n-Gorenstein Projective Modules and Dimensions over 
Frobenius Extensions

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Abstract In this paper, we study n-Gorenstein projective modules over Frobenius extensions 
and n-Gorenstein projective dimensions over separable Frobenius extensions. Let \( R \subset A \) be a 
Frobenius extension of rings and \( M \) any left \( A \)-module. It is proved that \( M \) is an n-Gorenstein 
projective left \( A \)-module if and only if \( A \otimes_R M \) and \( \text{Hom}_R(A, M) \) are n-Gorenstein projective 
left \( A \)-modules if and only if \( M \) is an n-Gorenstein projective left \( R \)-module. Furthermore, when 
\( R \subset A \) is a separable Frobenius extension, n-Gorenstein projective dimensions are considered.

Keywords Frobenius extensions; n-Gorenstein projective modules; n-Gorenstein projective 
dimensions

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1. Introduction

The study of Gorenstein homological algebra stems from finitely generated modules of G-
dimensions zero over any noetherian rings, introduced by Auslander and Bridger [1] in 1969 as 
a generalization of finite generated projective modules. In order to complete the analogy, in 
1995, Enochs and Jenda introduced the Gorenstein projective modules (not necessarily finitely 
generated) over any associative rings; and dually, Gorenstein injective modules were defined 
in [2]. In 2004, Holm further studied the properties of these modules in [3]. In 2015, n-Gorenstein 
projective modules and n-Gorenstein injective modules were introduced by Tang in [4] as a generalization 
of these modules. Tang used these two classes of modules to give a new characterization 
of Gorenstein rings in terms of top local cohomology modules of flat modules.

The theory of Frobenius extensions was developed by Kasch [5] in 1954 as a generalization 
of Frobenius algebras, and was further studied by Nakayama and Tsuzuku [6, 7] in 1960–1961, 
properties of modules and Gorenstein homological dimensions along Frobenius extensions of 
rings in [10, 11].

Inspired by above conclusions, in this paper, we intend to study the n-Gorenstein projective 
properties of modules and n-Gorenstein homological dimensions along Frobenius extensions of 
rings.

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Throughout this paper, let $R$ be an associative ring with identity. All modules will be unitary left $R$-modules. We write $P(R)$ and $GP(R)$ for the classes of projective and Gorenstein projective left $R$-modules, respectively. For each positive integer $n$ and the class of modules $\mathcal{X}$, we denote by $\perp^n \mathcal{X} := \{M|\text{Ext}^i_R(M, X) = 0\text{ for any }X \in \mathcal{X},\text{ and }1 \leq i \leq n\}$.

2. $n$-Gorenstein projective modules over Frobenius extensions

As a generalization of Gorenstein projective modules, Tang defined $n$-Gorenstein projective modules in [4].

**Definition 2.1** Suppose that $n$ is a positive integer. An $R$-module $M$ is said to be $n$-Gorenstein projective, if there exists an acyclic complex of projective modules $P = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow P_{-2} \rightarrow \cdots$ such that $M \cong \text{Im}(P_0 \rightarrow P_{-1})$ and such that for any projective module $Q$ the complex $\text{Hom}_R(P, Q) = \cdots \rightarrow P_1^* \rightarrow P_0^* \rightarrow P_{-1}^* \rightarrow P_{-2}^* \rightarrow \cdots$ is exact at $P_i^*$ for all $i \geq -n$, where $P_i^* = \text{Hom}_R(P_{i-1}, Q)$. The class of $n$-Gorenstein projective modules is denoted by $n\text{-GP}(R)$.

Clearly, by the definitions we have $P(R) \subseteq GP(R) \subseteq n\text{-GP}(R)$. However, there are $n$-Gorenstein projective modules which are not Gorenstein projective by [4, Example 2.4].

**Lemma 2.2** ([4, Proposition 2.2]) Suppose that $M$ is an $R$-module, and $m$, $n$ are positive integers such that $m < n$, then the following statements hold.

