Nonlinear Mixed Bi-Skew Jordan Triple Derivations on Prime $\ast$-Algebras

Fangfang ZHAO, Dongfang ZHANG, Changjing LI*

School of Mathematics and Statistics, Shandong Normal University, Shandong 250014, P. R. China

Abstract Let $\mathcal{A}$ be a unital prime $\ast$-algebra with a nontrivial projection. In this paper, it is proved that a map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ satisfies

$$\Phi([A, B]_\circ \circ C) = [\Phi(A), B]_\circ \circ C + [A, \Phi(B)]_\circ \circ C + [A, B]_\circ \circ \Phi(C)$$

for all $A, B, C \in \mathcal{A}$ if and only if $\Phi$ is an additive $\ast$-derivation, where $A \circ B = A^*B + B^*A$ and $[A, B]_\circ = A^*B - B^*A$.

Keywords mixed bi-skew Jordan triple derivations; $\ast$-derivations; prime $\ast$-algebras

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1. Introduction

Let $\mathcal{A}$ be a $\ast$-algebra over the complex field $\mathbb{C}$. For $A, B \in \mathcal{A}$, we call the product $A \circ B = A^*B + B^*A$ the bi-skew Jordan product and $[A, B]_\circ = A^*B - B^*A$ the bi-skew Lie product. These two new products have attracted many scholars to study [1–9]. Particular attention has been paid to understanding maps which preserve the bi-skew Jordan product and the bi-skew Lie product on $C^\ast$-algebras. Wang and Ji [1] proved that every bijective map preserving bi-skew Lie product between factor von Neumann algebras is a linear $\ast$-isomorphism or a conjugate linear $\ast$-isomorphism. Li et al. [9] proved that every bijective map preserving bi-skew Jordan product between von Neumann algebras with no central abelian projections is just the sum of a linear $\ast$-isomorphism and a conjugate linear $\ast$-isomorphism. Taghavi and Gholampoor [5] studied surjective maps preserving bi-skew Jordan product between $C^\ast$-algebras.

Recall that an additive map $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is said to be an additive derivation if $\Phi(AB) = \Phi(A)B + A\Phi(B)$ for all $A, B \in \mathcal{A}$. Furthermore, $\Phi$ is said to be an additive $\ast$-derivation if it is an additive derivation and satisfies $\Phi(A^*) = \Phi(A)^*$ for all $A \in \mathcal{A}$. We say that $\Phi : \mathcal{A} \rightarrow \mathcal{A}$ is a nonlinear bi-skew Lie derivation or bi-skew Jordan derivation if

$$\Phi([A, B]_\circ) = [\Phi(A), B]_\circ + [A, \Phi(B)]_\circ$$

or

$$\Phi(A \circ B) = \Phi(A) \circ B + A \circ \Phi(B)$$

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* Corresponding author
E-mail address: wanwanf2@163.com (Fangfang ZHAO); 1776767307@qq.com (Dongfang ZHANG); lcjbxh@163.com (Changjing LI)
for all $A, B \in \mathcal{A}$. Recently, many authors have studied nonlinear bi-skew Lie derivations and bi-skew Jordan derivations. For example, Kong and Zhang [4] proved that any nonlinear bi-skew Lie derivation on a factor von Neumann algebra $\mathcal{A}$ with $\dim \mathcal{A} \geq 2$ is an additive $\ast$-derivation. Taghavi and Razeghi [8] investigated nonlinear bi-skew Lie derivations on prime $\ast$-algebras. Let $\Phi$ be a nonlinear bi-skew Lie derivation on a unital prime $\ast$-algebra with a nontrivial projection. They proved that if $\Phi(I)$ and $\Phi(ii)$ are self-adjoint, then $\Phi$ is an additive $\ast$-derivation. Darvish et al. [2] proved any nonlinear bi-skew Jordan derivation on prime $\ast$-algebras is an additive $\ast$-derivation. Khan [3] proved that any nonlinear bi-skew Lie triple derivation on a factor von Neumann algebra $\mathcal{A}$ with $\dim \mathcal{A} \geq 2$ is an additive $\ast$-derivation.

