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Higher-Order $(F, \alpha, \beta, \rho, d, E)$-Convexity in Fractional Programming

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Abstract In this paper we define higher order $(F, \alpha, \beta, \rho, d, E)$-convex function with respect to E-differentiable function K and obtain optimality conditions for nonlinear programming problem (NP) from the concept of higher order $(F, \alpha, \beta, \rho, d)$-convexity. Here, we establish Mond-Weir and Wolfe duality for (NP) and utilize these duality in nonlinear fractional programming problem.

Keywords E-convexity; higher order $(F, \alpha, \beta, \rho, d, E)$-convexity; optimality conditions; duality; fractional programming

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1. Introduction

Applications of generalized convexity cover a broad area in mathematical programming, in which optimality criteria and duality relations make a dominant place. Hanson [1] considered sufficient conditions (Khun Tucker-conditions) for the existence of solution of programming problems with convexity. Then Hanson and Mond [2] obtained these conditions and duality results for generalized convexity. Vial [3] studied weakly and strongly convex sets and defined ρ-convex function. Preda [4] defined (F, ρ)-convex functions and obtained duality results under the assumption of (F, ρ)-convexity. Liang et al. [5] generalized convexity to (F, α, ρ, d)-convexity and founded optimality conditions and duality related results in nonlinear fractional programming. Yuan et al. [6] expanded the concept of (F, α, ρ, d)-convexity to (C, α, ρ, d)-convexity. Gulati and Saini [7] introduced higher order $(F, \alpha, \beta, \rho, d)$-convexity and applied its concept in fractional programming for obtaining duality results.

A well known class of generalized convexity, namely E-convexity performs a significant role in mathematical programming. Youness [8] gave the concept of E-convexity and designed some results of E-convex functions in programming problem. Then Youness [9,10] obtained necessary and sufficient optimality conditions for E-convex programming and discussed E-Fritz John and E-KT (E-Khun Tucker) conditions. Chen [11] considered semi E-convex functions and its related some properties. Syau and Lee [12] produced some properties of E-convex functions with the concept of E-quasiconvex functions. Megahed et al. [13] designed a combined interactive

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approach for multi-objective E-convex programming. Later, Megahed et al. [14] also defined E-differentiable function and used this definition with KT-conditions in producing optimal solutions of programming problems with E-differentiable function. Then Iqbal et al. [15] defined geodesic E-convex sets, geodesic E-convex functions and E-epigraphs.

2. Definitions and preliminaries

Some definitions and illustrative example, are given:

Definition 2.1 *E-convex set. A set $M \subseteq R^n$, n is said to be an E-convex set with respect to an operator $E : R^n \to R^n$ if*

$$tE(x) + (1-t)E(y) \in M,$$

for each $x, y \in M$ and $0 \le t \le 1$ (see [8]).

Definition 2.2 *E-convex function. A real valued function $\varphi : M \subseteq R^n \to R$, R is said to be an E-convex function with respect to an operator $E : R^n \to R^n$ on M. If M is an E-convex set and for each $x, y \in M$ and $0 \le t \le 1$*

$$\varphi(tE(x) + (1-t)E(y)) \le t\varphi(E(x)) + (1-t)\varphi(E(y)).$$

If $\varphi(tE(x) + (1-t)E(y)) \ge t\varphi(E(x)) + (1-t)\varphi(E(y))$ then φ is called E-concave function on M.

Definition 2.3 *A point $\bar{x}$ is an optimal solution of the problem (P) if and only if $\varphi(E(\bar{x})) \le \varphi(E(x)) \forall x \in M$, M is an E-convex set.*

