tau'}(E(\gamma))^{(1-\tau')}|\mathrm{d}\tau' + \\
&\quad (\ln E(\gamma) - \ln E(\alpha))\int_0^1 (E(\gamma)^{3\tau'}E(\alpha)^{3(1-\tau')})|g'(E(\gamma))^{\tau'}(E(\alpha))^{(1-\tau')}|\mathrm{d}\tau' \\
&\le (\ln E(\beta) - \ln E(\gamma))\int_0^1 ((E(\beta)^{3\tau' m}E(\gamma)^{3(1-\tau')m})\mathrm{d}\tau')^{\frac{1}{m}} \times \\
&\quad \Big(\int_0^1 |g'((E(\beta))^{\tau'}(E(\gamma)))^{(1-\tau')}|^n \mathrm{d}\tau'\Big)^{\frac{1}{n}} + \\
&\quad (\ln E(\gamma) - \ln E(\alpha))\int_0^1 ((E(\gamma)^{3\tau' m}E(\alpha)^{3(1-\tau')m})\mathrm{d}\tau')^{\frac{1}{m}} \times \\
&\quad \Big(\int_0^1 |g'((E(\gamma))^{\tau'}(E(\alpha)))^{(1-\tau')}|^n \mathrm{d}\tau'\Big)^{\frac{1}{n}} \\
&\le (\ln E(\beta) - \ln E(\gamma))((E(\gamma))^{3m}\int_0^1 (\frac{(E(\beta))^{3m}}{(E(\gamma))^{3m}})^{\tau'}\mathrm{d}\tau')^{\frac{1}{m}} \times \\
&\quad \Big(\int_0^1 |g'(E(\beta)|^{n\tau'}|g'(E(\gamma))|^{n(1-E(\gamma))}\mathrm{d}\tau'\Big)^{\frac{1}{n}} + \\
&\quad (\ln E(\gamma) - \ln E(\beta))((E(\beta))^{3m}\int_0^1 (\frac{(E(\gamma))^{3m}}{(E(\beta))^{3m}})^{\tau'}\mathrm{d}\tau')^{\frac{1}{m}} \times \\
&\quad \Big(\int_0^1 |g'(E(\gamma))|^{n\tau'}|g'(E(\beta))|^{n(1-E(\gamma))}\mathrm{d}\tau'\Big)^{\frac{1}{n}}.
\end{aligned}$$

Then we get the desired result. This theorem is applicable to all differentiable functions whose first derivative and its powers are *GG-E*-convex function.

Theorem 3.3 *Under the assumptions of Theorem 3.2, the following inequality holds:*

$$\begin{aligned}
&\Big|(E(\beta))^2 g(E(\beta) - (E(\alpha))^2 g(E(\alpha) - 2\int_{E(\alpha)}^{E(\beta)} E(\gamma)g(E(\gamma))\mathrm{d}(E(\gamma))\Big| \\
&\quad \le (\ln E(\beta) - \ln E(\gamma))((L((E(\beta))^{3n}|g'(E(\beta))|^n, (E(\gamma))^{3n}|g'(E(\gamma))|^n))^{\frac{1}{n}} +
\end{aligned}$$

$$(\ln E(\gamma) - \ln E(\alpha))((L((E(\gamma))^{3n}|g'(E(\gamma)|^n, (E(\alpha))^{3n}|g'(E(\alpha))|^n))^{\frac{1}{n}},$$

where $n > 1$ *and* $\frac{1}{m} + \frac{1}{n} = 1$.

