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Geodesic γ-Pre-E-Convex Functions on Riemannian Manifolds

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Abstract In this paper, we generalize geodesic E-convex function and define geodesic γ-pre-E-convex and geodesic γ-E-convex functions on Riemannian manifolds. The sufficient condition of equivalence class of geodesic γ-pre-E-convexity and geodesic γ-E-convexity for differentiable function on Riemannian manifolds is studied. We discuss the sufficient condition for E-epigraph to be geodesic E-convex set. At the end, we establish some optimality results with the aid of geodesic γ-pre-E-convex and geodesic γ-E-convex functions and discuss the mean value inequality for geodesic γ-pre-E-convex function.

Keywords geodesic E-convex set; geodesic γ-pre-E-convex function; geodesic γ-E-convex function; optimality conditions

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1. Introduction

In optimization theory, the applications of convexity play an effective role. The concept of convexity has been developed several forms of convexity as well as non-convexity, like geodesic convexity, invexity, preinvexity and E-convexity etc. by the researchers at different times. Day by day the outcome of such research is generating a new class of generalised convex and invex functions and providing the applications of these functions in programming problem to obtain optimality results. Hanson [1] investigated the sufficiency of Kuhn-Tucker conditions for an optimal solution. Today these conditions have become an important tool in applied and pure mathematics through programming. Generalisation of these conditions for generalised non-convex functions, called K-invex functions was obtained in [2]. Jeykumar [3] conformed invexity in weak and strong forms. Pini extended the concept of invexity on Riemannian manifolds in [4]. Later, Mititelu [5] investigated generalised invexity by introducing (ρ, η)-invex, (ρ, η)-pseudoinvex, (ρ, η)-quasiinvex functions and applied this invexity on vector programming defined on differential manifolds to obtain Kuhn-Tucker conditions. On the other hand, the concept of E-convexity was generated and some results of E-programming problem were established by Youness [6]. For the properties of E-convex functions, the readers may refer to [7]. Ferreira [8] presented a proximal point method in Euclidean space to solve the convex constrained problem. Generalization of convex and invex

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functions on Riemannian manifolds is investigated by researchers in [5, 9–12]. Motivated by invexity and preinvexity [2, 13–15], Antczak defined r-preinvex and r-invex functions, where $r \in \Re$ with optimal-dual theorems by using these functions and presented an alternative approach for modified r-preinvex functions with respect to function η in [16] and Antczak [17] proposed the generalisation of the mean value theorem in invexity sense. E-epigraph with respect to E-convex function was introduced in [18]. Fulga and Preda [19] established E-preinvexity. A generalisation of geodesic preinvexity like η-preinvexity [20], α-preinvexity [21] and r-preinvexity [22] on Riemannian manifolds was developed at different time. During this period geodesic E-convexity [23], E-epigraph [23] and geodesic semi local E-convexity [24] were studied. Its generlisation in invexity, geodesic (α, E)-preinvexity on Riemannian manifold was defined in [25]. A geodesic concept for E-convex function was obtained in [24].

2. Definitions and preliminaries

For some basic definitions and notions of Riemannian manifolds, we refer to [12, 26]. Let M be an n-dimensional differentiable manifold and T_{xM} be the tangent space to M at the point $x \in M$, then suppose that at each point $x \in M$, a positive inner product $g_x(m_1, m_2)$ on T_{xM} is given by $(m_1, m_2 \in T_{xM})$. A C^∞ mapping $g : x \to g_x$, which assigns a positive inner product g_x on T_xM to each point $x \in M$, is called a Riemannian metric and manifold M equipped with Riemannian metric g is called Riemannian manifold. We denote the tangent space to M by T_xM. Suppose that (M, g) is a complete n-dimensional Riemannian manifold with Riemannian connection ∇. Let $m_1, m_2 \in M$ and $\zeta_{(m_1,m_2)} : [0, 1] \to M$ be a geodesic joining the points m_1 and m_2 i.e., $\zeta_{(m_1,m_2)}(0) = m_2$, $\zeta_{(m_1,m_2)}(1) = m_1$. We consider the length of a piece-wise C^1 curve $\zeta : [a, b] \to M$ defined by

$$L(\zeta) := \int_a^b \|\zeta'(t)\mathrm{d}t\|.$$

For any two points $x, y \in M$, distance $d(x, y) := \inf\{L(\zeta) : \zeta$ is a piece-wise C^1 curve joining x and $y\}$ induces the topology on M. We know that on every Riemannian manifold, for any vector fields $M_1, M_2 \in M$ there exists one and only one covariant derivation, denoted by $\nabla M_1 M_2$, called Levicivita connection. We also recall that a geodesic is a C^∞ smooth path ζ, whose tangent is parallel along the path ζ i.e., ζ satisfies $\nabla_{d\zeta(t)/d(t)} d\zeta(t)/d(t) = 0$. Any path joining $x \in M$ and $y \in M$ such that $L(\zeta) = d(x, y)$, is a geodesic and called a minimal geodesic. Now we consider the following definitions:

