Dynamical Analysis of Toxin Producing Model with Delay

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Abstract In this paper, a toxin producing phytoplankton-zooplankton model with inhibitory substrate and time delay is investigated. A discrete time delay is induced to both of the consume response function and distribution of toxic substance term. Moreover, Tissiet type function is used for zooplankton grazing to account for the effect of toxication by the TPP population. The conditions to guarantee the coexistence of two species and stability of coexistence equilibrium are given. In particular, we show that there exist critical values of the delay parameters below which the coexistence equilibrium is stable and above which it is unstable. Hopf bifurcation occurs when the delay parameters cross their critical values. Some numerical simulations are executed to validate the analytical findings.

Keywords phytoplankton; zooplankton; toxin; delay; stability

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1. Introduction

As we know, plankton are microscopic organisms that float freely with oceanic currents and in other bodies of water. They are made up of tiny plants-phytoplankton and tiny animals-zooplankton. Phytoplankton is not only the basis for all aquatic food chain but also perform a very useful service by producing a huge mount of oxygen for human and other living animals after absorbing carbon-dioxide from surrounding environments [1]. Zooplankton are microscopic animals that eat other plankton and serve as a most favorable food source for fish and other aquatic animals. Plankton is the first link in the marine food chain; it is eaten by many organisms, such as fish, mussels and mammals (for example, baleen whales). In addition, some phytoplankton (e.g., nori, kelp and eucheuma) and some zooplankton (e.g., jellyfish, krill and acetes) are harvested for food.

Because of the difficulty in measuring plankton biomass, mathematical modeling of plankton population is an important alternative method of improving our knowledge of the physical and biological processes relating to plankton ecology [2–9]. Das and Ray [2] studied the effect of delay on nutrient cycling in phytoplankton-zooplankton interactions in the estuarine system. In [3], two plankton ecosystem models with explicit representation of viruses and virally infected phytoplankton are presented. Models of nutrient-plankton interaction with a toxic substance
that inhibits either the growth rate of phytoplankton, zooplankton or both trophic levels are proposed and studied by authors in [4]. In [5], authors have investigated a nutrient-plankton model in an aquatic environment in the context of phytoplankton bloom.

The reduction of grazing pressure due to the toxin is an important phenomena in the plankton ecology [10, 11]. In the aquatic system, the toxin-producing phytoplankton (TPP) may act as the controlling factor in the phytoplankton-zooplankton interaction dynamics. Saha and Bandyopadhyay [6] considered a TPP model in which the toxin liberation by phytoplankton species follows a discrete time variation. In [7], the effect of seasonality and periodicity on plankton dynamics is investigated. In [8], the authors have constructed a mathematical model for describing the interaction between a nontoxic and toxic phytoplankton with a single nutrient. In [12], authors considered a mathematical model consisting of two harmful phytoplankton and zooplankton where the mortality of zooplankton due to the liberation of toxic substances by harmful phytoplankton occurs after an incubation time. Chattopadhyay et al. [9] proposed a mathematical model for the TPP and zooplankton interaction and explored the role of the TPP behind harmful algal blooms. The general form system considered in [9] is as follows

\[ \begin{align*}
\frac{dP}{dt} &= rP(t)(1 - \frac{P(t)}{K}) - \beta f(P(t))Z(t), \\
\frac{dZ}{dt} &= \beta_1 f(P(t))Z(t) - \delta Z(t) - \rho g(P(t))Z(t)
\end{align*} \] (1.1)

where the variable \( P(t) \) is the density of TPP population and \( Z(t) \) is the density of the zooplankton at any instant of time \( t \). The parameter \( r (> 0) \) is the intrinsic growth rate and \( K \) is the environmental carrying capacity of the TPP population. The constant \( \beta (> 0) \) is the maximum uptake rate for the zooplankton species. \( \beta_1 (> 0) \) denotes the ratio of biomass conversion (satisfying the obvious restriction \( 0 < \beta_1 < \beta \)) and \( D (> 0) \) is the natural death rate of the zooplankton. The parameter \( \rho (> 0) \) denotes the rate of the toxic substances produced by per unite biomass of the phytoplankton population. The term \( f(P) \) describes the functional response for the grazing of phytoplankton by zooplankton and \( g(P) \) represents the distribution of toxic substance which ultimately contributes to the death of zooplankton population.

A system can be more realistic if a delay effect is considered in the conversion from one species to another species [13–15]. The delayed differential system exhibits much more rich dynamics than ordinary differential systems. In the phytoplankton-zooplankton system, the release of toxin from phytoplankton species is not an instantaneous process but is mediated by some time lag required for maturity of the species, and the zooplankton may die after some time lapse of the bloom of toxic phytoplankton (see http://www.mote.org/, http://www.mdsg.umd.edu/). Therefore, for different Holling type of \( f(p) \) and \( g(p) \), the model (1.1) was extensively studied by many authors [9,12–20]. Another fact is that very high substrate concentrations in the lakes actually inhabit the growth of phytoplankton cells. Moreover, with the substrate concentrations increasing unlimitedly, some kind of microorganism will die eventually [16]. Based on the above fact and the model 3 in [13], we intend to study a model system with the assumption that \( f(p) \) and \( g(p) \) are described by same type of function, namely, Tissiet functional response. Besides,
we also consider the harvesting for the zooplankton. Then our model takes the following form:

\[
\begin{aligned}
\frac{dP}{dt} &= rP(t)(1 - \frac{P(t)}{K}) - \frac{\beta P(t)Z(t)}{\gamma_1 + P(t)} e^{-\frac{P(t)}{K}}, \\
\frac{dZ}{dt} &= \frac{\beta_1 P(t - \tau_1)Z(t - \tau_1)}{\gamma_1 + P(t - \tau_1)} e^{-\frac{P(t - \tau)}{K}} - DZ(t) - \frac{\rho P(t - \tau_2)Z(t - \tau_2)}{\gamma_2 + P(t - \tau_2)} e^{-\frac{P(t - \tau)}{K}} - EZ(t).
\end{aligned}
\]

(1.2)

In model (1.2), the constant \(\tau_1 (\tau_1 > 0)\) denotes gestation delay of the zooplankton and \(\tau_2 (\tau_2 > 0)\) is the delay required for the maturity of the TPP. In general case, the gestation delay of the zooplankton is different from the delay which is required for the maturity of the TPP, but in this paper the special case is considered, that is \(\tau_1 = \tau_2\) (the case \(\tau_1 \neq \tau_2\) will be discussed in the future). \(\gamma_1\) and \(\gamma_2\) are the half-saturation constants. The constant \(E\) is the harvesting effort.