1. $M$ is $n$-Gorenstein projective if and only if $M$ belongs to the class $\perp_n P(R)$, and admits a co-proper right $P(R)$-resolution.

2. $n$-Gorenstein projective modules are $m$-Gorenstein projective modules.

3. $GP(R) = \bigcap_{n=1}^{\infty} n\text{-GP}(R)$.

4. If $M$ is $n$-Gorenstein projective, then there is an exact sequence $0 \rightarrow M \rightarrow P \rightarrow G \rightarrow 0$ such that $P$ is projective and $G$ is $(n+1)$-Gorenstein projective.

**Lemma 2.3** ([4, Proposition 2.6 and Corollary 2.7]) $n\text{-GP}(R)$ is closed under direct sums, direct summands and extensions.

**Lemma 2.4** ([4, Corollary 3.2]) Let $0 \rightarrow G_1 \rightarrow G \rightarrow M \rightarrow 0$ be an exact sequence, where $G$ and $G_1$ are $n$-Gorenstein projective and $\text{Ext}^1(M, Q) = 0$ for all projective modules $Q$. Then $M$ is $n$-Gorenstein projective.

**Definition 2.5** ([9, Definition 1.1 and Theorem 1.2]) A ring extension $R \subset A$ is a Frobenius extension, which provided that one of the following equivalent conditions holds:

1. The functors $T = A \otimes_R -$ and $H = \text{Hom}_R(A, -)$ are naturally equivalent.

2. $RA$ is finite generated projective and $\text{Ext}^1_A(R, A) = \text{Hom}_R(RA, A)$.

3. $RA$ is finite generated projective and $\text{Ext}^1_A(R, A) = \text{Hom}_R(RA, A)$.

**Example 2.6** ([9, Definition 1.1 and Theorem 1.2]) (1) For a finite group $G$, $\mathbb{Z} \subset \mathbb{Z}G$ is a Frobenius extension.

(2) ([11, Lemma 3.1]) Let $R$ be an arbitrary ring, and $A = R[x]/(x^2)$ is the quotient of the
polynomial ring, where \( x \) is a variable which is supposed to commute with all the elements of \( R \). Then the ring extension \( R \subseteq A \) is a Frobenius extension.

In [11], Ren studied the Gorenstein projective properties of modules along Frobenius extensions of rings. Let \( M \) be any left \( A \)-module. It is proved that \( M \) is a Gorenstein projective left \( A \)-module if and only if \( M \) is a Gorenstein projective left \( R \)-module if and only if \( A \otimes_R M \) and \( \text{Hom}_R(A, M) \) are Gorenstein projective left \( A \)-modules. Analogously, we have the following conclusions for \( n \)-Gorenstein projective modules.

**Proposition 2.7** Let \( R \subseteq A \) be a Frobenius extension of rings and \( M \) a left \( A \)-module. If \( A,M \) is \( n \)-Gorenstein projective, then the underlying left \( R \)-module \( R,M \) is also \( n \)-Gorenstein projective.

**Proof** Let \( M \) be an \( n \)-Gorenstein projective left \( A \)-module. There exists an acyclic complex of projective left \( A \)-module \( P = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow P_{-2} \rightarrow \cdots \) such that \( M \cong \text{Im}(P_0 \rightarrow P_{-1}) \) and for any projective left \( A \)-module \( Q \) the complex \( \text{Hom}_A(P,Q) = \cdots \rightarrow P_i^* \rightarrow P_0^* \rightarrow P_{-1}^* \rightarrow P_{-2}^* \rightarrow \cdots \) is exact at \( P_i^* \) for all \( i \geq -n \), where \( P_i^* = \text{Hom}_A(P_{-i},Q) \). Note that each \( P_i \) is a projective left \( R \)-module. Then by restricting \( P \) one gets an acyclic complex of projective \( R \)-modules.