Definition 2.1 ([6]) *A set $K \subseteq \Re^n$ is called an E-convex set for the function $E : \Re^n \to \Re^n$, if for all $m_1, m_2 \in K$ and $\theta \in [0, 1]$, $\theta E(m_1) + (1 - \theta)E(m_2) \in K$.*

Definition 2.2 ([6]) *Let $K \subseteq \Re^n$ be an E-convex set for the function $E : \Re^n \to \Re^n$. Then a function $\psi : K \to \Re$ is said to be an E-convex function, if for all $m_1, m_2 \in K$ and $\theta \in [0, 1]$, we have $\psi(\theta E(m_1) + (1 - \theta)E(m_2)) \leq \theta\psi(E(m_1)) + (1 - \theta)\psi(E(m_2))$.*

The set $\Re^n$ was replaced by Riemannian manifold M and defined geodesic E-convex set, geodesic E-convex function on Riemanniam manifold M, where $E : M \to M$ is a function defined on Riemaanian manifolds in [23].

Definition 2.3 ([23]) *Let $E : M \to M$ be a function. Then $D \subseteq M$ is said to be a geodesic E-convex set, if for all $m_1, m_2 \in D$, $\theta \in [0,1]$, there is a unique geodesic $\zeta_{E(m_1),E(m_2)} : [0,1] \to M$ such that*

$$\zeta_{E(m_1),E(m_2)}(\theta) \in D, \zeta_{E(m_1),E(m_2)}(0) = E(m_2), \zeta'_{E(m_1),E(m_2)}(0) = E(m_1) - E(m_2).$$

Definition 2.4 ([23]) *Let M be a Riemannian manifolds and $D \subseteq M$ be a geodesic E-convex set in M. Then function $\psi : D \to \Re$ is said to be geodesic E-convex on D, if for all $m_1, m_2 \in D$, $\theta \in [0,1]$,*

$$\psi(\zeta_{E(m_1),E(m_2)}(\theta)) \leq \theta\psi(E(m_1)) + (1-\theta)\psi(E(m_2)).$$

If for all $m_1, m_2 \in D$, $\theta \in [0,1]$, this inequality has strict sign, then ψ is a strictly geodesic E-convex function.

Remark 2.5 Let M be a Cartan-Hadamard manifold and $E : M \to M$ be a function defined on M then from [8], for all $m_1, m_2 \in M$, we can write $E(m_1) - E(m_2) = \zeta'_{E(m_1),E(m_2)}(0)$, where $\zeta_{E(m_1),E(m_2)}$ denotes the unique minimal geodesic joining $E(m_1)$ and $E(m_2)$ such that for all $\theta \in [0,1]$, $\zeta_{E(m_1),E(m_2)}(\theta) = \exp_{E(m_2)}(\theta \exp^{-1}_{E(m_2)}(E(m_1))$.

Remark 2.6 If $E(m) = m$, $\forall\, m \in M$, then Definitions 2.3 and 2.4 reduce to the definitions of geodesic convex set and geodesic convex function, respectively.

Now we consider Property (P) (see [4]) and Condition (C) (see [10]) then define Property (P) and Condition (C) in the following forms.

Definition 2.7 *Property (P). Let M be a Riemannian manifolds, $E : M \to M$ be a function and $\zeta_{E(m_1),E(m_2)} : [0,1] \to M$ be a geodesic such that,*

$$\zeta_{E(m_1),E(m_2)}(0) = E(m_2),\ \ \zeta_{E(m_1),E(m_2)}(1) = E(m_1).$$

Then $\zeta_{E(m_1),E(m_2)}$ is said to possess the Property (P) with respect to $E(m_1)$, $E(m_2)$ if for all $\lambda, \theta \in [0,1]$, we have

$$\zeta'_{E(m_1),E(m_2)}(\lambda)(\theta - \lambda) = \zeta_{E(m_1),E(m_2)}(\theta) - \zeta_{E(m_1),E(m_2)}(\lambda).$$

Condition (C) is defined as

$$\begin{cases} P^0_{\theta,\zeta_{E(m_1),E(m_2)}}[E(m_2) - \zeta_{E(m_1),E(m_2)}(\theta)] = -\theta(E(m_1) - E(m_2)), \\ P^0_{\theta,\zeta_{E(m_1),E(m_2)}}[E(m_1) - \zeta_{E(m_1),E(m_2)}(\theta)] = (1-\theta)(E(m_1) - E(m_2)), \end{cases}$$

for all $\theta \in [0,1]$.