Recently, many authors have studied derivations corresponding to some mixed products. Zhou, Yang and Zhang [10] proved any map $\Phi$ from a unital $\ast$-algebra $\mathcal{A}$ containing a nontrivial projection to itself satisfying

$$\Phi([[[A, B]_\ast, C]_\ast]) = [[[\Phi(A), B]_\ast, C]_\ast] + [[A, \Phi(B)]_\ast, C]_\ast + [[A, B]_\ast, \Phi(C)]_\ast$$

for all $A, B, C \in \mathcal{A}$, is an additive $\ast$-derivation, where $[A, B] = AB - BA$ is the usual Lie product of $A$ and $B$ and $[A, B]_\ast = AB - BA^\ast$ is the skew Lie product of $A$ and $B$. Zhou and Zhang [11] proved that any map $\Phi$ on a factor von Neumann algebra $\mathcal{A}$ satisfying

$$\Phi([[[A, B], C]_\ast]) = [[[\Phi(A), B], C]_\ast] + [[A, \Phi(B)], C]_\ast + [[A, B], \Phi(C)]_\ast$$

for all $A, B, C \in \mathcal{A}$, is also an additive $\ast$-derivation. Zhao and Fang [7] gave a similar result on finite von Neumann algebras with no central summands of type $I_1$. Pang, Zhang and Ma [12] proved that if $\Phi$ is a second nonlinear mixed Jordan triple derivable mapping on a factor von Neumann algebra $\mathcal{A}$, that is,

$$\Phi(A \circ B \cdot C) = \Phi(A) \circ B \cdot C + A \circ \Phi(B) \cdot C + A \circ B \cdot \Phi(C)$$

for all $A, B, C \in \mathcal{A}$, then $\Phi$ is an additive $\ast$-derivation, where $A \circ B = AB + BA$ is the usual Jordan product of $A$ and $B$ and $A \cdot B = AB + BA^\ast$ is the Jordan $\ast$-product of $A$ and $B$.

Motivated by the above mentioned works, in this paper, we will consider derivations corresponding to the new product of the mixture of the bi-skew Lie product and the bi-skew Jordan product. A map $\Phi : \mathcal{A} \to \mathcal{A}$ is said to be a nonlinear mixed bi-skew Jordan triple derivation if

$$\Phi([[[A, B]_\circ, C]_\circ]) = [[[\Phi(A), B]_\circ, C]_\circ] + [[A, \Phi(B)]_\circ, C]_\circ + [[A, B]_\circ, \Phi(C)]_\circ$$

for all $A, B, C \in \mathcal{A}$. Recall that an algebra $\mathcal{A}$ is prime if $AAB = \{0\}$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$. Let $\mathcal{A}$ be a unital prime $\ast$-algebra with a nontrivial projection. In this paper, we prove that $\Phi$ is a nonlinear mixed bi-skew Jordan triple derivation on $\mathcal{A}$ if and only if $\Phi$ is an additive $\ast$-derivation.

2. The main result and its proof

The main result in this paper reads as follows.

**Theorem 2.1** Let $\mathcal{A}$ be a unital prime $\ast$-algebra with a nontrivial projection $P$. Then a map
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\(\Phi : A \to A\) satisfies

\[\Phi([A, B]_\circ \circ C) = [\Phi(A), B]_\circ \circ C + [A, \Phi(B)]_\circ \circ C + [A, B]_\circ \circ \Phi(C)\]

for all \(A, B, C \in A\) if and only if \(\Phi\) is an additive \(\ast\)-derivation.

Let \(P_1 = P\) and \(P_2 = I - P\). Denote

\[A_{ij} = P_i A P_j, \quad i, j = 1, 2.\]

Let

\[M = \{A \in A : A^\ast = A\},\]

\[N = \{A \in A : A^\ast = -A\},\]

\[M_{12} = \{P_1 M P_2 + P_2 M P_1 : M \in M\}\]

and

\[M_{ii} = P_i M P_i, \quad i = 1, 2.\]

Thus, for any \(M \in M\), \(M = M_{11} + M_{12} + M_{22}\), where \(M_{11} \in M_{11}, M_{12} \in M_{12}, M_{22} \in M_{22}\).

Clearly, we only need to prove the necessity. We will complete the proof by several lemmas.

**Lemma 2.2** Let \(\Phi\) be a nonlinear mixed bi-skew Jordan triple derivation on \(A\). Then \(\Phi(0) = 0\).

**Proof** Indeed, we have

\[\Phi(0) = \Phi([0, 0]_\circ \circ 0) = [\Phi(0), 0]_\circ \circ 0 + [0, \Phi(0)]_\circ \circ 0 + [0, 0]_\circ \circ \Phi(0) = 0.\]