In order to reduce the number of the parameters, we perform the following scaling for model (1.2):

\[
\begin{aligned}
t &= rt, \quad \bar{P}(t) = \frac{P(t)}{K}, \quad \bar{Z}(t) = \frac{\beta Z(t)}{rK}, \quad \bar{\tau}_1 = \frac{\gamma_1}{K}, \quad \bar{\tau}_2 = \frac{\gamma_2}{K}, \quad \bar{\beta}_1 = \frac{\beta_1}{r}, \quad \bar{\rho} = \frac{\rho}{r}, \quad \bar{\tau} = \tau r, \quad \bar{D} = \frac{D}{r}, \quad \bar{E} = \frac{E}{r}.
\end{aligned}
\]

(1.3)

Then remove the “\(\bar{\cdot}\)”, and denote \(\tau_1 = \tau_2 = \tau\), one can rewrite the system (1.2) as

\[
\begin{aligned}
\frac{dP}{dt} &= P(t)(1 - P(t)) - \frac{P(t)Z(t)}{\gamma_1 + P(t)} \exp(-P(t)), \\
\frac{dZ}{dt} &= \frac{\beta_1 P(t - \tau)Z(t - \tau)}{\gamma_1 + P(t - \tau)} \exp(-P(t - \tau)) - DZ(t) - \frac{\rho P(t - \tau)Z(t - \tau)}{\gamma_2 + P(t - \tau)} \exp(-P(t - \tau)) - EZ(t).
\end{aligned}
\]

(1.4)

In accordance with the biological meaning, we are only interested in the dynamics of model (1.4) in the closed first quadrant \(\mathbb{R}^2_+\), and the initial conditions of the system (1.4) are taken as follows,

\[
P(\theta) = \varphi_1(\theta) \geq 0, \quad Z(\theta) = \varphi_2(\theta) \geq 0, \quad \theta \in [-\tau, 0],
\]

(1.5)

where \((\varphi_1, \varphi_2) \in C([-\tau, 0], \mathbb{R}^2_+)\).

For system (1.4), we make an assumption that \(\beta_1 > \rho\), that is, the ratio of biomass consumed by zooplankton is greater than the rate of toxic substance liberation by phytoplankton species.

2. Stability analysis

In this section, we give a qualitative analysis of system (1.4).

2.1. Positive and Boundedness

Regarding the positive and boundedness of the solution for the model system (1.4) we state the following theorem:

**Theorem 2.1** All the solutions \((P(t), Z(t))\) of system (1.4) with the initial condition (1.5) are
positive if \( \gamma_2 \geq \gamma_1 \) and satisfy
\[
\lim_{t \to +\infty} \sup_P(t) \leq 1, \quad \lim_{t \to +\infty} \sup_Z(t) \leq M,
\]
where \( M = \frac{\beta_1}{4(D + E)}(D + E + 1)^2 \). If \( \beta_1 < \frac{4(D + E)\gamma_1}{(D + E + 1)^2} \) holds, then \( \lim_{t \to +\infty} P(t) \geq m_0 \), \( m_0 = 1 - \frac{M}{\gamma_1} \). Further, the subset
\[
\Gamma = \{(\varphi_1(\theta), \varphi_2(\theta)) \in C | m_0 \leq \varphi_1(\theta) \leq 1, 0 < \varphi_2(\theta) \leq M \}
\]
is positively invariant with respect to (1.4).

**Proof** Suppose that \((P(t), Z(t))\) is a solution of system (1.4) for \( t \in [0, \sigma] \). Integrating the first equation of (1.4) gives
\[
P(t) = \varphi_1(0) \exp \left[ \int_0^t (1 - P(u)) - \frac{Z(u)}{\gamma_1 + P(u)} \exp(-P(u)) \right] > 0.
\]
To prove the \( Z(t) \geq 0 \) for any \( t \in [0, \sigma] \), using the method of contradiction. Suppose there exists a \( \hat{t} \in [0, \sigma] \) such that \( Z(\hat{t}) = 0 \), \( Z'(\hat{t}) \leq 0 \), \( Z(t) > 0 \) for any \( t \in [0, \hat{t}] \).

From the second equation of system (1.4), we have
\[
Z'(\hat{t}) = \frac{\beta_1 P(\hat{t} - \tau) Z(\hat{t} - \tau)}{\gamma_1 + P(\hat{t} - \tau)} \exp(-P(\hat{t} - \tau)) - \frac{\rho P(\hat{t} - \tau)}{\gamma_2 + P(\hat{t} - \tau)} \exp(-P(\hat{t} - \tau))
\]
\[
= \left( \frac{\beta_1}{\gamma_1 + P(\hat{t} - \tau)} - \frac{\rho}{\gamma_2 + P(\hat{t} - \tau)} \right) P(\hat{t} - \tau) Z(\hat{t} - \tau) \exp(-P(\hat{t} - \tau))
\]
\[
= \frac{\beta_1 \gamma_2 - \rho \gamma_1 + (\beta_1 - \rho) P(\hat{t} - \tau) \gamma_2 + P(\hat{t} - \tau)}{(\gamma_1 + P(\hat{t} - \tau))(\gamma_2 + P(\hat{t} - \tau))} P(\hat{t} - \tau) Z(\hat{t} - \tau) \exp(-P(\hat{t} - \tau))
\]
\[
> 0.
\]
This contradicts \( Z(\hat{t}) \leq 0 \). Hence \( Z(t) > 0 \) for any \( t \in [0, \sigma] \).