Remark 2.8 If we fix the function $E : M \to M$ such that $E(m) = m$, for all $m \in M$ in Definition 2.7, then putting $\eta(x,y) = x - y$ in [4] and [10], then (P) and (C) of [4] and [10] are obtained in special case, respectively.

Motivated by these definitions, we introduce geodesic γ-pre-E-convex function and geodesic γ-E-convex functions on geodesic E-convex set with respect to the function $E : M \to M$, where γ is any real number on Riemannian maninfold.

Definition 2.9 (Geodesic γ-pre-E-convex function) *Let M be a Riemannian manifolds and $D \subseteq M$ be a geodesic E-convex set. Then $\psi : D \to \Re$ is said to be a geodesic γ-pre-E-convex function on D, if the following inequality holds*

$$\psi(\zeta_{E(m_1),E(m_2)}(\theta)) \leq \begin{cases} \ln(\theta e^{\gamma\psi(E(m_1))} + (1-\theta)e^{\gamma\psi(E(m_2))})^{1/\gamma}, & \text{if } \gamma \neq 0; \\ \theta\psi(E(m_1)) + (1-\theta)\psi(E(m_2)), & \text{if } \gamma = 0, \end{cases} \tag{2.1}$$

for all $m_1, m_2 \in D$, $\theta \in [0,1]$. If (2.1) has strict sign for all $m_1, m_2 \in D$, $\theta \in [0,1]$, then function ψ is said to be a strict geodesic γ-pre-E-convex function. It is well true that every strict geodesic γ-pre-E-convex function (If (2.1) has a strict sign) is a geodesic γ-pre-E-convex function.

Example 2.10 Let the map $E : \Re \to \Re$ be defined as $E(m) = e^{cm}$, where $c \in \Re$ and the function $\psi : \Re \to \Re$ is defined as

$$\psi(m) = \begin{cases} m+1, & \text{if } m > 0; \\ -m - m^2, & \text{if } m \leq 0. \end{cases}$$

Suppose that the geodesic $\zeta_{E(m_1),E(m_2)}(\theta) = \theta e^{cm_1} + (1-\theta)e^{cm_2}$, then by (2.1), we see that ψ is geodesic γ-pre-E-convex function on $\Re$.

Example 2.11 Let the map $E : \Re \to \Re$ be defined as

$$E(m) = \begin{cases} m^2 - m, & \text{if } m \leq 0; \\ m, & \text{if } m > 0, \end{cases}$$

the function $\psi : \Re \to \Re$ be defined as

$$\psi(m) = \begin{cases} (m+1)^2, & \text{if } m > 0; \\ 0, & \text{if } m \leq 0, \end{cases}$$

and the geodesic $\zeta_{E(m_1),E(m_2)}(\theta)$ be defined as

$$\zeta_{E(m_1),E(m_2)}(\theta) = \begin{cases} \theta m_1 + (1-\theta)m_2, & \text{if } m_1, m_2 > 0; \\ \theta m_1 + (1-\theta)(m_2^2 - m_2), & \text{if } m_1 > 0, m_2 \leq 0; \\ \theta(m_1^2 - m_1) + (1-\theta)(m_2^2 - m_2), & \text{if } m_1, m_2 \leq 0. \end{cases}$$

For all cases of $m_1, m_2 \in \Re$, and $\gamma \in \Re$, Definition 2.9 is satisfied. Hence ψ is geodesic γ-pre-E-convex function on $\Re$.

Proposition 2.12 *Let $D \subseteq M$ be a geodesic E-convex set, $\psi : D \to \Re$ be a geodesic γ-pre-E-convex function on D and function $\phi : D \to \Re$ be defined as $\phi(m_1) := e^{\gamma\psi(m_1)}$ on D. Then ϕ is a geodesic 0-pre-E-convex (geodesic E-convex) function on D.*

Proof Since $\psi : D \to \Re$ is a geodesic γ-pre-E-convex function on D, then for $\gamma \neq 0$ by (2.1), for all $m_1, m_2 \in D$, $\theta \in [0,1]$, we have

$$\psi(\zeta_{E(m_1),E(m_2)}(\theta)) \leq \ln(\theta e^{\gamma\psi(E(m_1))} + (1-\theta)e^{\gamma\psi(E(m_2))})^{1/\gamma}.$$