Definition 2.4 *Let $E : R^n \to R^n$ be an operator and φ is an E-convex and E-differentiable function on an E-convex set M, then φ is said to be higher order $(F, \alpha, \beta, \rho, d, E)$-convex function at $\bar{x}$ on M if for all $x \in M$, then there exists a vector $p \in R^n$, real valued functions $\alpha, \beta : M \times M \to R^+ - \{0\}$ and a real valued function $d : M \times M \to R$ and a real number ρ such that*

$$\begin{aligned}\varphi(E(x)) - \varphi(E(\bar{x})) \ge & F(E(x), E(\bar{x}); \alpha(x,\bar{x})[\nabla\varphi(E(\bar{x})) + \nabla_p K(E(\bar{x}), p)]) + \\ & \beta(x,\bar{x})(K(E(\bar{x}), p) - \frac{1}{2}p^{\mathrm{T}}\nabla_p K(E(\bar{x}), p)) + \rho d^2(x, \bar{x}).\end{aligned}$$

We consider the following nonlinear programming problem:

$$\begin{aligned}&(NP_E) \quad \text{Minimize} \quad \varphi(E(x)),\\ &\text{Subject to } M = \{x \in R^n : h(E(x)) \le 0, \quad x \in M\},\end{aligned}$$

where the function φ and a set M are E-convex with respect to the map $E : R^n \to R^n$ and the functions $\varphi : M \to R$ and $h = (h_1, h_2, h_3, \ldots, h_m) : M \to R^m$ are E-differentiable on X. Let $S = x \in M : h(E(x)) \le 0$ denote the set of all feasible solutions for (NP_E).

Definition 2.5 *Let $E : R^n \to R^n$ be an operator and f be E-convex function on an E-convex* *set M. The functional $F : M \times M \times R^n \to R$ is said to be sublinear in the third variable, if for all $x, \bar{x} \in M$.*

(i) $F(E(x), E(\bar{x}); \eta_1 + \eta_2) \leq F(E(x), E(\bar{x}); \eta_1) + F(E(x), E(\bar{x}); \eta_2)$ for all $\eta_1, \eta_2 \in R^n$.

(ii) $F(E(x), E(\bar{x}); \alpha a) = \alpha F(E(x), E(\bar{x}); a)$ for all $\alpha \in R_+$ and $a \in R^n$.

From (ii) it is clear that $F(E(x), E(\bar{x}); 0) = 0$.

Based on the concept of the sublinear functional, we now introduce the class of higher-order $(F, \alpha, \beta, \rho, d, E)$-convex functions as follows:

Let M be an convex open set and $M \subset R^n$. Let $\varphi : M \to R$, $K : X \times R^n \to R$ be E-differentiable functions, $F : M \times M \times R^n \to R$ be a sublinear functional in the third variable and $d : M \times M \to R$. Further, let $\alpha, \beta : M \times M \to R_+ \backslash 0$ and $\rho \in R$.

Definition 2.6 *Let $E : R^n \to R^n$ be an operator and f be E-convex function on an E-convex set M. The function φ is said to be higher-order $(F, \alpha, \beta, \rho, d, E)$-convex at $\bar{x}$ with respect to K, if for all $x \in M$ and $\rho \in R^n$,*

$$\begin{aligned}\varphi(E(x)) - \varphi(E(\bar{x})) \geq & F(E(x), E(\bar{x}; \alpha(x, \bar{x})\{\nabla\varphi(E(\bar{x})) + \nabla_p K(\bar{x}, p\}) + \\ & \beta(x, \bar{x})\{K(E(\bar{x}), p) - p^{\mathrm{T}}\nabla_p K(E(\bar{x}), p)\} + \rho d^2(x, \bar{x}).\end{aligned}$$

Remark 2.7 Let $E(x) = x$. Then this gives results of [7].

Remark 2.8 Let $K(\bar{x}, p) = 0$.

(i) Then the above definition becomes that of $(F, \alpha, \beta, \rho, d, E)$-convex function.

(ii) If $\alpha(x, \bar{x}) = 1$, we obtain the definition of (F, ρ, E)-convex function.

(iii) If $\alpha(x, \bar{x}) = 1$, $\rho = 0$ and $F(E(x), E(\bar{x}); \nabla\varphi(E(\bar{x}))) = \zeta^{\mathrm{T}}(E(x), E(\bar{x})\nabla\varphi(E(\bar{x})))$ for a certain map $\zeta : M \times M \to R^n$, then $(F, \alpha, \beta, \rho, d, E)$-convexity reduces to the E-convexity.

