Global Stability of the Deterministic and Stochastic SIS Epidemic Models with Vaccination

Xu ZHAO$^{1,2}$, Wenshu ZHOU$^{2,*}$

1. School of Mathematics and Information Science, Beifang Minzu University, Ningxia 750021, P. R. China;
2. School of Science, Dalian Minzu University, Liaoning 116600, P. R. China

Abstract We study the stability of endemic equilibriums of the deterministic and stochastic SIS epidemic models with vaccination. The deterministic SIS epidemic model with vaccination was proposed by Li and Ma (2004), for which some sufficient conditions for the global stability of the endemic equilibrium were given in some earlier works. In this paper, we first prove by Lyapunov function method that the endemic equilibrium of the deterministic model is globally asymptotically stable whenever the basic reproduction number is larger than one. For the stochastic version, we obtain some sufficient conditions for the global stability of the endemic equilibrium by constructing a class of different Lyapunov functions.

Keywords SIS epidemic model; vaccination; global stability

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1. Introduction

Studies of epidemic models with vaccination have become an important area in the mathematical theory of epidemiology, and they have largely been inspired by the works [1–5]. The vaccination enables the vaccinated to acquire a permanent or temporary immunity. When the immunity is temporary, the immunity can be lost after a period of time. In [6], Li and Ma proposed the following SIS model with vaccination:

\[
\begin{align*}
\frac{dS}{dt} &= (1-q)A - \beta SI - (\mu + p)S + \gamma I + \varepsilon V, \\
\frac{dI}{dt} &= \beta SI - (\mu + \gamma + \alpha)I, \\
\frac{dV}{dt} &= qA + pS - (\mu + \varepsilon)V.
\end{align*}
\]

Here $S(t)$ denotes the number of members who are susceptible to an infection at time $t$. $I(t)$ denotes the number of members who are infective at time $t$. $V(t)$ denotes the number of members who are immune to an infection at time $t$ as a result of vaccination. $A$ stands for a constant input of new members into the population per unit time, $q$ denotes the fraction of vaccinated for new born; $\mu$ denotes the natural death rate of $S$, $I$, $V$ compartments; $\beta$ is the transmission coefficient between compartments $S$ and $I$; $p$ represents the proportional coefficient of vaccinated for the
Theorem 1.1 Let \( R_0 < 1 \). Then the endemic equilibrium \( E^* \) of system (1.1) is globally asymptotically stable.

However, the evolving process of the epidemic disease over the time is naturally subject to random and environmental perturbations. To understand the impacts due to such randomness and fluctuations, it is convenient and effective to model the disease spreading through a stochastic differential equation (SDE) approach \([11–18]\). In \([15]\), Zhao, Jiang and O'Regan take into account the effect of randomly fluctuating environment in model (1.1) by assuming that fluctuations in the environment will manifest themselves mainly as fluctuations in the parameter \( \beta \): \( \beta \to \beta + \sigma B \), where \( B \) is standard Brownian motion and \( \sigma^2 \) represents its intensity, and obtained the following SDE model:

\[
\begin{align*}
\frac{dS}{dt} &= [(1 - q)A - \beta SI - (\mu + p)S + \gamma I + \varepsilon V]dt - \sigma S dB, \\
\frac{dI}{dt} &= [\beta SI - (\mu + \gamma + \alpha)I]dt + \sigma I dB, \\
\frac{dV}{dt} &= [qA + pS - (\mu + \varepsilon)\gamma I]dt + \sigma dB.
\end{align*}
\] (1.2)

They proved that when the noise is large, the infective decays exponentially to zero regardless of the magnitude of \( R_0 \). When the noise is small, some sufficient conditions on extinction and persistence are established. Some further works on system (1.2) can be referred to \([19–23]\). In \([16]\), Zhao and Jiang considered the following stochastic system:

\[
\begin{align*}
\frac{dS}{dt} &= [(1 - q)A - \beta SI - (\mu + p)S + \gamma I + \varepsilon V]dt - \sigma_1 S dB_1, \\
\frac{dI}{dt} &= [\beta SI - (\mu + \gamma + \alpha)I]dt + \sigma_2 I dB_2, \\
\frac{dV}{dt} &= [qA + pS - (\mu + \varepsilon)\gamma I]dt + \sigma_3 dB_3.
\end{align*}
\] (1.3)