After simplification and by the definition of function ϕ, this follows that

$$\phi(\zeta_{E(m_1),E(m_2)}(\theta)) \leq \theta\phi(E(m_1)) + (1-\theta)\phi(E(m_2)).$$

This gives the required result. □

Proposition 2.13 *Let $D \subseteq M$ be a geodesic E-convex set for an idempotent function $E : M \to M$ and $\psi : D \to \Re$ be a geodesic γ-pre-E-convex function on D. Then for any $k \in \Re$, the lower set $L_k = \{E(m) \in D : \psi(E(m)) \leq k\}$ is geodesic E-convex.*

Proof For any $E(m_1), E(m_2) \in L_k$, $\theta \in [0,1]$, we have $\psi(E(m_1)), \psi(E(m_2)) \leq k$. Since ψ is a geodesic γ-pre-E-convex function, then by (2.1)

$$\psi(\zeta_{E(E(m_1)),E(E(m_2))}(\theta)) \leq \ln(\theta e^{\gamma\psi(E(E(m_1)))} + (1-\theta)e^{\gamma\psi(E(E(m_2)))})^{1/\gamma}.$$

Since $E : M \to M$ be an idempotent function on M, then this follows that

$$e^{\gamma\psi(\zeta_{E(E(m_1)),E(E(m_2))}(\theta))} \leq \theta e^{\gamma\psi(E(m_1))} + (1-\theta)e^{\gamma\psi(E(m_2))} = e^{\gamma k}.$$

Therefore, $\psi(\zeta_{E(E(m_1)),E(E(m_2))}(\theta)) \leq k$. Hence the lower set L_k is geodesic E-convex in D. □

3. Geodesic γ-pre-E-convex function and differentiability

For the differentiable function, we generalize geodesic γ-pre-E-convexity and define geodesic γ-E-convex function. Then we relate geodesic γ-pre-E-convex and geodesic γ-E-convex functions.

Definition 3.1 (Geodesic γ-E-convex function) *Let M be a Riemannian manifolds and $D \subseteq M$ be a geodesic E-convex set for the map $E : M \to M$. Then a differentiable function $\psi : D \to \Re$ is said to be geodesic γ-E-convex function at $m_2 \in D$, if for all $m_1 \in D$, the following inequalities*

$$\frac{1}{\gamma}e^{\gamma\psi(E(m_1))} - \frac{1}{\gamma}e^{\gamma\psi(E(m_2))} \geq e^{\gamma\psi(E(m_2))} d\psi_{E(m_2)}(E(m_1) - E(m_2)), \text{ if } \gamma \neq 0, \tag{3.1}$$

$$\psi(E(p)) - \psi(E(q)) \geq d\psi_{E(m_2)}(E(m_1) - E(m_2)), \text{ if } \gamma = 0 \tag{3.2}$$

hold. If (3.1) and (3.2) are satisfied for all $m_1, m_2 \in D$, then ψ is a geodesic γ-E-convex function on D. If (3.1) and (3.2) have strict sign for all $m_1, m_2 \in D$ with $E(m_1) \neq E(m_2)$, then ψ is a strict geodesic γ-E-convex function on D.

Theorem 3.2 *Let $D \subseteq M$ be a geodesic open E-convex set and $\psi : D \to \Re$ be a geodesic γ-E-convex function on D. If geodesic curve ζ satisfies the conditions (C), then ψ is a geodesic γ-pre-E-convex function on D.*

Proof Suppose that D is a geodesic E-convex set, then for all $m_1, m_2 \in D$, there exits a unique geodesic $\zeta_{E(m_1),E(m_2)}(0) = E(m_2)$, $\zeta'_{E(m_1),E(m_2)}(0) = E(m_1) - E(m_2)$, $\zeta_{E(m_1),E(m_2)}(\theta) \in D$ for all $\theta \in [0,1]$. Now we assume that for any $\theta \in [0,1]$ and $m_3 \in D$, $E(m_3) = \zeta_{E(m_1),E(m_2)}(\theta)$. Then by Definition 3.1, we have

$$\frac{1}{\gamma}e^{\gamma\psi(E(m_1))} - \frac{1}{\gamma}e^{\gamma\psi(E(m_3))} \geq e^{\gamma\psi(E(m_3))}d\psi_{E(m_3)}(E(m_1) - E(m_3)) \tag{3.3}$$

$$\frac{1}{\gamma}e^{\gamma\psi(E(m_2))} - \frac{1}{\gamma}e^{\gamma\psi(E(m_3))} \geq e^{\gamma\psi(E(m_3))}d\psi_{E(m_3)}(E(m_2) - E(m_3)). \tag{3.4}$$