Let \( F \) be a projective left \( R \)-module. It follows from isomorphisms \( \text{Hom}_R(A,F) \cong A \otimes_R F \) that \( \text{Hom}_R(A,F) \) is a projective left \( A \)-module. Then the complex

\[
\text{Hom}_A(P, \text{Hom}_R(A,F)) = \cdots \rightarrow T_i^* = \text{Hom}_A(P_{-i}, \text{Hom}_R(A,F)) \rightarrow T_{-1}^* \rightarrow T_{-2}^* \rightarrow \cdots
\]

is exact at \( T_i^* \) for all \( i \geq -n \), where \( T_i^* = \text{Hom}_A(P_{-i}, \text{Hom}_R(A,F)) \). Moreover, there are isomorphisms

\[
\text{Hom}_R(P,F) \cong \text{Hom}_R(A \otimes_A P,F) \cong \text{Hom}_A(P, \text{Hom}_R(A,F)).
\]

This implies that the complex \( \text{Hom}_R(P,F) \) is exact at \( H_i^* \) for all \( i \geq -n \), where \( H_i^* = \text{Hom}_R(P_{-i},F) \), and hence the underlying \( R \)-module \( M \) is \( n \)-Gorenstein projective.

**Proposition 2.8** Let \( R \subseteq A \) be a Frobenius extension of rings and \( M \) a left \( A \)-module. Then \( A \otimes_R M(\text{Hom}_R(A,M)) \) is an \( n \)-Gorenstein projective left \( A \)-module if and only if the underlying left \( R \)-module \( R,M \) is also \( n \)-Gorenstein projective.

**Proof** \( \Rightarrow \). Let \( M \) be an \( n \)-Gorenstein projective left \( R \)-module. There exists an acyclic complex of projective left \( R \)-module \( P = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow P_{-2} \rightarrow \cdots \) such that \( M \cong \text{Im}(P_0 \rightarrow P_{-1}) \) and for any projective left \( R \)-module \( Q \) the complex \( \text{Hom}_R(P,Q) = \cdots \rightarrow P_i^* \rightarrow P_0^* \rightarrow P_{-1}^* \rightarrow P_{-2}^* \rightarrow \cdots \) is exact at \( P_i^* \) for all \( i \geq -n \), where \( P_i^* = \text{Hom}_R(P_{-i},Q) \). It is easy to see that \( A \otimes_R P \) is an acyclic complex of projective \( A \)-modules, and

\[
A \otimes_R M \cong \text{Im}(A \otimes_R P_0 \rightarrow A \otimes_R P_{-1}).
\]

Moreover, for any projective left \( A \)-module \( P \), there are isomorphisms

\[
\text{Hom}_A(A \otimes_R P, P) \cong \text{Hom}_R(P,P).
\]

This implies that the complex \( \text{Hom}_A(A \otimes_R P, P) \) is exact at \( U_i^* \) for all \( i \geq -n \), where \( U_i^* = \text{Hom}_A(A \otimes_R P_{-i},P) \).
Moreover, since

\[ \text{Hom}_R(A, M) \cong A \otimes_R M. \]

This implies that the module \( \text{Hom}_R(A, M) \) is an \( n \)-Gorenstein projective left \( A \)-module.

\[ \Rightarrow \text{ Note that for the ring extension } R \subset A \text{ and any } A\text{-module } M, \text{ the module } M \text{ is a left } R\text{-module. By Proposition 2.7, it suffices to prove that when the left } A\text{-module } A \otimes_R M \text{ is } n\text{-Gorenstein projective, } A \otimes_R M \text{ is an } n\text{-Gorenstein projective left } R\text{-module. It is easy to see that the module } R M \text{ is a direct summand of the left } R\text{-module } A \otimes_R M. \text{ According to Lemma 2.3, } R M \text{ is an } n\text{-Gorenstein projective left } R\text{-module.} \]

**Theorem 2.9** Suppose \( M \) is any left \( A \)-module. Then \( A \otimes_R M \ (\text{Hom}_R(A, M)) \) is an \( n \)-Gorenstein projective left \( A \)-module if and only if \( M \) is an \( n \)-Gorenstein projective left \( A \)-module.