(iv) If F is E-convex with respect to the third argument, then we obtain the definition of $(F, \alpha, \beta, \rho, d, E)$-convex function.

Remark 2.9 Let $\beta(x, \bar{x}) = 1$.

(i) If $K(E(\bar{x}), p) = \frac{1}{2}p^{\mathrm{T}}\nabla^2\varphi(E(\bar{x}))$, then the above inequality reduces to the definition of second order (F, α, ρ, d, E)-convex function. And if $E(\bar{x}) = \bar{x}$ then it shows result of [16].

(ii) $\alpha(x, \bar{x}) = 1$, $\rho = 0$, $K(E(\bar{x}), p) = \frac{1}{2}p^{\mathrm{T}}\nabla^2\varphi(E(\bar{x}))$ and $F(E(x), E(\bar{x}); a) = \zeta^{\mathrm{T}}(E(x), E(\bar{x})a)$, where $\zeta : M \times M \to R^n$, the above definition becomes that of ζ-E-convexity. And if $E(\bar{x}) = \bar{x}$, then it shows result of [17].

Proposition 2.10 (Kuhn-Tucker Necessary Optimality Conditions [18]) *Let $\bar{x} \in M$ be an optimal solution of (NP_E) and let h satisfy a constraint qualification. Then there exists $\bar{v} \in R^m$ such that*

$$\nabla\varphi(E(\bar{x})) + \nabla h(E(\bar{x}))\bar{v} = 0, \tag{2.1}$$

$$\bar{v}^{\mathrm{T}}h(E(\bar{x}) = 0, \tag{2.2}$$

$$\bar{v} \geq 0, h(E(\bar{x})) \leq 0, \tag{2.3}$$

where $\nabla h(E(\bar{x}))$ *denotes the* $n \times m$ *matrix* $[\nabla h_1(E(\bar{x})), \nabla h_2(E(\bar{x})), \nabla h_3(E(\bar{x})), \ldots, \nabla h_m(E(\bar{x}))]$.

The following example illustrates our results.

Example 2.11 We consider the function $\varphi : M \subseteq R_+ \to R$ such that $\varphi(x) = x^n - 2x$. If

$$F(x, \bar{x}, \alpha) = \alpha(x - \bar{x}^2) - 3x,\ d(x, \bar{x}) = x - \bar{x},$$

$$\alpha(x, \bar{x}) = \frac{x + \bar{x}^2 + 1}{3},\ \beta(x, \bar{x}) = \frac{x + \bar{x}^2 + 1}{3}$$

and the operator $E(x) = x^2$, then for $\rho = 0$, φ is higher-order (F, α, ρ, d, E)-convex function at $\bar{x} = 0$ with respect to $p, -\infty < p \leq 1$.

3. Sufficient optimality conditions

In this section, we establish Kuhn-Tucker sufficient optimality conditions for (NP_E) under $(F, \alpha, \beta, \rho, d, E)$-convexity assumptions.

Theorem 3.1 *Let* $\bar{x} \in M$ *and* $\bar{v} \in R^m$ *satisfy* (2.1)–(2.3). *If*

(i) φ *is higher-order* $(F, \alpha, \beta, \rho_1, d, E)$*-convex at* $\bar{x}$ *with respect to* K,

(ii) $\bar{v}^{\mathrm{T}} h$ *is higher-order* $(F, \alpha, \beta, \rho_2, d, E)$*-convex at* $\bar{x}$ *with respect to* $-K$, *and*

(iii) $\rho_1 + \rho_2 \geq 0$,

then $\bar{x}$ *is an optimal solution of the problem* (NP_E).