where \( B_i \) (\( i = 1, 2, 3 \)) are independent Brownian motions and \( \sigma_i \) (\( i = 1, 2, 3 \)) are their intensities. When the perturbations and the disease-related death rate \( \alpha \) are small, they showed that there is a stationary distribution and it is ergodic when \( R_0 > 1 \), whereas the asymptotic behavior of the solution around the disease-free equilibrium prevails when \( R_0 \leq 1 \). In \([24]\), the sufficient conditions for extinction and persistence in mean are obtained, and a threshold of the stochastic model which determines the outcome of the disease was established when the white noises are small.
From the above we know that system (1.1) admits a unique endemic equilibrium $E^*$ when $R_0 > 1$. Furthermore, we assume stochastic perturbations are of white noise type, which are directly proportional to distances $S, I, V$ from values of $S^*, I^*, V^*$, influence on the $S, I, V$ respectively. Thus system (1.1) results in

$$
\begin{align*}
\frac{dS}{dt} &= [(1 - q)A - \beta SI - (\mu + p)S + \gamma + \varepsilon V]dt + \sigma_1(S - S^*)dB_1(t), \\
\frac{dI}{dt} &= [\beta SI - (\mu + \gamma + \alpha)I]dt + \sigma_2(I - I^*)dB_2(t), \\
\frac{dV}{dt} &= [qA + pS - (\mu + \varepsilon)V]dt + \sigma_3(V - V^*)dB_3(t),
\end{align*}
$$

(1.4)

where $B_i (i = 1, 2, 3)$ are independent standard Brownian motions and $\sigma_i (i = 1, 2, 3)$ represent their intensities. Obviously, stochastic system (1.4) has the same equilibrium points as system (1.1). In this paper, we will investigate asymptotic stability of the equilibrium $E^*$ of stochastic system (1.4). We obtain

**Theorem 1.2** Let $R_0 > 1$. Then the equilibrium $E^*$ of system (1.4) is stochastically asymptotically stable in the large if the following conditions are satisfied:

$$
\begin{align*}
\sigma_1^2 &< \frac{2\beta \eta^* I^*}{\theta + 1}, \\
\sigma_2^2 &< (\mu + \alpha)((1 + \frac{p + \alpha + 2\mu + \eta^*(\mu + \alpha)}{\beta})^{-1}, \\
\sigma_3^2 &< \frac{\varepsilon^2}{2\alpha(\mu + \alpha)(\theta + 1) I^*},
\end{align*}
$$

where $M^* = \frac{1}{\rho}, [2(\mu + \varepsilon)(\mu + p) - \rho \varepsilon]$ and $\eta^* = \frac{1}{4\alpha(\mu + \alpha)(\theta + 1)I^*} p^2 \varepsilon^2$.

This paper is organized as follows. In Section 2, we will prove Theorem 1.1 by using Lyapunov function method. In Section 3, the proof of Theorem 1.2 will be given by constructing a class of different Lyapunov functions.

**2. Proof of Theorem 1.1**

In the section, we will give the proof of Theorem 1.1 by using Lyapunov function method.

**Proof** Let us consider a nonnegative solution $(S, I, V)$ of system (1.1). Denote

$$
x = S - S^*, \; y = V - V^*, \; z = N - N^*, \; N = S + I + V,
$$

where $N^* = S^* + I^* + V^*$. Adding the three equations in (1.1) yields

$$
\frac{dN}{dt} = A - (\mu + \alpha)N + \alpha(S + V).
$$

(2.1)

Thus, $(x, I, y, z)$ satisfies

$$
\begin{align*}
\frac{dx}{dt} &= -\beta I x - (\mu + p)x - (\mu + \alpha)(I - I^*) + \varepsilon y, \\
\frac{dI}{dt} &= \beta I x, \\
\frac{dy}{dt} &= px - (\mu + \varepsilon)y, \\
\frac{dz}{dt} &= \alpha(x + y) - (\mu + \alpha)z.
\end{align*}
$$

(2.2)

Define the functions $V_i (i = 1, 2, 3)$ along the solution $(x, I, y, z)$ of system (2.2) by

$$
V_1 = \frac{1}{2}x^2 + \frac{\mu + \alpha}{\beta} (I - I^* - I^* \ln \frac{I}{I^*}), \; V_2 = \frac{1}{2}y^2, \; V_3 = \frac{1}{2}z^2.
$$
Consider the Lyapunov function $V_X(t) = V_1(t) + XV_2(t) + YV_3(t)$, where the positive constants $X$ and $Y$ will be determined later. Next, we calculate the derivatives $\frac{dV}{dt}$ along the solution of system (2.2). By the identities

$$
\begin{align*}
S^* &= \frac{a+q+a}{p}, \quad qA + pS^* - (\mu + \varepsilon)V^* = 0, \\
(1-q)A - \beta S^* I^* - (\mu + p)S^* + \gamma I^* + \varepsilon V^* = 0,
\end{align*}
$$

we have

$$
\begin{align*}
\frac{dV_1}{dt} &= -\beta I x^2 - (\mu + p)x^2 + \varepsilon xy, \\
\frac{dV_2}{dt} &= px - (\mu + \varepsilon)y^2, \\
\frac{dV_3}{dt} &= -((\mu + \alpha)z^2 + \alpha xz + \alpha yz).
\end{align*}
$$