For \( t \in [0, \sigma] \), let \((P(t), Z(t))\) be any solution of the system (1.4). It follows from the first equation of (1.4) that \( P'(t) \leq P(t)(1 - P(t)) \), which implies that \( \lim_{t \to +\infty} P(t) \leq 1 \). Define \( W(t) = Z(t) + \beta_1 P(t - \tau), t \geq 0 \), then it follows from the system (1.4) that
\[
W'(t) = -D + E)W(t) + (D + E)\beta_1 P(t - \tau) + \beta_1 P(t - \tau)(1 - P(t - \tau)) - \frac{\rho P(t - \tau) Z(t - \tau)}{\gamma_1 + P(t - \tau)}
\]
\[
\leq -(D + E)W(t) + \frac{\beta_1 (D + E + 1)^2}{4}.
\]
Therefore, by the comparison theorem, we have
\[
W(t) \leq W(0) \exp(-(D + E)t) + \frac{\beta_1}{4(D + E)}(D + E + 1)^2(1 - \exp(-(D + E)t)).
\]
From this we obtain that as \( t \to +\infty \), then \( 0 \leq W(t) \leq \frac{\beta_1}{4(D + E)}(D + E + 1)^2 \). Hence, \( \lim_{t \to +\infty} Z(t) \leq M, \) where \( M = \frac{\beta_1}{4(D + E)}(D + E + 1)^2 \). From the first equation of system (1.4), we get that for \( t \) large enough,
\[
P'(t) \geq P(t)(1 - P(t)) - \frac{M}{\gamma_1} P(t) = P(t)(1 - \frac{M}{\gamma_1} - P(t)),
\]
which implies that \( \lim_{t \to +\infty} P(t) = m_0, \) where \( m_0 = 1 - \frac{M}{\gamma_1}. \)
2.2. The existence of equilibria

It is easy to see that model (1.4) has two boundary equilibria $E_0 = (0, 0)$ and $E_1 = (1, 0)$. The positive equilibria of the system (1.4) satisfies the following equations:

$$
\begin{align*}
  P(1 - P) - \frac{PZ}{\gamma_1 + P} e^{-P} &= 0, \\
  \frac{\beta_1 P}{\gamma_1 + P} e^{-P} - \frac{\rho P}{\gamma_2 + P} e^{-P} - (D + E) &= 0.
\end{align*}
\tag{2.2}
$$

Denote the left-hand side of the second equation in (2.2):

$$
F(P) = \frac{\beta_1 P}{\gamma_1 + P} e^{-P} - \frac{\rho P}{\gamma_2 + P} e^{-P} - (D + E).
\tag{2.3}
$$

Then $F(0) = -(D + E) < 0$, $\lim_{P \to \infty} F(P) = -(D + E)$.

$$
F'(P) = \frac{\beta_1 \gamma_1 - \beta_1 \gamma_1 P - \beta_1 P^2}{(\gamma_1 + P)^2} e^{-P} - \frac{\rho \gamma_2 - \rho \gamma_2 P - \rho P^2}{(\gamma_2 + P)^2} e^{-P}
= -\frac{e^{-P}}{(\gamma_1 + P)^2(\gamma_2 + P)^2} G(P),
$$

where $G(P) = AP^4 + BP^3 + CP^2 + D_1 P + E_1$, $A = \beta_1 - \rho$, $B = \beta_1 \gamma_1 + 2\beta_1 \gamma_2 - \rho \gamma_2 - 2\gamma_2 \rho$, $C = -\beta_1 \gamma_1 + 2\beta_1 \gamma_1 \gamma_2 + \beta_1 \gamma_2^2 + \rho \gamma_2 - 2\rho \gamma_1 \gamma_2 - \rho \gamma_1^2$, $D_1 = \beta_1 \gamma_1 \gamma_2^2 - 2\beta_1 \gamma_1 \gamma_2 + 2\rho \gamma_1 \gamma_2 - \rho \gamma_1^2 \gamma_2$, $E_1 = \rho \gamma_2 \gamma_1^2 - \beta_1 \gamma_1 \gamma_2^2$.

Since $\beta_1 > \rho$, $\gamma_2 \geq \gamma_1$, it is easy to see that $A > 0$, $B > 0$ and $E_1 < 0$. Let $f(P) = AP^4 + BP^3 + CP^2$, $g(P) = -D_1 P - E_1$. Then $f(P) = g(P)$ implies $G(P) = 0$. We have $f(0) = f(P_01) = f(P_02) = 0$, where

$$
P_{01} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}, \quad P_{02} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}.
$$

According to the expression of $f(P)$, we obtain $f'(P) = P(4AP^2 + 3BP + 2C)$ and the roots of $f'(P) = 0$ are

$$
P_1 = \frac{-3B - \sqrt{(3B)^2 - 32AC}}{8A}, \quad P_2 = \frac{-3B + \sqrt{(3B)^2 - 32AC}}{8A}.
$$

We also know that $f''(P) = 2(6AP^2 + 3BP + C)$. From $f''(P) = 0$ we obtain

$$
P_{11} = \frac{-3B - \sqrt{(3B)^2 - 24AC}}{12A}, \quad P_{22} = \frac{-3B + \sqrt{(3B)^2 - 24AC}}{12A}.
$$

Case 1. $C < 0$.

In this case, $P_{01} < P_1 < P_{11} < 0 < P_{22} < P_2 < P_{02}$. From this we know that the function $f(P)$ is monotone decreasing and convex in the interval $(-\infty, P_1) \cup [P_{22}, P_2]$, monotone increasing and convex in the interval $[P_1, P_{11}] \cup [P_2, +\infty)$, monotone increasing and concave in the interval $[P_{11}, 0]$, monotone decreasing and concave in the interval $[0, P_{22}]$.

(i) Suppose $D_1 < 0$. Let $f_1$ and $f_2$ represent the straight lines which are across the point $(0, -E_1)$ and tangent to curve $f(P)$ at point $Q_{ij}$ and $Q_{21}$, respectively. The straight line $f_i$ and curve $f(P)$ intersect at three points $Q_{ij} (i = 1, 2, j = 1, 2, 3)$. The straight line, which is across $(0, -E_1)$ and its slope $-D_1$ satisfies $f_1(Q_{12}) < -D_1 < f_2(Q_{21})$, and curve $f(P)$ intersect at four points. The straight line, which is across $(0, -E_1)$ and its slope $-D_1$ is smaller than $f_1(Q_{12})$ or...
larger than \( f_2(Q_{21}) \), and curve \( f(P) \) intersect at two points. Therefore, if \( C < 0 \), \( D_1 < 0 \), then \( g(P) \) and \( f(P) \) intersect at least two points in which only one intersect point, says \( Q_0(a_0, b_0) \) satisfies \( a_0 > 0 \). If \( P < a_0 \), then \( f(P) - g(P) < 0 \) and \( f(P) - g(P) > 0 \) if \( P > a_0 \). Hence \( a_0 < 1 \) (Figure 1 (a)).