**Lemma 2.3** For any \(M \in M\), we have \(\Phi(M) \in M\).

**Proof** For any \(M \in M\), \(M = [M, \frac{i}{2} I]_\circ \circ \left(\frac{i}{2} I\right)\). Since \([A, B]_\circ \circ C \in M\) for all \(A, B, C \in A\), we obtain

\[\Phi(M) = \Phi([M, \frac{i}{2} I]_\circ \circ \left(\frac{i}{2} I\right)) = [\Phi(M), \frac{i}{2} I]_\circ \circ \left(\frac{i}{2} I\right) + [M, \Phi(\frac{i}{2} I)]_\circ \circ \left(\frac{i}{2} I\right) + [M, \frac{i}{2} I]_\circ \circ \Phi(\frac{i}{2} I) \in M.\]

**Lemma 2.4** For any \(A_{11} \in M_{11}, M_{12} \in M_{12}\) and \(A_{22} \in M_{22}\), we have

\[\Phi(A_{11} + M_{12}) = \Phi(A_{11}) + \Phi(M_{12})\]

and

\[\Phi(M_{12} + A_{22}) = \Phi(M_{12}) + \Phi(A_{22}).\]

**Proof** Let \(T = \Phi(A_{11} + M_{12}) - \Phi(A_{11}) - \Phi(M_{12})\). By Lemma 2.3, we have \(T^\ast = T\). We only need to prove

\[T = T_{11} + T_{12} + T_{22} = 0.\]

Since \([iP_2, A_{11}]_\circ = 0\), we obtain

\[[\Phi(iP_2), A_{11} + M_{12}]_\circ \circ (iI) + [iP_2, \Phi(A_{11} + M_{12})]_\circ \circ (iI) + [iP_2, A_{11} + M_{12}]_\circ \circ \Phi(iI) = \Phi([iP_2, A_{11} + M_{12}]_\circ \circ (iI))\]
Proof

Let \( \Phi(M_{12}) \circ (iI) \) be any \( T \) which implies that \( \Phi(M_{12}) \circ (iI) \) is the image of \( iP_2, A_{11} + M_{12} \circ (iI) + [iP_2, \Phi(A_{11}) + \Phi(M_{12})] \circ (iI) \) +

\[ iP_2, A_{11} + M_{12} \circ \Phi(iI). \]

From this, we get \( [iP_2, T] \circ (iI) = 0 \), and hence \( T_{12} = T_{22} = 0 \).

It follows from \( [i(P_1 - P_2), M_{12}] = 0 \) that

\[ [\Phi(i(P_1 - P_2), A_{11} + M_{12}) \circ (iI) + [i(P_1 - P_2), \Phi(A_{11} + M_{12})] \circ (iI) + \]

\[ i(P_1 - P_2), A_{11} + M_{12} \circ \Phi(iI) = \Phi([i(P_1 - P_2), A_{11} + M_{12}] \circ (iI)) \]

\[ = [\Phi(i(P_1 - P_2), A_{11} + M_{12}) \circ (iI) + [i(P_1 - P_2), M_{12}] \circ (iI)] \]

\[ = [\Phi(i(P_1 - P_2), A_{11} + M_{12}) \circ (iI) + [i(P_1 - P_2), \Phi(A_{11}) + \Phi(M_{12})] \circ (iI) + \]

\[ i(P_1 - P_2), A_{11} + M_{12} \circ \Phi(iI), \]

which implies that \( [i(P_1 - P_2), T] \circ (iI) = 0 \). So \( T_{11} = 0 \), and hence \( T = 0 \).

Similarly, we can show that \( \Phi(M_{12} + A_{22}) = \Phi(M_{12}) + \Phi(A_{22}) \). \( \square \)

**Lemma 2.5** For any \( A_{11} \in M_{11}, M_{12} \in M_{12} \) and \( C_{22} \in M_{22} \), we have

\[ \Phi(A_{11} + M_{12} + C_{22}) = \Phi(A_{11}) + \Phi(M_{12}) + \Phi(C_{22}). \]

**Proof** Let \( T = \Phi(A_{11} + M_{12} + C_{22}) - \Phi(A_{11}) - \Phi(M_{12}) - \Phi(C_{22}) \). By Lemma 2.3, we have \( T^* = T \). Since \( [iP_1, C_{22}] = 0 \), it follows from Lemma 2.4 that

\[ [\Phi(iP_1), A_{11} + M_{12} + C_{22}] \circ (iI) + [iP_1, \Phi(A_{11} + M_{12} + C_{22})] \circ (iI) + \]

\[ [iP_1, A_{11} + M_{12} + C_{22}] \circ \Phi(iI) = \Phi([iP_1, A_{11} + M_{12} + C_{22}] \circ (iI)) \]

\[ = [\Phi(iP_1, A_{11} + M_{12} + C_{22}) \circ (iI) + \Phi([iP_1, C_{22}] \circ (iI)] \]

\[ = [\Phi(iP_1, A_{11} + M_{12} + C_{22}) \circ (iI) + [iP_1, \Phi(A_{11}) + \Phi(M_{12}) + \Phi(C_{22})] \circ (iI) + \]

\[ [iP_1, A_{11} + M_{12} + C_{22}] \circ \Phi(iI), \]

which yields that \( [iP_1, T] \circ (iI) = 0 \). So \( T_{11} = T_{22} = 0 \). In the similar manner, we can get that \( T_{22} = 0 \). Hence \( T = 0 \). \( \square \)