Proof From Lemma 2.8, using the property of the modulus, GG-E-convexity of $|g'|^n$ and Holder integral inequality, we can write

$$\begin{aligned}
&\Big|(E(\beta))^2 g(E(\beta) - (E(\alpha))^2 g(E(\alpha) - 2\int_{E(\alpha)}^{E(\beta)} E(\gamma)g(E(\gamma))\mathrm{d}(E(\gamma))\Big| \\
&= (\ln E(\beta) - \ln E(\gamma))\int_0^1 (E(\beta)^{3\tau'} E(\gamma)^{3(1-\tau')})|g'(E(\beta))^{\tau'}(E(\gamma))^{(1-\tau')}|\mathrm{d}\tau' + \\
&\quad (\ln E(\gamma) - \ln E(\alpha))\int_0^1 (E(\gamma)^{3\tau'} E(\alpha)^{3(1-\tau')})|g'(E(\gamma))^{\tau'}(E(\alpha))^{(1-\tau')}|\mathrm{d}\tau' \\
&\le (\ln E(\beta) - \ln E(\gamma))\Big(\int_0^1 \mathrm{d}\tau'\Big)^{\frac{1}{m}} \times \\
&\quad \Big(\int_0^1 (E(\beta))^{3\tau' n}(E(\gamma))^{3(1-\tau')n}|g'((E(\beta))^{\tau'}(E(\gamma)))^{(1-\tau')}|^n \mathrm{d}\tau'\Big)^{\frac{1}{n}} + \\
&\quad (\ln E(\gamma) - \ln E(\alpha))\Big(\int_0^1 \mathrm{d}\tau'\Big)^{\frac{1}{m}} \times \\
&\quad \Big(\int_0^1 (E(\gamma))^{3\tau' n}(E(\alpha))^{3(1-\tau')n}|g'((E(\gamma))^{\tau'}(E(\alpha)))^{(1-\tau')}|^n \mathrm{d}\tau'\Big)^{\frac{1}{n}} \\
&\le (\ln E(\beta) - \ln E(\gamma))\Big(\int_0^1 \mathrm{d}\tau'\Big)^{\frac{1}{m}} \times \\
&\quad \Big(\int_0^1 (E(\beta))^{3\tau' n}(E(\gamma))^{3(1-\tau')n}|g'(E(\beta)|^{n\tau'}|g'(E(\gamma))|^{n(1-\tau')}\mathrm{d}\tau'\Big)^{\frac{1}{n}} + \\
&\quad (\ln E(\gamma) - \ln E(\alpha))\Big(\int_0^1 \mathrm{d}\tau'\Big)^{\frac{1}{m}} \times \\
&\quad \Big(\int_0^1 (E(\gamma))^{3\tau' n}(E(\alpha))^{3(1-\tau')n}|g'(E(\gamma)|^{n\tau'}|g'(E(\alpha))|^{n(1-\tau')}\mathrm{d}\tau'\Big)^{\frac{1}{n}},
\end{aligned}$$

If we calculate the above integral, we get the desired result. This theorem is generalization of theorem 3.1. □

Theorem 3.4 *Let* $g : S \subseteq R_+ = (0, \infty) \to R$ *be a differentiable function on* S^o *and* $\alpha, \beta \in S^o$ *with* $\alpha < \beta$ *and* $E : R \to R$ *is a non-decreasing function so* $E(\alpha) < E(\beta)$. *Suppose* $g' \in L([E(\alpha)], [E(\beta)])$. *If* $|g'|^n$ *is GG-E-convex on* $[E(\alpha), E(\beta)]$, *for all* $E(\gamma) \in [E(\alpha), E(\beta)]$, *then the following inequality holds:*

$$\begin{aligned}
&\Big|(E(\beta))^2 g(E(\beta) - (E(\alpha))^2 g(E(\alpha) - 2\int_{E(\alpha)}^{E(\beta)} E(\gamma)g(E(\gamma))\mathrm{d}(E(\gamma))\Big| \\
&\le (\ln E(\beta) - \ln E(\gamma))((L((E(\beta))^3, (E(\gamma))^3))^{(1-\frac{1}{n})} \times \\
&\quad (L(E(\beta))^3(|g'(E(\beta))|^n, (E(\gamma))^3|g'(E(\gamma))|^n)^{\frac{1}{n}} + \\
&\quad (\ln E(\gamma) - \ln E(\alpha))((L((E(\gamma))^3, (E(\alpha))^3))^{(1-\frac{1}{n})} \times \\
&\quad (L(E(\gamma))^3(|g'(E(\gamma))|^n, (E(\alpha))^3|g'(E(\alpha))|^n)^{\frac{1}{n}}
\end{aligned}$$

for $n \geq 1$.