Proof Let $\bar{x} \in M$ since φ is a higher-order $(F, \alpha, \beta, \rho_1, d, E)$-convex at $\bar{x}$ with respect to K, for all $x \in M$, we have

$$\begin{aligned}\varphi(E(x)) - \varphi(E(\bar{x})) \geq & F(E(x), E(\bar{x}); \alpha(x, \bar{x})[\nabla \varphi(E(\bar{x})) + \nabla_P K(E(\bar{x}), p)])+ \\ & \beta(x, \bar{x})(K(E(\bar{x}), p) - \frac{1}{2} p^{\mathrm{T}} \nabla_p K(E(\bar{x}), p)) + \rho_1 d^2(x, \bar{x}).\end{aligned} \tag{3.1}$$

Using (2.1), we get

$$\begin{aligned}\varphi(E(x)) - \varphi(E(\bar{x})) \geq & F(E(x), E(\bar{x}); \alpha(x, \bar{x})[\nabla h(E(\bar{x}))\bar{v} + \nabla_P K(E(\bar{x}), p)])+ \\ & \beta(x, \bar{x})(K(E(\bar{x}), p) - \frac{1}{2} p^{\mathrm{T}} \nabla_p K(E(\bar{x}), p)) + \rho_1 d^2(x, \bar{x}).\end{aligned} \tag{3.2}$$

Also $\bar{v}^{\mathrm{T}} h$ is higher-order $(F, \alpha, \beta, \rho_2, d, E)$-convex at $\bar{x}$ with respect to $-K$. Therefore,

$$\begin{aligned}\bar{v}^{\mathrm{T}} h(E(x)) - \bar{v}^{\mathrm{T}} h(E(\bar{x})) \geq & F(E(x), E(\bar{x}); \alpha(x, \bar{x})[\nabla \bar{v}^{\mathrm{T}} h(E(\bar{x})) - \nabla_P K(E(\bar{x}), p)])- \\ & \beta(x, \bar{x})(K(E(\bar{x}), p) - \frac{1}{2} p^{\mathrm{T}} \nabla_p K(E(\bar{x}), p)) + \rho_2 d^2(x, \bar{x}).\end{aligned} \tag{3.3}$$

Since $\bar{v}^{\mathrm{T}} h(E(\bar{x})) = 0$, $\bar{v} \geq 0$ and $h(E(x)) < 0$, we get

$$\begin{aligned}0 \geq & F(E(x), E(\bar{x}); \alpha(x, \bar{x})[\nabla \bar{v}^{\mathrm{T}} h(E(\bar{x})) - \nabla_P K(E(\bar{x}), p)])- \\ & \beta(x, \bar{x})(K(E(\bar{x}), p) - \frac{1}{2} p^{\mathrm{T}} \nabla_p K(E(\bar{x}), p)) + \rho_2 d^2(x, \bar{x}).\end{aligned} \tag{3.4}$$

Adding the inequalities (3.2) and (3.4), we obtain

$$\varphi(E(x)) - \varphi(E(\bar{x})) \geq (\rho_1 + \rho_2)d^2(x, \bar{x}),$$

which by Hypothesis (iii) implies, $\varphi(E(x)) \geq \varphi(E(\bar{x}))$. Hence $\bar{x}$ is an optimal solution of the problem (NP_E). □

4. Mond Weir duality

In this section, we establish weak and strong duality theorems for the following Mond Weir dual (MD_E) for (NP_E):

(MD) Maximize $\varphi(E(u))$,

$$\text{Subject to } \nabla\varphi(E(u)) + \nabla h(E(u))v = 0, \tag{4.1}$$

$$v^{\mathrm{T}} h(E(u)) \geq 0, \tag{4.2}$$

$$u \in X, \quad v \geq 0, \ v \in R^m. \tag{4.3}$$

Theorem 4.1 (Weak Duality) *Let x and (u, v) be feasible solutions of (NP_E) and (MD_E), respectively. Let*

(i) φ be higher-order $(F, \alpha, \beta, \rho_1, d, E)$-convex at $\bar{x}$ with respect to K.

(ii) $v^{\mathrm{T}} h$ be higher-order $(F, \alpha, \beta, \rho_2, d, E)$-convex at u with respect to $-K$, and

(iii) $\rho_1 + \rho_2 \geq 0$.

Then $\varphi(E(x)) \geq \varphi(E(u))$ is an optimal solution of the problem (NP_E).