**Lemma 2.6** For any \( M_{12}, B_{12} \in M_{12} \), we have

\[ \Phi(M_{12} + B_{12}) = \Phi(M_{12}) + \Phi(B_{12}). \]

**Proof** Let \( M_{12}, B_{12} \in M_{12} \). Then \( M_{12} = iU_{12} - iU_{12}^* \), \( B_{12} = iV_{12} - iV_{12}^* \), where \( U_{12}, V_{12} \in A_{12} \). Since

\[ [P_1 + U_{12} + U_{12}^* + P_2 + V_{12} + V_{12}^* \circ (-\frac{i}{2} I) = M_{12} + B_{12} + iM_{12}B_{12} - iB_{12}M_{12}, \]

where

\[ M_{12} + B_{12} \in M_{12} \]
and
\[ iM_{12}T_{12} - iB_{12}M_{12} = P_1(iU_{12}V_{12} - iV_{12}U_{12})P_1 + P_2(iU_{12}V_{12} - iV_{12}U_{12})P_2 \in M_{11} + M_{22}, \]
by Lemma 2.5, we have
\[
\Phi(M_{12} + B_{12}) + \Phi(iM_{12}B_{12} - iB_{12}M_{12}) = \Phi(M_{12} + B_{12} + iM_{12}B_{12} - iB_{12}M_{12})
\]
\[
= \Phi([P_1 + U_{12} + U_{12}^*, P_2 + V_{12} + V_{12}^*] \circ (-\frac{i}{2}I))
\]
\[
= \Phi(P_1) + \Phi(U_{12} + U_{12}^*) + P_2 + V_{12} + V_{12}^* \circ (-\frac{i}{2}I) +
\]
\[
\quad [P_1 + U_{12} + U_{12}^*, \Phi(P_2) + \Phi(V_{12} + V_{12}^*)] \circ (-\frac{i}{2}I) +
\]
\[
\quad [P_1 + U_{12} + U_{12}^*, P_2 + V_{12} + V_{12}^*] \circ \Phi(-\frac{i}{2}I)
\]
\[
= \Phi([P_1, P_2] \circ (-\frac{i}{2}I)) + \Phi([P_1, V_{12} + V_{12}^*] \circ (-\frac{i}{2}I)) +
\]
\[
\quad \Phi([U_{12} + U_{12}^*, P_2] \circ (-\frac{i}{2}I)) + \Phi([U_{12} + U_{12}^*, V_{12} + V_{12}^*] \circ (-\frac{i}{2}I))
\]
\[
= \Phi(B_{12}) + \Phi(M_{12}) + \Phi(iM_{12}B_{12} - iB_{12}M_{12}),
\]
which implies that \( \Phi(M_{12} + B_{12}) = \Phi(M_{12}) + \Phi(B_{12}) \). □

**Lemma 2.7** For any \( A_{ii}, B_{ii} \in M_{ii}, i = 1, 2 \), we have
\[
\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).
\]

**Proof** Let \( T = \Phi(A_{11} + B_{11}) - \Phi(A_{11}) - \Phi(B_{11}) \). By Lemma 2.3, we have \( T^* = T \). Since \( [iP_2, A_{11}]_\circ = [iP_2, B_{11}]_\circ = 0 \), we obtain
\[
[\Phi(iP_2), A_{11} + B_{11}]_\circ \circ (iI) + [iP_2, \Phi(A_{11} + B_{11})]_\circ \circ (iI) +
\]
\[
[iP_2, A_{11} + B_{11}]_\circ \circ \Phi(iI)
\]
\[
= \Phi([iP_2, A_{11} + B_{11}]_\circ \circ (iI))
\]
\[
= \Phi([iP_2, A_{11}]_\circ \circ (iI)) + \Phi([iP_2, B_{11}]_\circ \circ (iI))
\]
\[
= \Phi(iP_2, A_{11} + B_{11}]_\circ \circ (iI) + [iP_2, \Phi(A_{11}) + \Phi(B_{11})]_\circ \circ (iI) +
\]
\[
[iP_2, A_{11} + B_{11}]_\circ \circ \Phi(iI).
\]
So \( [iP_2, T]_\circ \circ (iI) = 0 \), and hence \( T_{12} = T_{22} = 0 \). Now we have \( T = T_{11} \).