**Proof** By Propositions 2.7 and 2.8, it suffices to prove that \( n \)-Gorenstein projective left \( R \)-module \( M \) is also an \( n \)-Gorenstein projective left \( A \)-module.

Let \( Q \) be any projective left \( A \)-module. Then \( Q \) is a projective left \( R \)-module. Note that for the ring extension \( R \subset A \) and any \( A \)-module \( M \), the module \( M \) is a left \( R \)-module. Therefore, by the isomorphisms

\[ \text{Hom}_A(M, A \otimes_R Q) \cong \text{Hom}_A(M, \text{Hom}_R(A, Q)) \cong \text{Hom}_R(A \otimes_A M, Q) \cong \text{Hom}_R(M, Q), \]

we get the cohomology isomorphisms

\[ \text{Ext}^i_A(M, A \otimes_R Q) \cong \text{Ext}^i_A(M, \text{Hom}_R(A, Q)) \cong \text{Ext}^i_R(A \otimes_A M, Q) \cong \text{Ext}^i_R(M, Q). \]

Since \( M \) is an \( n \)-Gorenstein projective left \( R \)-module, it follows from Lemma 2.2(1) that \( M \in \perp P(R) \), i.e., \( \text{Ext}^i_R(M, Q) = 0 \) for all \( 1 \leq i \leq n \). Then we have \( \text{Ext}^i_A(M, A \otimes_R Q) \cong \text{Ext}^i_R(M, Q) = 0 \). Moreover, since \( A Q \) is a direct summand of \( A \otimes_R Q \), and then \( \text{Ext}^i_A(M, Q) = 0). \)

Since \( \text{Hom}_R(A, M) \) is an \( n \)-Gorenstein projective left \( A \)-module, by Lemma 2.2(2), there is an exact sequence \( 0 \rightarrow \text{Hom}_R(A, M) \rightarrow L \rightarrow 0 \) of left \( A \)-modules, where \( P_0 \) is projective and \( L \) is \((n + 1)\)-Gorenstein projective. By Lemma 2.2(4), \( L \) is \( n \)-Gorenstein projective. There is a map \( \varphi : M \rightarrow \text{Hom}_R(A, M) \) given by \( \varphi(m)(a) = am \), which is an \( A \)-homomorphism, and split when we restrict it as an \( R \)-homomorphism. Hence we have an \( R \)-homomorphism \( \varphi' : \text{Hom}_R(A, M) \rightarrow M \) such that \( \varphi' \varphi = \text{id}_M \). Let \( P \) be any projective \( R \)-module, and \( g : M \rightarrow P \) be any \( R \)-homomorphism. Since \( L \) is also \( n \)-Gorenstein projective as an \( R \)-module, for the \( R \)-homomorphism \( g \varphi' : \text{Hom}_R(A, M) \rightarrow P \), there is an \( R \)-homomorphism \( h : P_0 \rightarrow P \), such that \( g \varphi' = h f \). That is, we have the following commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}_R(A, M) & \rightarrow & P_0 & \rightarrow & L & \rightarrow & 0 \\
& & ^f \downarrow & & ^g \varphi' \downarrow & & ^h \rightarrow & & \\
& & & & & & \exists P & & \\
0 & \rightarrow & P_0 & \rightarrow & L_0 & \rightarrow & 0 \\
& & ^f \downarrow & & & & ^{\text{Coker}(f \varphi)} & & \\
& & & & & & \end{array}
\]

Now we have an \( A \)-homomorphism \( f \varphi : M \rightarrow P_0 \). Consider the exact sequence \( 0 \rightarrow M \rightarrow L \rightarrow 0 \) of \( A \)-modules, where \( P_0 \) is projective, \( L_0 = \text{Coker}(f \varphi) \). Restricting
the sequence, we note that it is $\text{Hom}_R(\cdot, P)$-exact for any projective $R$-modules $P$, since for any $R$-homomorphism $g : M \to P$, there is an $R$-homomorphism $h : P_0 \to P$, such that $g = (g\phi')\varphi = h(f\varphi)$. Then, it follows from the exact sequence $\text{Hom}_R(P_0, P) \to \text{Hom}_R(M, P) \to \text{Ext}^1_R(L_0, P) \to 0$ that $\text{Ext}^1_R(L_0, P) = 0$. Moreover, $M$ and $P_0$ are $n$-Gorenstein projective left $R$-modules, it follows from Lemma 2.4 that $L_0$ is an $n$-Gorenstein projective left $R$-module.