Proof By Hypothesis (i), we have

$$\begin{aligned}\varphi(E(x)) - \varphi(E(u)) \geq & F(E(x), E(u); \alpha(x, \bar{u})[\nabla\varphi(E(u)) + \nabla_P K(E(u), p)])+ \\ & \beta(x, u)(K(E(u), p) - p^{\mathrm{T}}\nabla_p K(E(u), p)) + \rho_1 d^2(E(x), E(u)).\end{aligned} \tag{4.4}$$

Also Hypothesis (ii) yields

$$\begin{aligned}v^{\mathrm{T}} h(E(x)) - v^{\mathrm{T}} h(E(u)) \geq & F(E(x), E(u); \alpha(x, \bar{u})[\nabla v^{\mathrm{T}} h(E(u)) - \nabla_P K(E(u), p)])- \\ & \beta(x, u)(K(E(u), p) - p^{\mathrm{T}}\nabla_p K(E(u), p)) + \rho_2 d^2(x, u).\end{aligned}$$

By (4.2), (4.3) and $h(E(x)) \leq 0$ it follows that

$$\begin{aligned}0 \geq & F(E(x), E(u); \alpha(x, \bar{u})[\nabla v^{\mathrm{T}} h(E(u)) - \nabla_P K(E(u), p)])- \\ & \beta(x, u)(K(E(u), p) - p^{\mathrm{T}}\nabla_p K(E(u), p)) + \rho_2 d^2(x, u).\end{aligned} \tag{4.5}$$

Adding the inequalities (4.4), (4.5) and applying the properties of sublinear functional, we obtain

$$\begin{aligned}\varphi(E(x)) - \varphi(E(u)) \geq & F(E(x), E(u); \alpha(x, \bar{u})[\nabla\varphi(E(u)) + \nabla v^{\mathrm{T}} h(E(u))])+ \\ & \rho_1 d^2(x, u) + \rho_2 d^2(x, u)\end{aligned}$$

which in view of (4.1) implies

$$\varphi(E(x)) - \varphi(E(u)) \geq (\rho_1 + \rho_2)d^2(x,u).$$

Using Hypothesis (iii) in the above inequality, we get

$$\varphi(E(x)) \geq \varphi(E(u)). \quad \square$$

Theorem 4.2 (Strong Duality) *Let $\bar{x}$ be an optimal solution of the problem (NP_E) and let h satisfy a constraint qualification. Further, let Theorem 4.1 hold for the feasible solution $\bar{x}$ of (NP_E) and all feasible solutions (u,v) of (MD_E). Then there exists a $\bar{v} \in {R_+}^m$ such that $(\bar{x},\bar{v})$ is an optimal solution of (MD_E).*

Proof Since $\bar{x}$ is an optimal solution for the problem (NP_E) and h satisfies a constraint qualification, by Proposition 2.10 there exists a $\bar{v} \in {R_+}^m$ such that the Kuhn-Tucker conditions, (2.1)–(2.3) hold. Hence $(\bar{x},\bar{v})$ is feasible for (MD_E).

Now let (u,v) be any feasible solution of (MD_E). Then by weak duality (Theorem 4.1), we have

$$\varphi(\bar{x}) \geq \varphi(u).$$

Therefore, $(\bar{x},\bar{v})$ is an optimal solution of (MD_E). $\square$

5. Higher-order $(F,\alpha,\beta,\rho,d,E)$-convexity in fractional programming

Let Y be an E-convex set with respect to the map $E: R \to R$ and function $\varphi: Y \to R$ be defined as

$$\varphi = \frac{f(E(x))}{g(E(x))},$$

where $f,g: Y \to R$ is defined on Y with $f(E(x)) \geq 0$ and $g(E(x)) > 0$. Then we consider the following fractional programming problem (FP_E) from the nonlinear programming problem (NP_E)

$$\begin{aligned}&\text{Min}\frac{f(E(x))}{g(E(x))},\\ &\text{sub to } h(E(x)) \leq 0, \quad x \in X.\end{aligned}$$

Then we obtain some following results under the assumptions of higher-order $(F,\alpha,\beta,\rho,d,E)$-convexity of the ratio $\frac{f(E(x))}{g(E(x))}$.