(ii) Suppose \( D_1 > 0 \). In this case, by the similar argument as in case (i), we can conclude that \( g(P) \) and \( f(P) \) intersect at least at two points and all their positive abscissas are smaller than 1 (Figure 1 (b)).

\[ \begin{align*}
\text{Case II.} & \quad C > 0 \\
\text{(i) Suppose that } & \quad (B)^2 - 4AC > 0. \text{ In this case, we see that } P_{01} < P_{02} < 0, P_1 < P_2 < 0 \\
& \quad \text{and } P_{11} < P_{22} < 0. \text{ It is easy to verify that } P_{01} < P_1 < P_{11} < P_{02} < P_2 < P_{22} < 0. \text{ The function } f(P) \text{ is monotone decreasing in the interval } (-\infty, P_1] \cup [P_2, 0], \text{ monotone increasing in the interval } [P_1, P_2] \cup [0, +\infty). \text{ Thus, in both case } D_1 > 0 \text{ and } D_1 < 0, \text{ the straight line } g(P) \text{ and the curve } f(P) \text{ intersect at least at two points and in which only one abscissa of intersection point is positive and smaller than 1 (Figure 2 (a)).} \\
\text{(ii) Suppose that } & \quad (B)^2 - 4AC = 0. \text{ Then } f(P) = 0 \text{ has two roots } P = 0 \text{ and } P_{01} = P_{02} = -\frac{B}{2A} < 0, P_1 < P_2 < 0. \text{ It is easy to verify that } P_{01} = P_{02} = P_1 < P_2 < 0. \text{ The function } f(P) \text{ is monotone decreasing in the interval } (-\infty, P_1] \cup [P_2, 0], \text{ monotone increasing in the interval } [P_1, P_2] \cup [0, +\infty). \text{ Thus, in both case } D_1 > 0 \text{ and } D_1 < 0, \text{ the straight line } g(P) \text{ and the curve } f(P) \text{ intersect at least at two points and in which only one abscissa of intersection point is positive and smaller than 1 (Figure 2 (b)).} \\
\text{(iii) Suppose that } & \quad (B)^2 - 4AC < 0 \text{ and } (3B)^2 - 32AC > 0. \text{ Then } f(P) = 0 \text{ has only one root } P = 0. \text{ In this case, we also have } P_1 < P_2 < 0 \text{ and } P_{11} < P_{22} < 0. \text{ It is easy to verify that } P_1 < P_{11} < P_2 < P_{22} < 0. \text{ The function } f(P) \text{ is monotone decreasing in the interval } (-\infty, P_1] \cup [P_2, 0], \text{ monotone increasing in the interval } [P_1, P_2] \cup [0, +\infty). \text{ Thus, in both case } D_1 > 0 \text{ and } D_1 < 0, \text{ the straight line } g(P) \text{ and the curve } f(P) \text{ intersect at least at two points and in which only one abscissa of intersection point is positive and smaller than 1 (Figure 2 (c)).} \\
\text{(iv) Suppose that } & \quad (3B)^2 - 32AC = 0. \text{ Then } f(P) = 0 \text{ has only one root } P = 0, f'(P) \text{ has two roots } P = 0 \text{ and } P_1 = P_2 = -\frac{3B}{2A} < 0. \text{ It is easy to verify that } P_1 = P_2 = P_{11} < P_{22} < 0. \text{ The}
\end{align*} \]
function $f(P)$ is monotone decreasing and convex in the interval $(-\infty, P_{11}] \cup [P_{22}, 0]$, monotone decreasing and concave in the interval $[P_{11}, P_{22}]$, monotone increasing and convex in the interval $[0, +\infty)$. Thus, in both case $D_1 > 0$ and $D_1 < 0$, the straight line $g(P)$ and the curve $f(P)$ intersect at least at two points and in which only one abscissa of intersection point is positive and smaller than 1 (Figure 2 (d)).

![Graphs](image)

Figure 2 The graph of $f(P)$ and $g(P)$ intersect for $C > 0$

Discussion on case (i)–(iv) indicates that in both case $D_1 > 0$ and $D_1 < 0$, $g(P)$ and $f(P)$ intersect at least at two points in which only one abscissa of intersection point is positive and smaller than 1.

(v) Suppose that $(3B)^2 - 32AC < 0$ and $(3B)^2 - 24AC > 0$. Then both $f(P) = 0$ and $f'(P)$ have only one root $P = 0$. The roots of $f''(P) = 0$ satisfy $P_{11} < P_{22} < 0$. By this we know that the function $f(P)$ is monotone decreasing in the interval $(-\infty, 0]$ and monotone increasing in the interval $[0, +\infty)$. Thus, in both case $D_1 > 0$ and $D_1 < 0$, $g(P)$ and $f(P)$ intersect at two points in which only one abscissa of intersection point is positive and smaller than 1 (Figure 2 (e)).
(vi) Suppose $(3B)^2 - 24AC = 0$. Then both $f(P) = 0$ and $f'(P) = 0$ have only one root $P = 0$. $f''(P) = 0$ has only one root $P = \frac{-B}{14A} < 0$. By this we know that the function $f(P)$ is monotone decreasing in the interval $(-\infty, 0]$ and monotone increasing in the interval $[0, +\infty)$. Thus, in both case $D_1 > 0$ and $D_1 < 0$, $g(P)$ and $f(P)$ intersect at two points in which only one abscissa of intersection point is positive and smaller than 1 (Figure 2 (e)).

(vii) Suppose $(3B)^2 - 24AC < 0$. Then both $f(P) = 0$ and $f'(P) = 0$ have only one root $P = 0$. $f''(P) > 0$ for $p \in (-\infty, +\infty)$. By this we know that the function $f(P)$ is monotone decreasing in the interval $(-\infty, 0]$ and monotone increasing in the interval $[0, +\infty)$. Thus, in both case $D_1 > 0$ and $D_1 < 0$, $g(P)$ and $f(P)$ intersect at two points in which only one abscissa of intersection point is positive and smaller than 1 (Figure 2 (e)).