Multiplying (3.3) and (3.4) by θ and $(1-\theta)$, respectively, and then adding them, we have

$$\begin{aligned}&\theta\frac{1}{\gamma}e^{\gamma\psi(E(m_1))} + (1-\theta)\frac{1}{\gamma}e^{\gamma\psi(E(m_2))} - \frac{1}{\gamma}e^{\gamma\psi(E(m_3))}\\ &\geq e^{\gamma\psi(E(m_3))}d\psi_{E(m_3)}[\theta(E(m_1) - E(m_3)) + (1-\theta)(E(m_2) - E(m_3))].\end{aligned}$$

Now by Condition (C), we get

$$\begin{aligned}&\theta(E(m_1) - E(m_3)) + (1-\theta)(E(m_2) - E(m_3))\\ &= \theta(1-\theta)P^0_{\theta,\zeta}[E(m_1) - E(m_2)] - \theta(1-\theta)P^0_{\theta,\zeta}[E(m_1) - E(m_2)] = 0.\end{aligned}$$

Therefore,

$$\theta e^{\gamma\psi(E(m_1))} + (1-\theta)e^{\gamma\psi(E(m_2))} \geq e^{\gamma\psi(E(m_3))}. \tag{3.5}$$

This implies that for all $m_1, m_2 \in D, \theta \in [0,1]$,

$$\psi(E(m_3)) = \psi(\zeta_{E(m_1),E(m_2)}(\theta)) \leq ln(\theta e^{\gamma\psi(E(m_1))} + (1-\theta)e^{\gamma\psi(E(m_2))})^{\frac{1}{\gamma}}.$$

Hence ψ is geodesic γ-pre-E-convex function on D. □

Theorem 3.3 *Let $D \subseteq M$ be a geodesic open E-convex set and $\psi : D \to \Re$ be a differentiable geodesic γ-pre-E-convex function on D. Then ψ is also a geodesic γ-E-convex function on D.*

Proof Suppose that D is a geodesic E-convex set, then for all $m_1, m_2 \in D$, there exits a unique geodesic $\zeta_{E(m_1),E(m_2)}(0) = E(m_2)$, $\zeta'_{E(m_1),E(m_2)}(0) = E(m_1) - E(m_2)$, $\zeta_{E(m_1),E(m_2)}(\theta) \in D$ for all $\theta \in [0,1]$. Then by the differentiability of ψ at $E(m_2) \in D$, we get

$$d\psi_{E(m_2)}(E(m_1) - E(m_2)) = \lim_{\theta\to 0}\frac{1}{\theta}[\psi(\zeta_{E(m_1),E(m_2)}(\theta)) - \psi(E(m_2))].$$

Therefore,

$$\psi(E(m_2)) + d\psi_{E(m_2)}(E(m_1) - E(m_2))\theta + O^2(\theta) = \psi_{E(m_2)}(E(m_1) - E(m_2)).$$

Since ψ is geodesic γ-pre-E-convex for all $\theta \in [0,1]$, we have

$$\psi(E(m_2)) + d\psi_{E(m_2)}(E(m_1) - E(m_2))\theta + O^2(\theta) \leq \ln(\theta e^{\gamma\psi(E(m_1))} + (1-\theta)e^{\gamma\psi(E(m_2))})^{1/\gamma}.$$

This implies that

$$e^{\gamma\psi(E(m_2))+\gamma d\psi_{E(m_2)}(E(m_1)-E(m_2))\theta+\gamma O^2(\theta)} - e^{\gamma\psi(E(m_2))} \leq \theta(e^{\gamma\psi(E(m_1))} - e^{\gamma\psi(E(m_2))}).$$

Taking lim $t \to 0$, we have

$$e^{\gamma\psi(E(m_2))}d\psi_{E(m_2)}(E(m_1) - E(m_2)) \leq \frac{1}{\gamma}(e^{\gamma\psi(E(m_1))} - e^{\gamma\psi(E(m_2))}).$$

This shows that ψ is geodesic γ-E-convex function on D. □

Now we obtain the sufficient condition for E-epigraph $\mathrm{epi}_E(\psi)$ to be geodesic E-convex set from geodesic γ-pre-E-convexity.