Theorem 5.1 *Let $f(x)$ and $-g(x)$ be two higher-order $(F,\alpha,\beta,\rho,d,E)$-convex functions at $\bar{x}$ with respect to the same function K. Then the ratio function $\frac{f(x)}{g(x)}$ is also a higher-order $(F,\alpha,\beta,\rho,d,E)$-convex function at $\bar{x}$ with respect $\bar{K}$, where*

$$\bar{\alpha}(x,\bar{x}) = \alpha(x,\bar{x})\frac{g(E(\bar{x}))}{g(E(x))},$$

$$\bar{\beta}(x,\bar{x}) = \beta(x,\bar{x})\frac{g(E(\bar{x}))}{g(E(x))},$$
$$\bar{K}(E(\bar{x}),p) = [\frac{1}{g(E(\bar{x}))} + \frac{f(E(\bar{x}))}{g^2(E(\bar{x}))}]K(E(\bar{x}),p),$$
$$\bar{d}(x,\bar{x}) = [\frac{1}{g(E(x))} + \frac{f(E(\bar{x}))}{g(E(\bar{x}))g(E(x))}]^{1/2}d(x,\bar{x}).$$

Proof Since $f(x)$ and $-g(x)$ are higher-order $(F, \alpha, \beta, \rho, d, E)$-convex functions at $\bar{x}$ with respect to the function K, we have

$$\begin{aligned} f(E(x) - f(E(\bar{x})) \geq & F(x,\bar{x};\alpha(x,\bar{x})\{\nabla f(E(\bar{x})) + \nabla_p K(E(\bar{x}),p)\})+ \\ & \beta(x,\bar{x})\{K(E(\bar{x}),p) - p^{\mathrm{T}}\nabla_p K(E(\bar{x}),p)\} + \rho d^2(x,\bar{x}) \end{aligned}$$

and

$$\begin{aligned} -g(E(x)) + g(E(\bar{x})) \geq & F(x,\bar{x};\alpha(x,\bar{x})\{\nabla g(E(\bar{x})) + \nabla_p K(E(\bar{x}),p)\})+ \\ & \beta(x,\bar{x})\{K(E(\bar{x}),p) - p^{\mathrm{T}}\nabla_p K(E(\bar{x}),p)\} + \rho d^2(x,\bar{x}), \end{aligned}$$

also

$$\begin{aligned} \frac{f(E(x))}{g(E(x))} - \frac{f(E(\bar{x}))}{g(E(\bar{x}))} = & \frac{1}{g(E(x))}[f(E(x)) - f(E(\bar{x}))]+ \\ & \frac{f(E(\bar{x}))}{g(E(x))g(E(\bar{x}))}[-g(E(x)) + g(E(\bar{x}))]. \end{aligned}$$