Discussion on case (v)-(vii) indicates that in both case $D_1 > 0$ and $D_1 < 0$, $g(P)$ and $f(P)$ intersect at two points in which only one abscissa of intersection point is positive and smaller than 1.

Figure 3  The graph of function $F(P)=0$
From above analysis we know that whether $C > 0$ and $C < 0$, in case $D_1 < 0$, the straight line $g(P)$ and curve $f(P)$ intersect at only one positive point. For the simplicity, in what follows, we assume that $D_1 < 0$. Then based on the above discussion, there is only one point $Q_0(a_0, b_0)$, which is an intersection point of $f(P)$ and $g(P)$, satisfying $0 < a_0 < 1$, that is $G(P) = 0$ has only one root in the interval $(0, 1)$ which implies that the function $F(P)$ is monotone increasing in the interval $[0, a_0]$, and monotone decreasing in the interval $[a_0, +\infty)$. By this, we obtain the following results:

**Theorem 2.2** Assume that $\gamma_2 \geq \gamma_1$ and $D_1 < 0$. Then the following statements hold.

1) If $F(a_0) < 0$, then equation $F(P) = 0$ has no roots in the interval $[0, +\infty)$. In this case, for system (1.4), there is no positive equilibrium (Figure 3 (a)).

2) If $F(a_0) > 0$ and $F(1) > 0$, then equation $F(P) = 0$ has only one root, $P = P_1^*$, in the interval $(0, 1)$, where $0 < P_1^* < 1$. In this case, for system (1.4), there is a single positive equilibrium $E^*(P_1^*, Z_1^2)$ (Figure 3 (b)).

3) If $F(a_0) > 0$ and $F(1) < 0$, then equation $F(P) = 0$ has two different roots, $P = P_2^*$ and $P = P_3^*$, in the interval $(0, 1)$, where $0 < P_2^* < a_0 < P_3^* < 1$. In this case, for system (1.4), there are two distinct positive equilibria $E^*(P_2^*, Z_2^2)$ and $E^*(P_3^*, Z_3^2)$ (Figure 3 (c)).

4) If $F(a_0) = 0$, then equation $F(P) = 0$ has a unique root, $P_4^* = a_0$, in the interval $(0, 1)$, where $0 < P_4^* = a_0 < 1$. In this case, for system (1.4), there is a unique positive equilibrium $E^*(P_4^*, Z_4^2)$ (Figure 3 (d)).

5) If $F(a_0) > 0$ and $F(1) = 0$, then equation $F(P) = 0$ has only one root, $P = P_5^*$, in the interval $(0, 1)$. In this case, for system (1.4), there is also a unique positive equilibrium $E^*(P_5^*, P_5^2)$ (Figure 3 (e)).

2.3. Local stability and Hopf bifurcation

It is easy to know that $E_0$ is always a saddle point. Let $\tilde{E}(\tilde{P}, \tilde{Z})$ denote any one of the equilibrium point of system (1.4). Linearizing (1.4) about $\tilde{E}$ gives the following equation:

$$
P'(t) = [1 - 2\tilde{P} - \frac{\gamma_1 \tilde{Z}}{(\gamma_1 + P)^2}]P(t) - \frac{\tilde{Z}}{\gamma_1 + P}e^{-\tilde{P}}Z(t),$$

$$
Z'(t) = [\frac{\beta_1 \gamma_1 \tilde{Z}}{(\gamma_1 + P)^2} - \frac{\beta_1 \tilde{P}^2 \tilde{Z}}{\gamma_1 + P} - \frac{\rho \gamma_1 \tilde{P}^2 \tilde{Z}}{\gamma_2 + P} + \frac{\rho \tilde{P}^2 \tilde{Z}}{\gamma_2 + P}]e^{-\tilde{P}}P(t - \tau) + (\frac{\beta_1}{\gamma_1 + P} - \frac{\rho}{\gamma_2 + P})\tilde{P}e^{-\tilde{P}}Z(t - \tau) - (D + E)Z(t).$$

(2.4)

The characteristic equation of system (2.4) at equilibrium point $\tilde{E}(\tilde{P}, \tilde{Z})$ is

$$
\lambda^2 + b_0 \lambda + b_1 + [b_2 - (\frac{\beta_1}{\gamma_1 + P} - \frac{\rho}{\gamma_2 + P})\tilde{P}e^{-\tilde{P}}\gamma_1]e^{-\lambda \tau} = 0,
$$

(2.5)

where $b_0 = D + E - \frac{\tilde{P}}{\gamma_1 + P}L(\tilde{P}), b_1 = -(D + E)\frac{\tilde{P}}{\gamma_1 + P}L(\tilde{P}), b_2 = \frac{\tilde{P}}{\gamma_1 + P}L(\tilde{P})(\frac{\beta_1}{\gamma_1 + P} - \frac{\rho}{\gamma_2 + P})\tilde{P}e^{-\tilde{P}} - \frac{\tilde{P}(1 - \tilde{P})e^{-\tilde{P}}}{(\gamma_1 + P)^2(\gamma_2 + P)^2}G(\tilde{P}), L(\tilde{P}) = -((\tilde{P})^2 - (1 + \gamma_1)\tilde{P} + 1).

In the following, stability of the equilibria of system (1.4) is investigated.
Theorem 2.3 Assume that $\gamma_2 \geq \gamma_1$. For any time delay $\tau \geq 0$, $E_1(1,0)$ is locally asymptotically stable if $F(1) < 0$ and is unstable if $F(1) > 0$.

Proof At $E_1(1,0)$, the (2.5) becomes

$$\lambda = b + (b_0 - (D + E)) \lambda + b_1 + b_2 = 0,$$  \hspace{1cm} (2.8)

where $b_0 - (D + E) = \frac{p}{\gamma_1 + \gamma_2}, b_1 + b_2 = \frac{(1-p_i) p_i \exp(-p_i)}{(\gamma_1 + \gamma_2)}, G(p_i)$, and $L(p_i)$ for $i = 1, 2, 3, 4$. $L(p_i) > 0$ if $0 < p_i < \Delta$ and $L(p_i) < 0$ if $\Delta < p_i < 1$. $i = 1, 2, 3, 4$.