Let $F$ be any projective left $A$-module. There is a split epimorphism $\psi : A \otimes_R F \to F$ of $A$-modules given by $\psi(a \otimes_R x) = ax$ for any $a \in A$ and $x \in F$, and then there exists an $A$-homomorphism $\psi' : F \to A \otimes_R F$ such that $\psi\psi' = \text{id}_F$. Note that $F$ is also projective as an $R$-module. Then, it follows from $\text{Ext}^1_A(L_0, A \otimes_R F) \cong \text{Ext}^1_R(L_0, F) = 0$ that the exact sequence $0 \to M \xrightarrow{f\phi} P_0 \to L_0 \to 0$ remains exact after applying $\text{Hom}_A(\cdot, A \otimes_R F)$. For any $A$-homomorphism $\alpha : M \to F$, we consider the following diagram

$$
\begin{array}{ccc}
0 & \xrightarrow{f\phi} & M \\
\downarrow{\alpha} & & \downarrow{\beta} \\
F & \xrightarrow{\psi'} & A \otimes_R F \\
\end{array}
$$

For $\psi'\alpha : M \to A \otimes_R F$, there exists an $A$-map $\beta : P_0 \to A \otimes_R F$ such that $\psi'\alpha = \beta(f\varphi)$. And then, we have $\psi\beta : P_0 \to F$, such that $\alpha = (\psi\psi')\alpha = (\psi\beta)(f\varphi)$. This implies that the sequence $0 \to M \xrightarrow{f\phi} P_0 \to L_0 \to 0$ is $\text{Hom}_A(\cdot, F)$-exact.

Note that $L_0$ is an $n$-Gorenstein projective left $R$-module, and then $\text{Hom}_R(A, L_0)$ is an $n$-Gorenstein projective left $A$-module. Repeating the process we followed with $M$, we inductively construct an exact sequence $0 \to M \to P_0 \to P_1 \to P_2 \to \cdots$ of $A$-modules, with each $P_i$ projective and which is also exact after applying $\text{Hom}_A(\cdot, F)$ for any projective left $A$-module $F$. It follows from Lemma 2.2(1) that $M$ is an $n$-Gorenstein projective left $A$-module. □

3. $n$-Gorenstein projective dimensions over Frobenius extensions

In [10], Ren studied the Gorenstein projective dimensions along Frobenius extensions of rings. In this section, we consider similar conclusions for $n$-Gorenstein projective dimensions.

**Definition 3.1** Let $R$ be a ring. The $n$-Gorenstein projective dimension of a left $R$-module $M$, denote by $n\text{-Gpd}_R M$, is defined as $\inf\{m | \text{there exists an exact sequence } 0 \to G_m \to \cdots \to G_1 \to G_0 \to M \to 0 \text{ of } R\text{-modules, where } G_i \text{ is an } n\text{-Gorenstein projective left } R\text{-module} \}$. If such $m$ does not exist, then $n\text{-Gpd}_R M = \infty$. Obviously, $M$ is an $n$-Gorenstein projective left $R$-module if and only if $n\text{-Gpd}_R M = 0$.

**Lemma 3.2** ([4, Proposition 3.1]) Let $M$ be an $R$-module with finite $n$-Gorenstein projective dimension $m$. Then there exists an exact sequence $0 \to K \to G \to M \to 0$, where $G$ is $n$-Gorenstein projective and $\text{pd}_R K = m - 1$.

**Proposition 3.3** Let $0 \to K \to G \to M \to 0$ be an exact sequence of left $R$-module, where $G$ is $n$-Gorenstein projective. If $1 \leq n\text{-Gpd}_R M < \infty$, then $n\text{-Gpd}_R K = n\text{-Gpd}_R M - 1$. 