By sub-linearity of function F and above inequalities, we have

$$\begin{aligned} & \frac{f(E(x))}{g(E(x))} - \frac{f(E(\bar{x}))}{g(E(\bar{x}))} \geq \frac{1}{g(E(x))}F(x,\bar{x};\alpha(x,\bar{x})\{\nabla f(E(\bar{x})) + \nabla_p K(E(\bar{x}),p)\})+ \\ & \quad \frac{1}{g(E(x))}(\beta(x,\bar{x})\{K(E(\bar{x}),p) - p^{\mathrm{T}}\nabla_p K(E(\bar{x}),p)\} + \rho d^2(x,\bar{x}))+ \\ & \quad \frac{f(E(\bar{x}))}{g(E(\bar{x}))g(E(x))}F(x,\bar{x};\alpha(x,\bar{x})\{-\nabla g(E(\bar{x})) + \nabla_p K(E(\bar{x}),p)\})+ \\ & \quad \frac{f(E(\bar{x}))}{g(E(\bar{x}))g(E(x))}(\beta(x,\bar{x})\{K(E(\bar{x}),p) - p^{\mathrm{T}}\nabla_p K(E(\bar{x}),p)\} + \rho d^2(x,\bar{x})) \\ & = F(x,\bar{x};\frac{\alpha(x,\bar{x})}{g(E(x))}\{\nabla f(E(\bar{x})) + \nabla_p K(E(\bar{x}),p)\})+ \\ & \quad F(x,\bar{x};\alpha(x,\bar{x})\frac{f(E(\bar{x}))}{g(E(\bar{x}))g(E(x))}\{-\nabla g(E(\bar{x})) + \nabla_p K(E(\bar{x}),p)\})+ \\ & \quad \beta(x,\bar{x})[\frac{1}{g(E(x))} + \frac{f(E(\bar{x}))}{g(E(\bar{x}))g(E(x))}]\{K(E(\bar{x}),p) - p^{\mathrm{T}}\nabla_p K(E(\bar{x}),p)\}+ \\ & \quad \rho[\frac{1}{g(E(x))} + \frac{f(E(\bar{x}))}{g(E(\bar{x}))g(E(x))}]d^2(x,\bar{x}) \\ & = F(x,\bar{x};\alpha(x,\bar{x})\frac{g(E(\bar{x}))}{g(E(x))}\{\nabla\frac{f(E(\bar{x}))}{g(E(\bar{x}))} + [\frac{1}{g(E(x))} + \frac{f(E(\bar{x}))}{g^2(E(\bar{x}))}]\nabla_p K(E(\bar{x}),p)\})+ \\ & \quad \beta(x,\bar{x})\frac{g(E(\bar{x}))}{g(E(x))}[\frac{1}{g(E(\bar{x}))} + \frac{f(E(\bar{x}))}{g^2(E(\bar{x}))}]\{K(E(\bar{x}),p) - p^{\mathrm{T}}\nabla_p K(E(\bar{x}),p)\}+ \end{aligned}$$

$$\rho[\frac{1}{g(E(x))} + \frac{f(E(\bar{x}))}{g(E(\bar{x}))g(E(x))}]d^2(x, \bar{x}).$$

Therefore,

$$\frac{f(E(x))}{g(E(x))} - \frac{f(E(\bar{x}))}{g(E(\bar{x}))} \geq F(x, \bar{x}; \alpha(x, \bar{x})[\frac{f(E(\bar{x}))}{g(E(\bar{x}))} + \nabla_p K(E(\bar{x}), p)]) + \bar{\beta}(x, \bar{x})\{K(E(\bar{x}), p) - p^{\mathrm{T}} \nabla_p K(E(\bar{x}), p)\} + \rho d^2(x, \bar{x}).$$

This shows that the ratio function $\frac{f(x)}{g(x)}$ is a higher-order $(F, \alpha, \beta, \rho, d, E)$-convex function at $\bar{x}$ with respect to $\bar{K}$. It follows from the relation between (FP_E) and its Mond-Weir dual (MFD_E)

$$\text{Max } \frac{f(E(x))}{g(E(x))},$$

$$\text{subject to } \nabla(\frac{f(E(x))}{g(E(x))}) + \nabla h(E(x))v = 0,$$

$$v^{\mathrm{T}} h(E(u)) \geq 0,$$

$$u \in M, \ v \geq 0, \ v \in R^n. \ \square$$

Theorem 5.2 (Weak Duality) *Let $E(x)$ and $(E(u), E(v))$ be feasible solutions of (FP_E) and (MFD_E), respectively, and*

(i) $f(x)$ and $-g(x)$ be two higher-order $(F, \alpha, \beta, \rho_1, d, E)$-convex functions at u with respect to K.

(ii) $f(x)$ and $v^{\mathrm{T}} h$ be higher-order $(F, \alpha, \beta, \rho_2, d, E)$-convex functions at u with respect to $-\bar{K}$, where $\bar{\alpha}, \bar{\beta}, \bar{K}$ and $\bar{d}$ are as given in Theorem 5.1, and