By argument in Subsection 2.2, as long as $E^*(p_i^*, z_i^*)$ exists, there must be $G(p_i^*) < 0$, $i = 1, 2$, thus $b_1 + b_2 > 0$; as long as $E^*(p_i^*, z_i^*)$ exists, there must be $G(p_i^*) > 0$, thus, $b_1 + b_2 < 0$; as long as $E^*(p_i^*, z_i^*)$ exist, $G(p_i^*) = 0$, thus, $b_1 + b_2 = 0$. Hence, according to Routh-Hurwitz theorem, we have that if $p_i^* > \Delta$, $i = 1, 2$, then $E^*(p_i^*, z_i^*)$ and $E^*(p_i^*, z_i^*)$ are locally asymptotically stable; if $0 < p_i^* < \Delta$, $i = 1, 2$, then $E^*(p_i^*, z_i^*)$ and $E^*(p_i^*, z_i^*)$ are unstable; if $p_i^* = \Delta$, $i = 1, 2$, then $E^*(p_i^*, z_i^*)$ and $E^*(p_i^*, z_i^*)$ are nonhyperbolic equilibria; $E^*(p_i^*, z_i^*)$ is unstable; $E^*(p_i^*, z_i^*)$ is a critical case.

Assume that for $\tau > 0$, $\lambda = i \omega$ ($\omega > 0$) is a root of the characteristic Eq. (2.5). Substituting it into (2.5), and separating the real and imaginary parts yields

$$
\begin{align*}
-\omega^2 + b_1 + b_2 \cos(\omega \tau) - (D + E) \omega \sin(\omega \tau) &= 0, \\
\omega b_0 - b_2 \sin(\omega \tau) - (D + E) \omega \cos(\omega \tau) &= 0,
\end{align*}
$$  \hspace{1cm} (2.9)

Theorem 2.4 For the system (1.4), we suppose that the positive equilibria following equality:

$$\omega = \frac{-\beta_1 - \gamma_1 - \gamma_2 - D}{\gamma_1 + \gamma_2}.$$  \hspace{1cm} (2.10)

One of the roots of (2.6) is $\lambda_1 = -1 < 0$, and other roots are determined by solution of the following equality:

$$\lambda = (D + E) e^{-\omega \tau} = \frac{\beta_1}{\gamma_1 + \gamma_2} = 1.$$  \hspace{1cm} (2.11)

It follows from [21] that

(i) If $F(1) < 0$, then all roots of (2.7) have negative real parts for any time delay $\tau \geq 0$.

Hence, $E_1(1,0)$ is locally asymptotically stable for any time $\tau \geq 0$.

(ii) If $F(1) > 0$, then (2.7) has roots which have positive real parts for any time delay $\tau \geq 0$.

Hence, $E_1(1,0)$ is unstable for any time $\tau \geq 0$. □
and from (2.9), we can also obtain
\[
\begin{align*}
\sin \omega T &= \frac{-\omega^2 (D + E) + b_1 \omega (D + E) + b_0 b_2 \omega}{\omega^2 (D + E)^2 + b_2^2}, \\
\cos \omega T &= \frac{b_0 \omega^2 + b_2 (D + E) \omega^2 - b_1 b_2}{\omega^2 (D + E)^2 + b_2^2}.
\end{align*}
\] (2.10)

Eliminating \( \tau \) from (2.10), we obtain
\[\omega^4 + (b_0^2 - 2b_1 - (D + E)^2)\omega^2 + b_1^2 - b_2^2 = 0.\] (2.11)

Solving (2.11) gives
\[\omega^2 = \frac{1}{2} \{(D + E)^2 + 2b_1 - b_0^2 \pm \sqrt{[b_0^2 - 2b_1 - (D + E)^2]^2 - 4(b_1^2 - b_2^2)}\}.\] (2.12)

From expression of \( b_i, i = 0, 1, 2, \) we obtain
\[
\begin{align*}
b_0^2 - 2b_1 - (D + E)^2 &= \left(\frac{p^*_i}{\gamma_i + p^*_i} L(p^*_i)\right)^2 > 0, \\
b_1^2 - b_2^2 &= [-2(D + E)\frac{p^*_i}{\gamma_i + p^*_i} L(p^*_i)] + \left(\frac{1-p^*_i}{\gamma_i + p^*_i}\right) G(p^*_i) + \left(\frac{1-p^*_i}{\gamma_i + p^*_i}\right) G(p^*_i),
\end{align*}
\]
where \( i = 1, 2, 3, 4. \)

(1) We consider the stability of \( E^*(p^*_i, z^*_i), i = 1, 2. \)

Since \( G(p^*_i) < 0, \) if \( -2(D + E)\frac{p^*_i}{\gamma_i + p^*_i} L(p^*_i) + \left(\frac{1-p^*_i}{\gamma_i + p^*_i}\right) G(p^*_i) < 0, \) then Eq. (2.12) possesses a unique positive root, say, \( \omega_+, \omega_+ > 0, \) such that
\[\omega^2 = \frac{1}{2} \{(D + E)^2 + 2b_1 - b_0^2 \pm \sqrt{[b_0^2 - 2b_1 - (D + E)^2]^2 - 4(b_1^2 - b_2^2)}\}.\] (2.13)

That is, (2.5) has one imaginary solution, \( \lambda = i\omega, \omega_+ > 0. \) By (2.11) we obtain the following set of \( \tau \) for which there are imaginary roots:
\[
\tau = \frac{1}{\omega_+} \arctan \left(\frac{-\omega_+^2 (D + E) + b_1 \omega_+(D + E) + b_0 b_2 \omega_+}{b_2 \omega_+^2 + b_0 (D + E) \omega_+^2 - b_1 b_2} + \frac{2k\pi}{\omega_+}, i = 1, 2, k = 0, 1, 2, \ldots, (2.14)\]

If \( p^*_i > \Delta, \) then \( L(p^*_i) < 0, \) when \( \tau = 0, \) \( E^*(p^*_i, z^*_i), i = 1, 2 \) are locally asymptotically stable.