Definition 3.4 ([3]) *E-epigraph $\mathrm{epi}_E(\psi)$ of ψ is defined as*

$$\mathrm{epi}_E(\psi) = \{(E(m), \alpha) : m \in D, \alpha \in \Re, \psi(E(m)) \leq \alpha\}.$$

Theorem 3.5 *Let $D \subseteq M$ be a geodesic E-convex set for an idempotent map $E : M \to M$, $\psi : D \to \Re$ be a geodesic γ-pre-E-convex function on D. Then E-epigraph $\mathrm{epi}_E(\psi)$ is geodesic E-convex set.*

Proof Let $m_1, m_2 \in D$ and D is a geodesic E-convex set, then for any $\theta \in [0, 1]$, $\zeta_{E(m_1),E(m_2)}(\theta) \in D$ and $(E(m_1), \alpha), (E(m_2), \alpha) \in \mathrm{epi}_E(\psi)$. Then by the definition of E-epigraph, we have

$$\psi(E(m_1)), \psi(E(m_2)) \leq \alpha. \tag{3.6}$$

Since ψ is a geodesic γ-pre-E-convex function on D, then by (2.1) and (3.6), we have

$$\psi(\zeta_{E(m_1),E(m_2)}(\theta)) \leq \alpha. \tag{3.7}$$

If E is an idempotent map, then (3.7) can be written as

$$\psi(\zeta_{E^2(m_1),E^2(m_2)}(\theta)) = \psi(\zeta_{E(E(m_1)),E(E(m_2))}(\theta)) \leq \alpha.$$

Hence $\mathrm{epi}_E(\psi)$ is a geodesic E-convex set. □

4. Optimality conditions

Now we apply geodesic γ-pre-E-convex function and geodesic γ-E-convex function on the mathematical programming problem to obtain some optimality results.

Let $D \subseteq M$ be a geodesic E-convex set. Then consider the following programming Problem (P):

$$\begin{aligned} &\mathrm{Min.}\psi(E(m)) \\ \text{subject to } \quad &\omega_j(E(m)) \leq 0, \; j = 1, 2, \ldots, m, \end{aligned}$$

where $\psi, \omega_j : D \to \Re$, $j = 1, 2, \ldots, m$ need not be differentiable functions on geodesic E-convex set D. If

$$\Lambda = \{m \in D : \omega_j(E(m)) \leq 0, \; j = 1, 2, ..., m\}$$

represents the feasible set of the above problem, then we establish following optimality results.

Theorem 4.1 *Let $D \subseteq M$ be a geodesic E-convex set, $\omega : D \to \Re$ be a geodesic γ-pre-E-convex function on D. Then the feasible set Λ is geodesic E-convex set.*

Proof Let $m^0, m^* \in \Lambda$. Then $\omega(E(m^0)), \omega(E(m^*)) \leq 0$. If ω is geodesic γ-pre-E-convex function, then by (2.1) and $\omega(E(m^0)), \omega(E(m^*)) \leq 0$, we have

$$\psi(\zeta_{E(m^o),E(m^*)}(\theta)) \leq 0. \tag{4.1}$$

Hence the feasible set Λ is geodesic E-convex set. □

Theorem 4.2 *Let $D \subseteq M$ be a geodesic E-convex set, $\psi : D \to \Re$ be a geodesic γ-pre-E-convex function on D and m^* be a local optimal solution of Problem (P). Then m^* is also a global optimal solution of Problem (P).*

Proof Suppose that the local solution $m^* \in \Lambda$ is not global optimal in (P), then there is a $m^0 \in \Lambda$, such that $\psi(E(m^0)) < \psi(E(m^*))$. Since ψ is a geodesic γ-pre-E-convex function on D, then by Definition 2.9, for all m^0, $m^* \in D$, we have

$$\psi(\zeta_{E(m^0),E(m^*)}(\theta)) \leq \ln(\theta e^{\gamma\psi(E(m^0))} + (1-\theta)e^{\gamma\psi(E(m^*))})^{1/\gamma}.$$

Since $\psi(E(m^0)) < \psi(E(m^*))$, we have

$$\psi(\zeta_{E(m^0),E(m^*)}(\theta)) < \ln(\theta e^{\gamma\psi(E(m^0))} + (1-\theta)e^{\gamma\psi(E(m^0))})^{1/\gamma} = \psi(E(m^0)).$$

This follows that, for all $m, m^0 \in D$ and $\theta \in [0,1]$,

$$\psi(\zeta_{E(m^0),E(m^*)}(\theta)) < \psi(E(m^0)).$$

This contradicts the optimality of $m^0 \in \Lambda$. Hence m^* is a global optimal solution of Problem (P). □

Theorem 4.3 *Let $D \subseteq M$ be a geodesic E-convex set, $\psi : D \to \Re$ be a geodesic γ-E-convex function on D. Then every stationary point of function ψ is a global minimum of Problem (P).*

Proof Let any $m_2 \in D$ be a stationary point of ψ. Then $d\psi_{E(m_2)}(E(m_1) - E(m_2)) = 0$ and ψ is a geodesic γ-E-convex function on D. Then we have

$$\frac{1}{\gamma}e^{\gamma\psi(E(m_1))} - \frac{1}{\gamma}e^{\gamma\psi(E(m_2))} \geq 0,$$

for all $m_1 \in D$. Therefore, $\psi(E(m_1)) \geq \psi(E(m_2))$. This shows that $m_2 \in D$ is a global minimum of Problem (P). □

Theorem 4.4 *Let $D \subseteq M$ be a geodesic E-convex set and $\psi : D \to \Re$ be a strictly geodesic γ-pre-E-convex function on D. Then*

(1) A local optimal of (P) is a global optimal of (P).