(iii) $\rho_1 + \rho_2 \geq 0$. Then

$$\frac{f(E(x))}{g(E(x))} \geq \frac{f(E(u))}{g(E(u))}.$$

Theorem 5.3 (Strong Duality) *Let $E(\bar{x})$ be an optimal solution of (FP_E) and let h satisfy a constraint qualification. Further, if Theorem 5.2 holds for the feasible solution $E(\bar{x})$ of (FP_E) and all feasible solutions $(E(u), E(v))$ of (MFD_E), then there is a $\bar{v} \in R^n_+$ such that $(E(\bar{x}), E(\bar{v}))$ is an optimal solution of (MFD_E).*

6. Wolfe duality

The Wolfe dual (NP_E) and (FP_E) are, respectively,

$$\text{Max } \varphi(E(u)) + v^t h(E(u)),$$

$$\text{subject to } \nabla\varphi(E(u))) + \nabla h(E(u))v = 0,$$

$$u \in X, \ v \geq 0, \ v \in R^n$$

and

$$\text{Max } \frac{f(E(u))}{g(E(u))},$$

$$\text{subject to } \nabla\left(\frac{f(E(u))}{g(E(u))}\right) + \nabla h(E(u))v = 0,$$
$$u \in X, \ v \geq 0, \ v \in R^n.$$

Now we consider duality relations for the primal problem (NP_E), (WD_E) and their Wolfe, respectively, in higher-order $(F, \alpha, \beta, \rho, d, E)$-convexity sense.

Theorem 6.1 (Weak Duality) *Let $E(x)$ and $(E(u), E(v))$ be feasible solutions of (NP_E) and (WD_E), respectively, and*

(i) φ be higher-order $(F, \alpha, \beta, \rho_1, d, E)$-convex functions at u with respect to K.

(ii) $v^t h$ be higher-order $(F, \alpha, \beta, \rho_2, d, E)$-convex functions at u with respect to $-K$.

(iii) $\rho_1 + \rho_2 \geq 0$.

Then $\varphi(E(x)) \geq \varphi(E(u)) + v^{\mathrm{T}} h(E(u))$.

Theorem 6.2 (Strong Duality) *Let $E(\bar{x})$ be an optimal solution of (NP_E) and let h satisfy a constraint qualification. Further, if Theorem 6.1 holds for the feasible solution $E(\bar{x})$ of (NP_E) and all feasible solutions $(E(u), E(v))$ of (WD_E), then there is a $\bar{v} \in R^n_+$ such that $(E(\bar{x}), E(\bar{v}))$ is an optimal solution of (WD_E) and values of optimal objective functions of (NP_E) and (WD_E) are equal.*

Theorem 6.3 (Weak Duality) *Let $E(x)$ and $(E(u), E(v))$ be feasible solutions of (FP_E) and (WFD_E), respectively, and*

(i) $f(x)$ and $-g(x)$ be two higher-order $(F, \alpha, \beta, \rho_1, d, E)$-convex functions at u with respect to K,

(ii) $v^t h$ be higher-order $(F, \alpha, \beta, \rho_2, d, E)$-convex at u with respect to $-\bar{K}$, where $\bar{\alpha}, \bar{\beta}, \bar{K}$ and $\bar{d}$ are as given in Theorem 5.1, and

(iii) $\rho_1 + \rho_2 \geq 0$.

Then

$$\frac{f(E(x))}{g(E(x))} \geq \frac{f(E(u))}{g(E(u))} + v^{\mathrm{T}} h(E(u)).$$

Theorem 6.4 (Strong Duality) *Let $E(\bar{x})$ be an optimal solution of (FP_E) and let h satisfy a constraint qualification. Further, if Theorem 6.3 holds for the feasible solution $E(\bar{x})$ of (FP_E) and all feasible solutions $(E(u), E(v))$ of (WFD_E), then there is a $\bar{v} \in R^n_+$ such that $(E(\bar{x}), E(\bar{v}))$ is an optimal solution of (WFD_E) and values of optimal objective functions of (FP_E) and (WFD_E) are equal.*

7. Conclusions

This work generates a new form of E-convexity from the concept of higher-order $(F, \alpha, \beta, \rho_2, d)$-convexity. If E is an identity map, then this work makes a correspondence to [7] and generalises the result related to the optimality criteria and duality of [7] for E-convexity in fractional programming.

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