Hence, if \( p^*_i > \Delta \) and \( -2(D + E)\frac{p^*_i}{\gamma_i + p^*_i} L(p^*_i) + \left(\frac{1-p^*_i}{\gamma_i + p^*_i}\right) G(p^*_i) < 0, \) then \( E^*(p^*_i, z^*_i) \) are locally asymptotically stable for \( \tau < \tau_0 \) \( (k = 0, i = 1, 2). \)

Differentiating (2.5) with respect to \( \tau, \) we get
\[(2\lambda + b_0 - (D + E)e^{-\lambda \tau} - (b_2 - (D + E)\lambda)e^{-\lambda \tau}) \frac{d\lambda}{d\tau} = (b_2 - (D + E)\lambda)\lambda e^{-\lambda \tau}.\] (2.15)

Then
\[\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda + b_0)e^{\lambda \tau} - (D + E)\lambda}{(b_2 - (D + E)\lambda)\lambda} = \frac{\tau}{\lambda}.\] (2.16)

It follows from (2.16) that
\[
\text{Re}[\left(\frac{d\lambda}{d\tau}\right)^{-1}] = \text{Re}\left[\frac{(2\lambda + b_0)e^{\lambda \tau} - (D + E)}{(b_2 - (D + E)\lambda)\lambda}\right] = \text{Re}\left[\frac{1}{b_2 \omega_+^2 + (D + E)^2 \omega_+^2 + 2(2D + E)^2 \omega_+^2 + (b_0^2 - 2b_1 (D + E)^2)^2 + 2(D + E)^2 \omega_+^2 + (b_0^2 + b_2^2) \omega_+^4 + b_0^2 (D + E)^2 \omega_+^2 + b_2^2 (D + E)^2 \omega_+^2}{b_0^2 (D + E)^2 \omega_+^2 + b_2^2 (D + E)^2 \omega_+^2}\right].
\]
Substituting $\lambda$ into (2.4) and separating real and imaginary parts, we have

$$\exp(\frac{d\lambda}{dt})_{\lambda=i\omega_i} = \exp(\frac{d\lambda}{dt})^{-1}_{\lambda=i\omega_i} > 0.$$ 

From this we obtain that there is a Hopf bifurcation at $\omega = \omega_+$. Therefore, if $p^*_1 > \Delta$ and $-2(D+E)p^*_1 L(p^*_1) + \frac{(1-p^*_1)p^*_1 \exp(-p^*_1)}{(\gamma_1+p^*_1)(\gamma_2+p^*_1)} G(p^*_1) < 0$, then $E^*(p^*_1, z^*_1)$ is locally asymptotically stable for $\tau < \tau_0^{(i)}$, $E^*(p^*_1, z^*_1)$ is unstable for $\tau > \tau_0^{(i)}$, and there is a period solution around $E^*(p^*_1, z^*_1)$ for $\tau = \tau_0^{(i)}$. If $p^*_1 > \Delta$ and $-2(D+E)p^*_1 L(p^*_1) + \frac{(1-p^*_1)p^*_1 \exp(-p^*_1)}{(\gamma_1+p^*_1)(\gamma_2+p^*_1)} G(p^*_1) > 0$ or $0 < p^*_1 < \Delta$, then $E^*(p^*_1, z^*_1)$ is unstable for any $\tau \geq 0, i = 1, 2$.

(2) We consider the stability of $E^*(p^*_3, z^*_2)$.

Since $G(p^*_3) > 0$, if $p^*_3 > \Delta$, then Eq. (2.5) has only one imaginary root. From proof of $E^*(p^*_3, z^*_2)$, $\exp(\frac{d\lambda}{dt})_{\lambda=i\omega_i} > 0$. If $0 < p^*_3 < \Delta$ and

$$-2(D+E)p^*_1 L(p^*_1) + \frac{(1-p^*_1)p^*_1 \exp(-p^*_1)}{(\gamma_1+p^*_1)(\gamma_2+p^*_1)} G(p^*_1) > 0,$$

then (2.5) has only one imaginary solution. If $0 < p^*_3 < \Delta$ and $-2(D+E)p^*_1 L(p^*_1) + \frac{(1-p^*_1)p^*_1 \exp(-p^*_1)}{(\gamma_1+p^*_1)(\gamma_2+p^*_1)} G(p^*_1) < 0$, then (2.5) does not have imaginary solution. When $\tau = 0$, $E^*(p^*_3, z^*_2)$ is unstable. Therefore, $E^*(p^*_3, z^*_2)$ is unstable for any $\tau \geq 0$.

(3) We consider the stability of $E^*(p^*_4, z^*_3)$.

$\lambda(\tau) = 0$ is one of the roots of (2.5) for all $\tau \geq 0$. Assume $\lambda = u + iv$ is a root of (2.5). Substituting $\lambda = u + iv$ into (2.5) and separating real and imaginary parts, we have

$$u^2 - v^2 + b_0 u + b_1 + [b_2 - (D+E)u]e^{-u\tau} \cos(v\tau) - (D+E)ve^{-u\tau} \sin(v\tau) = 0,$$

$$2uv + b_0 v - [b_2 - (D+E)u]e^{-u\tau} \sin(v\tau) - (D+E)ve^{-u\tau} \sin(v\tau) = 0.$$ 

From above two equations, we obtain

$$(u^2 - v^2 + b_0 u + b_1)^2 + v^2(2u + b_0)^2 = [(D+E)^2 v^2 + (b_2 - (D+E)u)^2] e^{-2u\tau}.$$ 

Suppose $0 < p^*_4 < \Delta$. Let $h(\lambda, \tau) = \lambda^2 + b_0 \lambda + b_1 + (b_2 - (D+E))e^{-\lambda \tau}$. It is easy to see that $h(0, \tau) = 0$ and $\lim_{\lambda \to +\infty} h(\lambda, \tau) = +\infty$. There is a $\lambda_0 > 0$ such that $h(\lambda, \tau) \geq 0$ if $\lambda \geq \lambda_0$, we also have

$$\frac{\partial h(\lambda, \tau)}{\partial \lambda} = 2\lambda + b_0 - [\tau(b_2 - (D+E)\lambda) + D+E]e^{-\lambda \tau},$$

$$\frac{\partial h(0, \tau)}{\partial \lambda} = b_0 - \tau b_2 - (D+E) = -\frac{p^*_4}{\gamma_1 + p^*_4} L(p^*_4)(1 + \tau(D+E)).$$

Since $0 < p^*_4 < \Delta$ implies $L(p^*_4) > 0$, we can get $\frac{\partial h(0, \tau)}{\partial \lambda} < 0$ ($\tau \geq 0$). Hence, there is a $\sigma(\tau) > 0$ such that when $0 < \lambda < \sigma(\tau)$, $h(\lambda, \tau) < 0$. Therefore, there exists at least one $\lambda$, $\sigma(\tau) < \lambda < \lambda_0$, such that $f(\lambda, \tau) = 0$. This implies equation (2.5) has at least a positive root. That is, the trivial solution of the linearized system of (2.4) about $E^*(p^*_4, z^*_3)$ is unstable. □

3. Numerical simulation
In this section, in order to facilitate the interpretation of our mathematical results, we present some numerical results for some particular values of the parameters associated with the model system (1.4).