For any \( d_{12} \in A_{12} \), let \( M = D_{12} + D_{12}^* \). Then
\[
[A_{11}, iM]_\circ \circ (iI), \quad [B_{11}, iM]_\circ \circ (iI) \in M_{12}.
\]
It follows from Lemma 2.6 that
\[
[\Phi(A_{11} + B_{11}), iM]_\circ \circ (iI) + [A_{11} + B_{11}, \Phi(iM)]_\circ \circ (iI) +
\]
\[
[A_{11} + B_{11}, iM]_\circ \circ \Phi(iI)
\]
\[
= \Phi([A_{11} + B_{11}, iM]_\circ \circ (iI))
\]
Let $\Phi$ be a nonlinear mixed bi-skew Jordan triple derivation on $\mathcal{A}$. Then $\Phi(iI) = 0$.

**Proof**  
For any $M \in \mathcal{M}$, it follows from Lemma 2.3 and Remark 2.8 that

$$
4\Phi(M) = \Phi(4M) = \Phi([M,iI]_o \circ (iI)) \\
= \Phi([M,iI]_o \circ (iI)) + [M,\Phi(iI)]_o \circ (iI) + [M, iI]_o \circ \Phi(iI) \\
= 4\Phi(M) + 2i(\Phi(iI)^*M - M\Phi(iI)) + 2i(\Phi(iI)^*M - M\Phi(iI)) \\
= 4\Phi(M) + 4i(\Phi(iI)^*M - M\Phi(iI)).
$$

So $\Phi(iI)^*M - M\Phi(iI) = 0$ for all $M \in \mathcal{M}$. Let $M = I$. Then $\Phi(iI) = \Phi(iI)^* \in \mathcal{M}$. Now we have $\Phi(iI)M = M\Phi(iI)$ for all $M \in \mathcal{M}$. Since for any $B \in \mathcal{A}$, $B = M_1 + iM_2$ with $M_1 = \frac{B+B^*}{2} \in \mathcal{M}$ and $M_2 = \frac{B-B^*}{2i} \in \mathcal{M}$, it follows that $\Phi(iI)B = B\Phi(iI)$ for all $B \in \mathcal{A}$. Hence

$$
\Phi(iI) \in \mathcal{Z}(\mathcal{A}) \cap \mathcal{M}. \tag{2.1}
$$

For any $M \in \mathcal{M}$, from Lemma 2.3, we see that

$$
0 = \Phi([M,iI]_o \circ I) = [M,iI]_o \circ \Phi(I) = 2i(\Phi(I)^*M - M\Phi(I)).
$$

In the same manner, we obtain

$$\Phi(I) \in \mathcal{Z}(\mathcal{A}) \cap \mathcal{M}. \tag{2.2}$$

Let $\Phi(iP_1) = W_1 + iW_2$, where $W_1, W_2 \in \mathcal{N}$. It follows from Eq. (2.1) that

$$
0 = \Phi([iI,iP_1]_o \circ (\frac{i}{2}I)) \\
= \Phi([iI,iP_1]_o \circ (\frac{i}{2}I) + [iI,\Phi(iP_1)]_o \circ (\frac{i}{2}I) \\
= 2\Phi(iI)P_1 - 2iW_2,
$$

which implies that $iW_2 = \Phi(iI)P_1$, and so

$$\Phi(iP_1) = W_1 + \Phi(iI)P_1. \tag{2.3}$$

In view of Eqs. (2.2) and (2.3), we find that

$$
4\Phi(P_1) = \Phi([I,iP_1]_o \circ (iI)) = \Phi([I, iP_1]_o \circ (iI) + [I, \Phi(iP_1)]_o \circ (iI) \\
= 4\Phi(I)P_1 - 4iW_1, \tag{2.4}
$$
which yields that
\[ \Phi(P_1) = \Phi(I)P_1 - iW_1. \] (2.5)

On the other hand, by Eqs. (2.3) and (2.5), we obtain
\[
4\Phi(P_1) = \Phi([P_1, iP_1]_\circ \circ (iI)) \\
= [\Phi(P_1), iP_1]_\circ \circ (iI) + [P_1, \Phi(iP_1)]_\circ \circ (iI) \\
= 4\Phi(I)P_1 - 4i(P_1W_1 + W_1P_1).
\] (2.6)