Then for any positive constants $X$ and $Y$, we have

$$
\frac{dV_{X,Y}}{dt} \leq - (\mu + p)x^2 + (pX + \varepsilon)xy - X(\mu + \varepsilon)y^2 - \\
Y(\mu + \alpha)z^2 + Y\alpha xz + Y\alpha yz \\
=: -x\mathcal{M}(X,Y)x^T,
$$

where $x = (x, y, z)$, $T$ denotes the transpose, and the matrix $\mathcal{M}(X,Y)$ is defined by

$$
\mathcal{M}(X,Y) = \begin{pmatrix}
\mu + p & -\frac{pX + \varepsilon}{2} & -\frac{Y\alpha}{2} \\
-\frac{pX + \varepsilon}{2} & X(\mu + \varepsilon) & -\frac{Y\alpha}{2} \\
-\frac{Y\alpha}{2} & -\frac{Y\alpha}{2} & Y(\mu + \alpha)
\end{pmatrix}.
$$

It is easy to see that by some elementary transformations, $\mathcal{M}(X,Y)$ can be transferred into

$$
\tilde{\mathcal{M}}(X,Y) = \begin{pmatrix}
\mu + p & -\frac{pX + \varepsilon}{2} & -\frac{Y\alpha}{2} \\
0 & X(\mu + \varepsilon) - \frac{(pX + \varepsilon)^2}{4(\mu + p)} & -\frac{Y\alpha}{2} - \frac{Y\alpha(\mu + \varepsilon)}{4(\mu + p)} \\
0 & -\frac{Y\alpha}{2} - \frac{Y\alpha(\mu + \varepsilon)}{4(\mu + p)} & Y(\mu + \alpha) - \frac{(Y\alpha)^2}{4(\mu + p)}
\end{pmatrix}.
$$

Clearly, $\mathcal{M}(X,Y)$ is positive definite if the following conditions are satisfied:

$$
\begin{align*}
\Delta_1(X) &= X(\mu + \varepsilon) - \frac{(pX + \varepsilon)^2}{4(\mu + p)} > 0, \\
\Delta_2(X,Y) &= \Delta_1(X) \cdot [Y(\mu + \alpha) - \frac{(Y\alpha)^2}{4(\mu + p)}] - \left[\frac{Y\alpha}{2} + \frac{Y\alpha(\mu + \varepsilon)}{4(\mu + p)}\right]^2 > 0.
\end{align*}
$$

Note that

$$
\Delta_2(X,Y) = Y\left\{(\mu + \alpha)\Delta_1(X) - Y\left[\frac{X(\mu + \varepsilon)\alpha^2}{4(\mu + p)} + \frac{\alpha^2(pX + \varepsilon)}{4(\mu + p)}\right]\right\} \\
=: Y[(\mu + \alpha)\Delta_1(X) - Y\Delta_3(X)],
$$

and

$$
\Delta_1(X) = -\frac{1}{4(\mu + p)}\left\{p^2X^2 - 22(\mu + \varepsilon)(\mu + p) - \varepsilon X + \varepsilon^2\right\}.
$$

Since $2(\mu + \varepsilon)(\mu + p) - \varepsilon > 0$ and $2(\mu + \varepsilon)(\mu + p) - \varepsilon^2 > 0$, the equation $\Delta_1(X) = 0$ has two positive roots $X_1$ and $X_2$ with $X_1 < X_2$. Taking $X^* = \frac{1}{2}(X_1 + X_2)$ yields $\Delta_1(X^*) > 0$. By choosing $Y = \frac{\alpha(\mu + \varepsilon)\alpha^2}{2\Delta_3(X^*)}$, one has $\Delta_2(X^*, Y^*) > 0$ and hence, the matrix $\mathcal{M}(X^*, Y^*)$ is positive definite. It follows from (2.4) that $\frac{dV_{X,Y}}{dt}$ is negative-definite. On the other hand,
it is clear that \( \frac{dV_i}{dt} = 0 \) if and only if \((S, I, V) = (S^*, I^*, V^*)\). According to the LaSalle’s invariant principle [10], \(E^*\) is globally asymptotically stable. The proof is completed. \(\square\)

3. Proof of Theorem 1.2

In the section, we study the stochastically asymptotic stability of \(E^*\) of system (1.4). To this end, we will construct a class of different Lyapunov functions to finish the proof of Theorem 1.2.