(2) The global optimal of Problem (P) is unique.

Proof For the part (1) of Theorem 4.4, we can consider the proof of Theorem 4.2, with all the inequalities of strict sign. Now for the uniqueness of global optimal solution of (P), we assume that $m^0, m^* \in \Lambda$ be two global optimal solution of (P) such that $\psi(E(m^0)) = \psi(E(m^*))$. Then by the definition of strictly geodesic γ-E-convex function for the function ψ, we have for all $m^0, m^* \in \Lambda$, $\theta \in [0,1]$

$$\begin{aligned}\psi(\zeta_{E(m^0),E(m^*)}(\theta)) &< \ln(\theta e^{\gamma\psi(E(m^0))} + (1-\theta)e^{\gamma\psi(E(m^*))})^{1/\gamma} \\ &= \ln(\theta e^{\gamma\psi(E(m^*))} + (1-\theta)e^{\gamma\psi(E(m^*))})^{1/\gamma} = \psi(E(m^*)).\end{aligned}$$

This follows that

$$\psi(\zeta_{E(m^0),E(m^*)}(\theta)) < \psi(E(m^*)).$$

This contradicts the global optimality of $m^* \in \Lambda$, hence the global optimal solution of Problem (P) is unique. $\square$

Now we define the mean value inequality for Cartan-Hadamard manifold. At first we consider the following definition [6] in E-convexity sense:

Definition 4.5 *Let M be a Riemannian manifolds, $E: M \to M$ be a function, $D \subseteq M$ be a nonempty geodesic E-convex set and any $m_1, m^* \in D$. Let for all $\theta \in [0,1]$, $\zeta : [0,1] \to M$ be a unique geodesic such that $\zeta(0) = E(m^*), \zeta'(0) = E(m_1) - E(m^*)$ and $\zeta(\theta) \in D$. Then*

- *A set $P_{E(m^*)E(\bar{m})}$ such that $P_{E(m^*)E(\bar{m})} = \{E(m_2) : E(m_2) = \zeta(\theta), \theta \in [0,1]\}$, is a closed path joining the points $E(m^*) = \zeta(0)$ and $E(\bar{m}) = \zeta(1)$.*
- *A set $P_{E(m^*)E(\bar{m})}$ such that $P^0_{E(m^*)E(\bar{m})} = \{E(m_2) : E(m_2) = \zeta(\theta), \theta \in (0,1)\}$, is an open path joining the points $E(m^*) = \zeta(0)$ and $E(\bar{m}) = \zeta(1)$.*

Here $P^0_{E(m^*)E(\bar{m})} = \phi$, if $E(m^*) = E(\bar{m})$.

Theorem 4.6 *Let M be a Cartan-Hadamard manifold and $D \subseteq M$ be a nonempty geodesic E-convex set such that $E(m^*) \neq E(m^0)$, for all $m^*, m^0 \in D$ with $m^* \neq m^0$. If $\zeta_{E(m^0),E(m^*)}(\theta) = \exp_{E(m^*)}(\theta(E(m^0) - E(m^*)))$, for all $m^*, m^0 \in D, \theta \in [0,1]$ and $E(m') = \zeta_{E(m^0),E(m^*)}(1)$. Then a necessary and sufficient condition for the function $\psi : D \to \Re$ to be geodesic γ-pre-E-convex is that, for all $E(m_1) \in P_{E(m')E(m^*)}$, the following inequality*

$$\begin{aligned} e^{\gamma\psi(E(m_1))} \leq & e^{\gamma\psi(E(m^*))} + \frac{e^{\gamma\psi(E(m^0))} - e^{\gamma\psi(E(m^*))}}{\langle E(m^0) - E(m^*), E(m^0) - E(m^*)\rangle_{E(m^*)}} \times \\ & \langle \exp^{-1}_{E(m*)}(E(m_1)), E(m^0) - E(m^*)\rangle_{E(m^*)} \end{aligned} \tag{4.2}$$

is satisfied.