Comparing Eqs. (2.4) and (2.6), we have \( P_1W_1 + W_1P_1 = W_1, \) and so
\[ P_1W_1P_1 = P_2W_1P_2 = 0. \] (2.7)

From Eqs. (2.3) and (2.7), we get that
\[ \Phi(iP_1) = W_1 + \Phi(iI)P_1 = \Phi(iI)P_1 + P_1W_1P_2 + P_2W_1P_1. \] (2.8)

For any \( A_{12} \subseteq A_{12}, \) putting \( M = A_{12} + A_{12}^* \), then \( M \subseteq M. \) It follows from Lemma 2.3 and Remark 2.8 that
\[
-2\Phi(M) = \Phi([iP_1, M]_\circ \circ (iI)) \\
= [\Phi(iP_1), M]_\circ \circ (iI) + [P_1, \Phi(M)]_\circ \circ (iI) \\
= -2i\Phi(iP_1)M - iM\Phi(iP_1) + \Phi(M)P_1 + P_1\Phi(M). \] (2.9)

Multiplying Eq. (2.9) by \( P_1 \) from the left and by \( P_2 \) from the right, then by Eq. (2.8), we have \( \Phi(iI)A_{12} = 0. \) It follows from the primeness of \( A \) that \( \Phi(iI)P_1 = 0. \) On the other hand, by Eq. (2.1), we also get \( \Phi(iI)A_{12}^* = 0. \) By the primeness of \( A, \) \( \Phi(iI)P_2 = 0. \) Now we obtain \( \Phi(iI) = \Phi(iI)P_1 + \Phi(iI)P_2 = 0. \) □

**Lemma 2.10**

(1) For any \( N \subseteq N, \) we have \( \Phi(N)^* = -\Phi(N) \) and \( \Phi(iN) = i\Phi(N) + i\Phi(I)N; \)

(2) \( \Phi \) is additive on \( N; \)

(3) For any \( H, K \subseteq N, \) we have \( \Phi(H + iK) = \Phi(H) + i\Phi(K) + i\Phi(I)K. \)

**Proof**

(1) For any \( N \subseteq N, \) it follows from Lemma 2.9 that
\[
0 = \Phi([iI, N]_\circ \circ (iI)) = [iI, \Phi(N)]_\circ \circ (iI) = -2(\Phi(N)^* + \Phi(N)).
\]
So \( \Phi(N)^* = -\Phi(N) \) for all \( N \subseteq N. \)

For any \( N \subseteq N, \) by Remark 2.8, Lemma 2.9 and Eq. (2.2), we get
\[
4\Phi(iN) = \Phi([N, I]_\circ \circ (iI)) = [\Phi(N), I]_\circ \circ (iI) + [N, \Phi(I)]_\circ \circ (iI) = 4i(\Phi(N) + \Phi(I)N).
\]
That is,
\[ \Phi(iN) = i\Phi(N) + i\Phi(I)N \] (2.10)
for all \( N \subseteq N. \)

(2) For any \( H, K \subseteq N, \) we can get from Remark 2.8 and Eq. (2.10) that
\[ i\Phi(H + K) + i\Phi(I)(H + K) = \Phi(i(H + K)) \]
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\[ \Phi(iH) + \Phi(iK) = \Phi(H + K). \]

Hence \( \Phi(H + K) = \Phi(H) + \Phi(K) \) for all \( H, K \in \mathcal{N} \).

(3) For any \( H, K \in \mathcal{N} \), by Remark 2.8, Lemma 2.9 and Eq. \((2.10)\), we have

\[ 4(\Phi(K) + \Phi(I)K) = \Phi(4iK) = \Phi([H + iK, iI]_\circ (iI)) \]

and

\[ 4(\Phi(iH) + \Phi(iI)H) = \Phi(4iH) = \Phi([H + iK, iI]_\circ (iI)) \]

for all \( H, K \in \mathcal{N} \).

In view of Eqs. \((2.11)\) and \((2.12)\), we obtain

\[ \Phi(H + iK) = \Phi(H) + i\Phi(K) + i\Phi(I)K. \]

**Lemma 2.11**

\begin{enumerate}
\item For any \( A \in \mathcal{A} \), we have \( \Phi(\Phi(A)) = \Phi(A) \);
\item \( \Phi \) is additive on \( \mathcal{A} \).
\end{enumerate}

**Proof**

(1) For any \( A \in \mathcal{A} \), \( A = A_1 + iA_2 \), where \( A_1, A_2 \in \mathcal{N} \). Then we can get from Eq. \((2.2)\) and Lemma 2.10 that

\[ \Phi(A) = \Phi(A_1 + iA_2) = \Phi(A_1) + i\Phi(A_2) + i\Phi(I)A_2 \]

\[ = -\Phi(A_1) + i\Phi(A_2) + i\Phi(I)A_2 = \Phi(-A_1 + iA_2) = \Phi(A^*) \]

for all \( A \in \mathcal{A} \).