**Proof** Let $1 \leq n\text{-}\text{Gpd}_R M < \infty$. On the one hand, by Lemma 3.2 and inclusion relation $P(R) \subseteq n\text{-}GP(R)$, we have an inequality $n\text{-}\text{Gpd}_R K \leq \text{pd}_R K = n\text{-}\text{Gpd}_R M - 1$.

On the other hand, let $n\text{-}\text{Gpd}_R K = s < \infty$. Then there exists an exact sequence $0 \rightarrow K_s \rightarrow K_{s-1} \rightarrow \cdots \rightarrow K_1 \rightarrow K_0 \rightarrow K \rightarrow 0$, where $K_j \in n\text{-}GP(R), j = 0, 1, \ldots, s-1, s$. There exists another exact sequence $0 \rightarrow K_s \rightarrow K_{s-1} \rightarrow \cdots \rightarrow K_1 \rightarrow K_0 \rightarrow G \rightarrow M \rightarrow 0$. So, we have an inequality $n\text{-}\text{Gpd}_R M \leq s + 1 = n\text{-}\text{Gpd}_R K + 1$, i.e., $n\text{-}\text{Gpd}_R K \geq n\text{-}\text{Gpd}_R M - 1$. □

**Proposition 3.4** Let $R$ be a ring. If $(M_i)_{i \in I}$ is any family of left $R$-module, then we have an equality,

$$n\text{-}\text{Gpd}_R (\oplus_{i \in I} M_i) = sup\{n\text{-}\text{Gpd}_R M_i | i \in I\}.$$

**Proof** The inequality ‘$\leq$’ is clear since $n\text{-}GP(R)$ is closed under direct sums by Lemma 2.3. For the converse inequality ‘$\geq$’, it suffices to show that if $M_1$ is any direct summand of an $R$-module $M$, then $n\text{-}\text{Gpd}_R M_1 \leq n\text{-}\text{Gpd}_R M$. Naturally we may assume that $n\text{-}\text{Gpd}_R M = m$ is finite, and then proceed by induction on $m$.

The induction start is clear, because if $M$ is $n$-Gorenstein projective, then so is $M_1$, by Lemma 2.3. If $m \geq 1$, we write $M = M_1 \oplus M_2$ for some module $M_2$. Suppose that when we have equality $n\text{-}\text{Gpd}_R^M = m - 1$, there is an inequality $n\text{-}\text{Gpd}_R^M \geq n\text{-}\text{Gpd}_R M_1$. Naturally we have $n\text{-}\text{Gpd}_R M_1 < \infty$, where $i = 1, 2$. By Lemma 3.2, there are exact sequences $0 \rightarrow K_1 \rightarrow G_1 \rightarrow M_1 \rightarrow 0$ and $0 \rightarrow K_2 \rightarrow G_2 \rightarrow M_2 \rightarrow 0$ of left $R$-modules, where $G_1$ and $G_2$ are $n$-Gorenstein projective. We get commutative diagram with split-exact rows.

![Diagram 1](image)

In Diagram 1, $G_1 \oplus G_2$ is $n$-Gorenstein projective. Applying Proposition 3.3 to the middle column in Diagram 1, we get that

$$n\text{-}\text{Gpd}_R^M (K_1 \oplus K_2) = n\text{-}\text{Gpd}_R^M (M_1 \oplus M_2) - 1 = m - 1.$$ 

Hence the induction hypothesis yields that $n\text{-}\text{Gpd}_R^M K_1 \leq m - 1$, and thus the short exact sequence $0 \rightarrow K_1 \rightarrow G_1 \rightarrow M_1 \rightarrow 0$ shows that $n\text{-}\text{Gpd}_R^M M_1 \leq m$, as desired. □
\textbf{Proposition 3.5} Let \( R \subset A \) be a Frobenius extension of rings. For any left \( R \)-module \( M \), if \( n\text{-Gpd}_RM < \infty \), then
\[ n\text{-Gpd}_RM = n\text{-Gpd}_A(A \otimes_R M) = n\text{-Gpd}_R(A \otimes_R M). \]