Let us verify the results of Theorem 2.4, we consider \( \tau \) as bifurcation parameter and for case (i) take parameters as \( \beta_1 = 0.9, \rho = 0.3, \gamma_1 = 2.5, \gamma_2 = 3, D = 0.05, E = 0.005 \). For this choice, \( F(a_0) \approx 0.0698 > 0, F(1) \approx 0.012 > 0 \). The stability of the positive equilibrium point \( E_1^*(0.3405, 2.6332) \) depends upon the magnitude of delay \( \tau \). From (2.14) and (2.15) we obtain \( \omega^2_1 = 0.3364, \tau_0 = 3.46 \). It is easy to verify that \( \Delta \approx 0.2656 < 0.3405, -2(D + E)\frac{p_1^* p_1}{\gamma_1 + p_1} L(p_1^*) + \frac{(1-p_1^*) p_1^* \exp(-p_1^*)}{\gamma_2 + p_2^*} G(p_2^*) \approx -0.0153 < 0 \). Then the conditions of case of (i) in Theorem 2.4 are satisfied. Our numerical simulations show equilibrium \( E_1^*(0.3405, 2.6332) \) is stable for \( \tau = 1.8 < \tau_0 = 3.46 \) and the initial value is \((0.8,1.8)\), which is shown in Figure 4((a)-(b)). When \( \tau \) increases to \( \tau = 5 \), \( E_1^*(0.3405, 2.6332) \) becomes unstable. A stable periodic solution appears when \( \tau = 3.46 \) and the initial value is \((0.33,2.8)\) (Figure 5((a)-(b))).

![Figure 4](image1.png)  
**Figure 4** The behavior and phase portrait of system (1.4) at equilibrium point \( E_1^* \) for \( \tau = 1.8 < \tau_0 \)

![Figure 5](image2.png)  
**Figure 5** The behavior and phase portrait of system (1.4) at equilibrium point \( E_1^* \) for \( \tau = \tau_0 \)

For case (ii) in Theorem 2.4, consider the following set of parametric values: \( \beta_1 = 0.98, \rho = 0.08, \gamma_1 = 1, \gamma_2 = 1.2, D = 0.17, E = 0.01 \). Then we have \( F(a_0) \approx 0.0071 > 0, F(1) \approx -0.0131 < 0 \). By Theorem 2.2, the system (1.4) has two equilibria \( E_2^*(0.4498, 1.2508) \) and \( E_2^*(0.8108, 0.7708) \). From direct calculation, we get that \( \tau_0 = 5.15, \Delta = 0.4142 \) and \(-2(D + E)\frac{p_1^* p_1}{\gamma_1 + p_1} L(p_1^*) + \frac{(1-p_1^*) p_1^* \exp(-p_1^*)}{\gamma_2 + p_2^*} G(p_2^*) \approx -0.0121 < 0 \). Our numerical simulations show that for \( \tau = 0.1 < \tau_0 \) and the initial value is \((0.4,1.1)\), interior equilibrium \( E_2^* \) is stable (Figure
When $\tau$ increases, $\tau = 8 > \tau_0$ and the initial value is $(0.45, 1.25)$, the equilibrium $E^*_2(0.4498, 1.2508)$ becomes unstable (Figure 8 ((a)–(b))). There is a periodic solution around the equilibrium $E^*_2$ when $\tau = \tau_0 = 5.15$ and the initial value is $(0.55, 1.17)$ (Figure 7 ((a)–(b))).

For case (iii) take parametric values as $\beta_1 = 1$, $\rho = 0.1$, $\gamma_1 = 0.4$, $\gamma_2 = 0.4$, $D = 0.27$, $E = 0.0024$. Then we have $F(a_0) = 0$, $p^*_i = a_0 = 0.4633$, $z^*_i = 0.7374$, and $p^*_i < \Delta = 0.5207$. By Theorem 2.2, the system (1.4) has only one equilibrium $E^*_i(p^*_i, z^*_i)$ and from Theorem 2.4 we know that the equilibrium $E^*_i(p^*_i, z^*_i)$ is unstable for any $\tau \geq 0$, where the initial value is $(0.41, 1.25)$ (Figure 9).
4. Discussion and conclusion

The factors effecting plankton dynamics and the abundance of harmful phytoplankton species have been the subject of intensive research recently. In this paper, we analyze a two component model consisting of the dynamical behaviors of zooplankton and phytoplankton system. In order to make the model more realistic and exhibits much more rich dynamics, in this paper, we introduce time delay to the model with exponential substrate uptake and exponential distribution of toxic substance term. By using comparison principle for functional differential equations and traditional analysis technique for transcendental equations [21], we give a detailed analysis on local asymptotic stability of the equilibria of system (1.4). Our results show that time delay is factually harmless for the local asymptotic stability of the zooplankton free equilibrium of (1.4), but it is not always harmless for the positive equilibrium; that is to say, because of the time delay the positive equilibrium becomes unstable (Theorem 2.4). From analytical results and the numerical study, we observed that the system exhibits a Hopf bifurcation around its interior equilibrium and the delay \( \tau \) plays crucial role in determining the asymptotic behavior of solution of model (1.4). It has been concluded that system exhibits stable behavior when \( \tau < \tau_0 \), and a switch from stability to instability is induced as \( \tau \) passes through its threshold value \( \tau_0 \), and stable oscillations appear when \( \tau = \tau_0 \) and the Hopf bifurcation periodic solution occurs. Thus, there is a range of gestation delay (liberation delay) parameter below which bloom does not appear and above that phenomenon of bloom formation occurs.

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References