(2) For any \( A, B \in \mathcal{A} \), \( A = A_1 + iA_2, B = B_1 + iB_2 \), where \( A_i, B_i \in \mathcal{N} \). It follows from Lemma 2.10 that

\[ \Phi(A + B) = \Phi((A_1 + B_1) + i(A_2 + B_2)) \]

\[ = \Phi(A_1 + B_1) + i\Phi(A_2 + B_2) + i\Phi(I)(A_2 + B_2) \]

\[ = (\Phi(A_1) + i\Phi(A_2)) + i\Phi(I)A_2 + (\Phi(B_1) + i\Phi(B_2)) + i\Phi(I)B_2 \]

\[ = \Phi(A) + \Phi(B). \]

Hence \( \Phi \) is additive on \( \mathcal{A} \).

**Lemma 2.12**

\begin{enumerate}
\item \( \Phi(i) = 0 \);
\item For any \( A \in \mathcal{A} \), we have \( \Phi(iA) = i\Phi(A) \).
\end{enumerate}

**Proof**

(1) In view of Eqs. \((2.5)\) and \((2.7)\), we have

\[ \Phi(P_1) = \Phi(i)P_1 - iP_1W_1P_2 - iP_2W_1P_1. \]

(2.13)

For any \( A_{12} \in \mathcal{A}_{12} \), it follows from Lemmas 2.9–2.11 that

\[ 2(i(\Phi(A_{12})^* - \Phi(A_{12}))) + i\Phi(I)(A_{12}^* - A_{12}) \]
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$$= 2\Phi(i(A_{12}^* - A_{12})) = \Phi([P_1, A_{12} + A_{12}^*]_\circ (iI))$$

$$= [\Phi(P_1), A_{12} + A_{12}^*]_\circ (iI) + [P_1, \Phi(A_{12} + A_{12}^*)]_\circ (iI)$$

$$= 2i([A_{12} + A_{12}^*](\Phi(I)P_1 - iP_1W_1P_2 - iP_2W_1P_1) - (\Phi(I)P_1 - iP_1W_1P_2 - iP_2W_1P_1)(A_{12} + A_{12}^*) + (\Phi(A_{12})^* + \Phi(A_{12}))P_1 - P_1(\Phi(A_{12}) + \Phi(A_{12})^*)].$$

Multiplying by $P_1$ from the left and by $P_2$ from the right, we obtain

$$P_1\Phi(A_{12})^*P_2 = 0. \quad \text{(2.14)}$$

On the other hand, we also have

$$2(\Phi(A_{12}) + \Phi(A_{12})^*) = \Phi([P_1, i(A_{12} - A_{12}^*)]_\circ (iI))$$

$$= [\Phi(P_1), i(A_{12} - A_{12}^*)]_\circ (iI) + [P_1, \Phi(i(A_{12} - A_{12}^*))]_\circ (iI)$$

$$= 2i([A_{12} - A_{12}^*](\Phi(I)P_1 - iP_1W_1P_2 - iP_2W_1P_1) + (\Phi(I)P_1 - iP_1W_1P_2 - iP_2W_1P_1)(A_{12} - A_{12}^*) + (\Phi(A_{12})^* - \Phi(A_{12}) + (A_{12}^* - A_{12})(\Phi(I)P_1 + P_1(\Phi(A_{12}) - \Phi(A_{12})^* + \Phi(I)(A_{12} - A_{12}^*))).$$

Multiplying by $P_1$ from the left and by $P_2$ from the right, we obtain $\Phi(I)A_{12} = 0$ by the Eq. (2.14). It follows from the primeness of $A$ that $\Phi(I)P_1 = 0$. On the other hand, by Eq. (2.2), we also get $\Phi(I)A_{12}^* = 0$. So $\Phi(I)P_2 = 0$. Now we obtain $\Phi(I) = \Phi(I)P_1 + \Phi(I)P_2 = 0$.