\textbf{Proof} It follows from Proposition 2.7 that \( n\text{-Gpd}_R(A \otimes_R M) \leq n\text{-Gpd}_A(A \otimes_R M) \). For any \( n\)-Gorenstein projective left \( R \)-module \( M \), it follows from Proposition 2.8 that \( A \otimes_R M \) is an \( n\)-Gorenstein projective left \( A \)-module. Then \( n\text{-Gpd}_A(A \otimes_R M) \leq n\text{-Gpd}_R M \). As \( R \)-modules, \( M \) is a direct summand of \( A \otimes_R M \). It follows immediately from Proposition 3.4 that \( n\text{-Gpd}_RM \leq n\text{-Gpd}_R(A \otimes_R M) \). Hence, we get the desired equality. \( \Box \)

\textbf{Definition 3.6} ([11, Definition 2.8]) A ring extension \( R \subset A \) is separable provided that the multiplication map \( \varphi : A \otimes_R A \to A(a \otimes_R b \mapsto ab) \) is a split epimorphism of \( A \)-bimodules. If \( R \subset A \) is simultaneously a Frobenius and separable extension, then it is called a separable Frobenius extension.

\textbf{Example 3.7} (1) ([11, Example 2.10]) For a finite group \( G \), the integral group ring extension \( \mathbb{Z} \subset \mathbb{Z}G \) is a separable Frobenius extension.

(2) ([9, Example 2.7]) Let \( F \) be a field and set \( A = M_4(F) \). Let \( R \) be the subalgebra of \( A \) with \( F \)-basis consisting of the idempotents and matrix units \( e_1 = e_{11} + e_{44}, e_2 = e_{22} + e_{33}, e_2, e_3, e_4, e_5, e_{45} \). Then \( R \subset A \) is a separable Frobenius extension.

\textbf{Lemma 3.8} ([11, Lemma 2.9]) The following are equivalent:

(1) \( R \subset A \) is a separable extension.

(2) For any \( A \)-bimodule \( M \), \( \theta : A \otimes_R M \to M \) is a split epimorphism of \( A \)-bimodules.

\textbf{Proposition 3.9} Let \( R \subset A \) be a separable Frobenius extension of rings. For any left \( A \)-module \( M \), if \( n\text{-Gpd}_AM < \infty \), then \( n\text{-Gpd}_AM = n\text{-Gpd}_RM \).

\textbf{Proof} By Proposition 2.7, any \( n\)-Gorenstein projective left \( A \)-module is also \( n\)-Gorenstein projective left \( R \)-module. It is easy to see that \( n\text{-Gpd}_RM \leq n\text{-Gpd}_AM < \infty \). For the converse, we can assume that \( n\text{-Gpd}_RM = m < \infty \), then there exists an exact sequence \( 0 \to G_m \to G_{m-1} \to \cdots \to G_1 \to G_0 \to M \to 0 \) of \( R \)-modules, where \( G_i \) is \( n\)-Gorenstein projective. By Proposition 2.8, \( A \otimes_R G_i \) is \( n\)-Gorenstein projective left \( A \)-modules, where \( i = 0, 1, \ldots, m - 1, m \).

Then there exists an exact sequence \( 0 \to A \otimes_R G_m \to A \otimes_R G_{m-1} \to \cdots \to A \otimes_R G_1 \to A \otimes_R G_0 \to A \otimes_R M \to 0 \) of left \( A \)-modules. Then \( n\text{-Gpd}_A(A \otimes_R M) \leq m \). By Lemma 3.8, left \( A \)-module \( M \) is direct summand of \( A \otimes_R M \). By Proposition 3.4, we have inequalities \( n\text{-Gpd}_AM \leq n\text{-Gpd}_A(A \otimes_R M) \leq m \). \( \Box \)

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\textbf{References}