Proof From Lemma 2.8, using the property of the modulus, GG-E-convexity of $|g'|^n$ and Power mean integral inequality, we can write

$$\begin{aligned}
&\Big|(E(\beta))^2 g(E(\beta) - (E(\alpha))^2 g(E(\alpha) - 2\int_{E(\alpha)}^{E(\beta)} E(\gamma) g(E(\gamma)) \mathrm{d}(E(\gamma))\Big| \\
&= (\ln E(\beta) - \ln E(\gamma)) \int_0^1 (E(\beta)^{3\tau'} E(\gamma)^{3(1-\tau')}) |g'(E(\beta))^{\tau'} (E(\gamma))^{(1-\tau')}| \mathrm{d}\tau' + \\
&\quad (\ln E(\gamma) - \ln E(\alpha)) \int_0^1 (E(\gamma)^{3\tau'} E(\alpha)^{3(1-\tau')}) |g'(E(\gamma))^{\tau'} (E(\alpha))^{(1-\tau')}| \mathrm{d}\tau' \\
&\leq (\ln E(\beta) - \ln E(\gamma)) \Big[\Big(\int_0^1 ((E(\beta))^{3\tau'} (E(\gamma))^{3(1-\tau')}) \mathrm{d}\tau' \Big)^{(1-\frac{1}{n})} \Big] \times \\
&\quad \Big[\Big(\int_0^1 (E(\beta)^{3\tau'} E(\gamma)^{3(1-\tau')}) |g'(E(\beta))^{\tau'} (E(\gamma))^{(1-\tau')}|^n \mathrm{d}\tau' \Big)^{\frac{1}{n}} \Big] + \\
&\quad (\ln E(\gamma) - \ln E(\alpha)) \Big[\Big(\int_0^1 ((E(\gamma))^{3\tau'} (E(\alpha))^{3(1-\tau')}) \mathrm{d}\tau' \Big)^{(1-\frac{1}{n})} \Big] \times \\
&\quad \Big[\Big(\int_0^1 (E(\gamma)^{3\tau'} E(\alpha)^{3(1-\tau')}) |g'(E(\gamma))^{\tau'} (E(\alpha))^{(1-\tau')}|^n \mathrm{d}\tau' \Big)^{\frac{1}{n}} \Big] \\
&\leq (\ln E(\beta) - \ln E(\gamma)) \Big[\Big(\int_0^1 ((E(\beta))^{3\tau'} (E(\gamma))^{3(1-\tau')}) \mathrm{d}\tau' \Big)^{(1-\frac{1}{n})} \Big] \times \\
&\quad \Big[\Big(\int_0^1 (E(\beta)^{3\tau'} E(\gamma)^{3(1-\tau')}) |g'(E(\beta)|^{n\tau'} |g'(E(\gamma))|^{n(1-\tau')} \mathrm{d}\tau' \Big)^{\frac{1}{n}} \Big] + \\
&\quad (\ln E(\gamma) - \ln E(\alpha)) \Big[\Big(\int_0^1 ((E(\gamma))^{3\tau'} (E(\alpha))^{3(1-\tau')}) \mathrm{d}\tau' \Big)^{(1-\frac{1}{n})} \Big] \times \\
&\quad \Big[\Big(\int_0^1 (E(\gamma)^{3\tau'} E(\alpha)^{3(1-\tau')}) |g'(E(\gamma)|^{n\tau'} |g'(E(\alpha))|^{n(1-\tau')} \mathrm{d}\tau' \Big)^{\frac{1}{n}} \Big].
\end{aligned}$$

If we calculate the above integral, we get the desired result. This theorem is a generalization of Theorem 3.2 and applicable to all differentiable functions whose first derivative and its powers are GG-E-convex functions.

4. Conclusion

This paper is generalization of [2]. If we take $E(x) = x$, then it shows the result of [13].

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