(2) For any $N \in N$, by Lemma 2.10 (1) and $\Phi(I) = 0$, we have

$$\Phi(iN) = i\Phi(N). \quad \text{(2.15)}$$

For any $A \in A, A = A_1 + iA_2$, where $A_1, A_2 \in N$. From Lemma 2.11 (2) and Eq. (2.15), we have

$$\Phi(iA) = \Phi(i(A_1 + iA_2)) = \Phi(iA_1 - A_2) = i\Phi(A_1) + i\Phi(A_2) = i\Phi(A)$$

for all $A \in A$. □

**Lemma 2.13** $\Phi$ is a derivation on $A$.

**Proof** For any $A, B \in A$, by Lemmas 2.11 (2) and 2.12 (2), we have

$$2(\Phi(A^*B + B^*A)) = \Phi([A, iB]_\circ (iI))$$

$$= [\Phi(A), iB]_\circ (iI) + [A, i\Phi(B)]_\circ (iI)$$

$$= 2(\Phi(A)^*B + B^*\Phi(A) + A^*\Phi(B) + \Phi(B)^*A),$$

which implies that

$$\Phi(A^*B + B^*A) = \Phi(A)^*B + B^*\Phi(A) + A^*\Phi(B) + \Phi(B)^*A. \quad \text{(2.16)}$$

On the other hand, we also have

$$-2i(\Phi(A^*B - B^*A)) = \Phi([iA, iB]_\circ (iI))$$
which yields that
\[
\Phi(A^*B - B^*A) = \Phi(A)^*B - B^*\Phi(A) + A^*\Phi(B) - \Phi(B)^*A.
\]  
(2.17)

By summing Eqs. (2.16) and (2.17), we obtain
\[
\Phi(A^*B) = \Phi(A)^*B + A^*\Phi(B).
\]
Then we can get from Lemma 2.11 (1) that
\[
\Phi(AB) = \Phi(A)B + A\Phi(B).
\]

Now, from Lemmas 2.11 and 2.13, we obtain that \( \Phi \) is an additive \( \ast \)-derivation on \( \mathcal{A} \). This completes the proof of Theorem 2.1.

3. Corollaries

Let \( \mathcal{B}(\mathcal{H}) \) be the algebra of all bounded linear operators on a complex Hilbert space \( \mathcal{H} \), and \( \mathcal{A} \subseteq \mathcal{B}(\mathcal{H}) \) be a von Neumann algebra. \( \mathcal{A} \) is a factor if its center is \( \mathbb{C}I \). It is well known that a factor von Neumann algebra is prime. Now we can get the following corollary.

**Corollary 3.1** Let \( \mathcal{A} \) be a factor von Neumann algebra with \( \dim(\mathcal{A}) \geq 2 \). Then \( \Phi : \mathcal{A} \to \mathcal{A} \) is a nonlinear mixed bi-skew Jordan triple derivation if and only if \( \Phi \) is an additive \( \ast \)-derivation.

We denote the subalgebra of all bounded finite rank operators by \( \mathcal{F}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H}) \). We call a subalgebra \( \mathcal{A} \) of \( \mathcal{B}(\mathcal{H}) \) a standard operator algebra if it contains \( \mathcal{F}(\mathcal{H}) \). Now we have the following corollary.

**Corollary 3.2** Let \( \mathcal{H} \) be an infinite dimensional complex Hilbert space and \( \mathcal{A} \) be a standard operator algebra on \( \mathcal{H} \) containing the identity operator \( I \). Suppose that \( \mathcal{A} \) is closed under the adjoint operation. Then \( \Phi : \mathcal{A} \to \mathcal{A} \) is a nonlinear mixed bi-skew Jordan triple derivation if and only if \( \Phi \) is a linear \( \ast \)-derivation. Moreover, there exists an operator \( T \in \mathcal{B}(\mathcal{H}) \) satisfying \( T + T^* = 0 \) such that \( \Phi(A) = AT - TA \) for all \( A \in \mathcal{A} \), i.e., \( \Phi \) is inner.

**Proof** Since \( \mathcal{A} \) is prime, we know that \( \Phi \) is an additive \( \ast \)-derivation. It follows from [13] that \( \Phi \) is a linear inner derivation, i.e., there exists an operator \( S \in \mathcal{B}(\mathcal{H}) \) such that \( \Phi(A) = AS - SA \). Using the fact \( \Phi(A^\ast) = \Phi(A)^\ast \), we have
\[
A^\ast S - SA^\ast = \Phi(A^\ast) = \Phi(A)^\ast = -A^\ast S^\ast + S^\ast A^\ast
\]
for all \( A \in \mathcal{A} \). This leads to \( A^\ast(S + S^\ast) = (S + S^\ast)A^\ast \). Hence, \( S + S^\ast = \lambda I \) for some \( \lambda \in \mathbb{R} \). Let us set \( T = S - \frac{1}{2}\lambda I \). One can check that \( T + T^* = 0 \) such that \( \Phi(A) = AT - TA \).

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References