**Proof** Let \(x(t) = S(t) - S^*, y(t) = I(t) - I^*\) and \(z(t) = V(t) - V^*\). Noticing (2.3), we have

\[
\left\{
\begin{aligned}
\frac{dx}{dt} &= -\beta xy - \beta I^*x - (\mu + \alpha)x - (\mu + p)x + \varepsilon z \ dt + \sigma_1 x dB_1, \\
\frac{dy}{dt} &= (\beta xy + \beta I^*x) dt + \sigma_2 y dB_2, \\
\frac{dz}{dt} &= [p x - (\mu + \varepsilon) z] dt + \sigma_3 z dB_3.
\end{aligned}
\right.
\]

To prove the theorem, it suffices to show that the zero solution of system (3.1) is stochastically asymptotically stable in the large. Let \(x = (x, y, z)\). Define the Lyapunov function

\[
V(x) = \sum_{i=1}^{4} a_i V_i(x),
\]

where \(a_i\) are positive constants to be chosen later, and \(V_i\) \((i = 1, 2, 3, 4)\) are defined as follows:

\[
V_1(x) = \frac{1}{2} x^2, \quad V_2(y) = \frac{1}{2} y^2, \quad V_3(z) = \frac{1}{2} z^2, \quad V_4(x, y) = \frac{1}{2} (x + y)^2.
\]

Let \(L\) be the differential operator associated with (3.1). By Itô formula [25, Theorem 6.2 of Chapter 1], we have

\[
L(V_1 + \frac{\mu + \alpha}{\beta I^*} V_2 + M^* V_3) = -\beta I^* x^2 - [(\mu + p)x^2 - (M^* p + \varepsilon) x z + M^*(\mu + \varepsilon) z^2] - \beta x^2 y + \mu + \alpha \frac{x^2 y}{I^*} + \frac{1}{2} \sigma_1^2 x^2 + \frac{\mu + \alpha}{\beta I^*} \sigma_2^2 y^2 + \frac{M^*}{2} \sigma_3^2 z^2
\]

\[
\begin{aligned}
&\leq -\beta I^* x^2 - \frac{f(M^*)}{4(\mu + p)} z^2 - \beta x^2 y + \mu + \alpha \frac{x^2 y}{I^*} + \\
&\quad \frac{1}{2} \sigma_1^2 x^2 + \frac{\mu + \alpha}{2\beta I^*} \sigma_2^2 y^2 + \frac{M^*}{2} \sigma_3^2 z^2,
\end{aligned}
\]

where \(f(M^*) = -\{p^2(M^*)^2 - 2[2(\mu + \varepsilon)(\mu + p) - p\varepsilon M^* + \varepsilon^2]\} > 0\). Similarly, we have

\[
L(V_4 + \frac{p + \alpha + 2\mu}{\beta I^*} V_2 + M^* V_3)
\]

\[
= -[(\mu + p)x^2 - (M^* p + \varepsilon) x z + M^*(\mu + \varepsilon) z^2] - (\mu + \alpha)y^2 + \varepsilon y z + \\
\frac{p + \alpha + 2\mu}{I^*} y^2 + \frac{1}{2} \sigma_1^2 x^2 + \frac{1}{2} (1 + \frac{p + \alpha + 2\mu}{\beta I^*}) \sigma_2^2 y^2 + \frac{M^*}{2} \sigma_3^2 z^2
\]
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\[ \begin{align*}
\leq -\frac{1}{4(\mu + p)} f(M^*) z^2 - (\mu + \alpha) y^2 + \varepsilon y z + \\
\frac{p + \alpha + 2\mu}{I^*} x y^2 + \frac{1}{2} \sigma_1^2 x^2 + \frac{1}{2} \left(1 + \frac{p + \alpha + 2\mu}{\beta I^*}\right) \sigma_2^2 y^2 + \frac{M^*}{2} \sigma_3^2 z^2.
\end{align*} \]

(3.3)