Proof Let ψ be a geodesic γ-pre-E-convex function on D, $m^*, m^0 \in D$ and $E(m_1) \in P_{E(m')E(m^*)}$. For $m^* = m_1$ or $m^0 = m_1$, then (4.2) is trivially satisfied. Now $E(m_1) \in P^0_{E(m')E(m^*)}$, then $E(m_1) = \exp_{E(m*)}(\theta(E(m^0) - E(m^*)))$, for some $\theta \in (0,1)$. Then by geodesic E-convexity of D, we have $E(m_1) \in D$ and

$$\theta = \frac{\langle \exp^{-1}_{E(m*)}(E(m_1)), E(m^0) - E(m^*)\rangle_{E(m^*)}}{\langle E(m^0) - E(m^*), E(m^0) - E(m^*)\rangle_{E(m^*)}}.$$

Since ψ is geodesic γ-pre-E-convex function on D, we have

$$\psi(E(m_1)) = \psi(\exp_{E(m*)}(\theta(E(m^0) - E(m^*)))) \leq \ln(\theta e^{\gamma\psi(E(m^0))} + (1-\theta)e^{\gamma\psi(E(m^*))})^{1/\gamma}.$$

This follows that

$$e^{\gamma\psi(E(m_1))} \leq \theta e^{\gamma\psi(E(m^0))} + (1-\theta)e^{\gamma\psi(E(m^*))} = e^{\gamma\psi(E(m^*))} + \theta(e^{\gamma\psi(E(m^0))} - e^{\gamma\psi(E(m^*))}).$$

By the value of θ, we have

$$
\begin{aligned}
e^{\gamma\psi(E(m_1))} \leq & e^{\gamma\psi(E(m^*))} + \frac{e^{\gamma\psi(E(m^0))} - e^{\gamma\psi(E(m^*))}}{\langle E(m^0) - E(m^*), E(m^0) - E(m^*)\rangle_{E(m^*)}} \times \\
& \langle \exp^{-1}_{E(m*)}(E(m_1)), E(m^0) - E(m^*)\rangle_{E(m^*)}.
\end{aligned}
$$

For the sufficient condition, we assume that (4.2) is satisfied and for some $\theta \in [0, 1]$, $m^*, m^0 \in D$, $E(m_1) = \exp_{E(m^*)}(\theta(E(m^0) - E(m^*)))$. Then $\psi(E(m_1)) = \psi(\exp_{E(m^*)}(\theta(E(m^0) - E(m^*))))$, $E(m_1) \in D$ and by (4.2)

$$
\begin{aligned}
e^{\gamma\psi(E(m_1))} \leq & e^{\gamma\psi(E(m^*))} + \frac{e^{\gamma\psi(E(m^0))} - e^{\gamma\psi(E(m^*))}}{\langle E(m^0) - E(m^*), E(m^0) - E(m^*)\rangle_{E(m^*)}} \times \\
& \langle \exp^{-1}_{E(m*)}(E(m_1)), E(m^0) - E(m^*)\rangle_{E(m^*)} \\
= & e^{\gamma\psi(E(m^*))} + \frac{e^{\gamma\psi(E(m^0))} - e^{\gamma\psi(E(m^*))}}{\langle E(m^0) - E(m^*), E(m^0) - E(m^*)\rangle_{E(m^*)}} \times \\
& \langle \exp^{-1}_{E(m*)}(\exp(\theta(E(m^0) - E(m^*)))), E(m^0) - E(m^*)\rangle_{E(m^*)} \\
= & e^{\gamma\psi(E(m^*))} + \theta(e^{\gamma\psi(E(m^0))} - e^{\gamma\psi(E(m^*))}).
\end{aligned}
$$

Therefore,

$$
\psi(E(m_1)) = \psi(\exp(\theta(E(m^0) - E(m^*)))) \leq \ln(\theta e^{\gamma\psi(E(m^0))} + (1-\theta)e^{\gamma\psi(E(m^*))})^{1/\gamma}.
$$

Hence ψ is a geodesic γ-pre-E-convex function on D. □

5. Conclusion

In this paper, we show that the class of geodesic γ-pre-E-convexity and geodesic γ-E-convexity is equivalent for the differentiable functions on Riemannian manifolds. The sufficient condition of E-epigraph is presented and optimality results are established by using geodesic γ-pre-E-convex and geodesic γ-E-convex functions. As some results can be obtained in the special case, if we take $E(m) = m$, for all $m \in M$, then put $\eta(m_1, m_2) = m_1 - m_2$, $\forall\, m_1, m_2 \in M$ in [12]. Therefore, in Riemannian manifolds the geodesic γ-pre-E-convex functions (geodesic γ-E-convex functions) are more general than the geodesic γ-pre-convex functions (geodesic γ-convex functions).

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