Note that \( \eta^* = \frac{2c^2(\mu + p)}{(\mu + \alpha)\beta M^*} \). It follows from (3.2) and (3.3) that

\[ \begin{align*}
L(V_4 + M^* V_3 + \frac{p + \alpha + 2\mu}{\beta I^*} V_2 + \eta^* (V_1 + M^* V_3 + \frac{\mu + \alpha}{\beta I^*} V_2)) \\
\leq -((\mu + \alpha) y^2 - \varepsilon y z + \eta^* f(M^*) z^2) - \frac{f(M^*)}{4(\mu + p)} z^2 - \beta \eta^* I^* x^2 + \\
\frac{p + \alpha + 2\mu}{I^*} x y^2 + \frac{1}{2} \sigma_1^2 x^2 + \frac{1}{2} \left(1 + \frac{p + \alpha + 2\mu}{\beta I^*}\right) \sigma_2^2 y^2 + \frac{M^*}{2} \sigma_3^2 z^2 - \\
\beta \eta^* I^* x^2 y + \eta^* (\frac{\mu + \alpha}{I^*} x y^2 + \frac{1}{2} \sigma_1^2 x^2 + \frac{M^*}{2} \sigma_3^2 z^2) + \frac{\mu + \alpha}{2} \sigma_2^2 y^2) \\
= -\frac{\mu + \alpha}{2} \left[(y - \frac{\varepsilon}{\mu + \alpha} z) - \frac{\mu + \alpha}{2} y^2 - \frac{f(M^*)}{4(\mu + p)} z^2 - \beta \eta^* I^* x^2 - \right. \\
\beta \eta^* x^2 y + \frac{p + \alpha + 2\mu + \eta^*(\mu + \alpha)}{I^*} x y^2 + \\
\left. \frac{\eta^* + 1}{2} \sigma_1^2 x^2 + \frac{1}{2} \left(1 + \frac{p + \alpha + 2\mu + \eta^*(\mu + \alpha)}{\beta I^*}\right) \sigma_2^2 y^2 + \frac{\eta^* + 1}{2} M^* \sigma_3^2 z^2 \right] \\
\leq -\beta \eta^* I^* x^2 - \frac{\mu + \alpha}{2} y^2 - \frac{f(M^*)}{4(\mu + p)} z^2 - \beta \eta^* x^2 y + \frac{p + \alpha + 2\mu + \eta^*(\mu + \alpha)}{I^*} x y^2 + \\
\frac{\eta^* + 1}{2} \sigma_1^2 x^2 + \frac{1}{2} \left(1 + \frac{p + \alpha + 2\mu + \eta^*(\mu + \alpha)}{\beta I^*}\right) \sigma_2^2 y^2 + \frac{\eta^* + 1}{2} M^* \sigma_3^2 z^2.
\end{align*} \]

By choosing \( a_i \) \( (i = 1, 2, 3, 4) \) as follows:

\[ a_1 = \eta^*, \quad a_2 = \frac{p + \alpha + 2\mu + \eta^*(\mu + \alpha)}{\beta I^*}, \quad a_3 = (\eta^* + 1)M^*, \quad a_4 = 1, \]

we have

\[ LV \leq -(Ax^2 + By^2 + Cz^2) - \beta \eta^* x^2 y + \frac{p + \alpha + 2\mu + \eta^*(\mu + \alpha)}{I^*} x y^2, \]

where

\[ \begin{align*}
A &= \beta \eta^* I^* - \frac{\eta^* + 1}{2} \sigma_1^2 > 0, \\
B &= \frac{\mu + \alpha}{2} - \frac{1}{2} \left(1 + \frac{p + \alpha + 2\mu + \eta^*(\mu + \alpha)}{\beta I^*}\right) \sigma_2^2 > 0, \\
C &= \frac{(M^*)}{4(\mu + p)} - \frac{\eta^* + 1}{2} M^* \sigma_3^2 > 0.
\end{align*} \]

Let \( \lambda = \min\{A, B, C\} \). Then \( LV \leq -\lambda \|x(t)\|^2 + o(\|x(t)\|^2) \), where \( \|x\| = \sqrt{x^2 + y^2 + z^2} \), and \( o(\|x(t)\|^2) \) is an infinitesimal of higher order of \( \|x(t)\|^2 \) for \( t \geq 0 \). Hence \( LV \) is negative-definite in a sufficiently small neighborhood of \( x = 0 \) for \( t \geq 0 \). Besides, it is clear that \( V(x) \) is positive-definite decreasent. According to [25, Theorem 2.4 of Chapter 4], we therefore conclude that the zero solution of system (3.1) is stochastically asymptotically stable in the large. The proof is completed. \( \square \)

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References