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Lewy方程的局部可解性问题*

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本文引入一种关于部分变数为$\mathscr{D}$空间，关于部分变数为整函数空间Z的所谓F_{DZ}空间，利用富氏变换方法，找出Lewy方程在这种基本空间的对偶空间中解的表达式注1)。

一、Lewy方程的形式基本解

先从形式的运算出发.

$$Lu \equiv \left[\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} - 2i(x_1 + ix_2)\frac{\partial}{\partial x_3}\right]u = \delta(x_1 - y_1,\ x_2 - y_2,\ x_3).$$

关于x_3作富氏变换：

$$\left[\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + 2(x_1 + ix_2)\xi_3\right]\hat{u}(x_1, x_2, \xi_3) = \delta(x_1 - y_1, x_2 - y_2)$$

即：

$$\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)\left[\hat{u}(x_1, x_2\xi_3)e^{(x_1^2+x_2^2)\xi_3}\right] = \delta(x_1 - y_1, x_2 - y_2)\, e^{(x_1^2+x_2^2)\xi_3}$$

$$= \delta(x_1 - y_1, x_2 - y_2)e^{(y_1^2+y_2^2)\xi_3}$$

所以：

$$\hat{u}(x_1, x_2, \xi_3) = \frac{1}{2\pi}\,\frac{e^{[(y_1^2+y_2^2)-(x_1^2+x_2^2)]\xi_3}}{x_1 - y_1 + i(x_2 - y_2)}$$

$$u(x_1, x_2, x_3) = \frac{1}{2\pi}\cdot\frac{1}{(x_1 - y_1) + i(x_2 - y_2)}$$

$$\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i[x_3 + i(x_1^2+x_2^2-y_1^2-y_2^2)]\xi_3}\cdot d\xi_3$$

$$= \frac{1}{2\pi}\cdot\frac{\delta[x_3 + i(x_1^2 + x_2^2 - y_1^2 - y_2^2)]}{(x_1 - y_1) + i(x_2 - y_2)}$$

可以形式地核验 $E(x_1, y_1, x_2, y_2, x_3) \equiv \dfrac{1}{2\pi}\cdot\dfrac{\delta[x_3 + i(x_1^2 + x_2^2 - y_1^2 - y_2^2)]}{(x_1 - y_1) + i(x_2 - y_2)}$ 为基本解.

事实上：

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注1) 此文在讨论班报告后，看到F. Traves曾讨论了具解析系数的方程在“超分布”空间的哥西问题[3]. 可以看出，此文所取的基本空间和那里的基本空间是相似的. 但求解的范围、方法是不同的.

$$\left[\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}-2i(x_1+ix_2)\frac{\partial}{\partial x_3}\right]E(x_1,y_1,\ x_2,y_2,\ x_3)$$

$$=\delta[x_3-i(x_1^2+x_2^2-y_1^2-y_2^2)]\left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right)\left(\frac{1}{2\pi}\cdot\frac{1}{(x_1-y_1)+i(x_2-y_2)}\right)$$

$$+\frac{1}{2\pi}\cdot\frac{1}{(x_1-y_1)+i(x_2-y_2)}\left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right)\delta[x_3+i(x_1^2+x_2^2-y_1^2-y_2^2)]$$

$$-\frac{1}{2\pi}\cdot\frac{2i(x_1+ix_2)}{(x_1-y_1)+i(x_2-y_2)}\cdot\frac{\partial}{\partial x_3}\delta[x_3+i(x_1^2+x_2^2-y_1^2-y_2^2)]$$

$$=\delta(x_1-y_1,x_2-y_2,x_3)$$

以上的一切运算都是形式的。我们称E为形式基本解注2)。下面引入F_{DZ}空间，使在F_{DZ}上$E(x_1,y_1,\ x_2,y_2,\ x_3)$确为基本解。

二、F_{DZ} 空 间

函数$\varphi(x_1,x_2,x_3)\in F_{DZ}(x_1,x_2;x_3)$。这里$\varphi$关于$x_1,x_2\in\mathscr{D}(x_1,x_2)$，而关于$x_3$可延拓为$x_3+i\eta_3$的整函数。关于$x_1,x_2$的拓扑为$\mathscr{D}(x_1,x_2)$中拓扑，而关于$x_3$的拓扑为$\mathscr{S}$注3)，且关于$\zeta=x_3+i\eta_3$的任何阶导数在$\zeta$平面的任何有界域上为一致收敛。(一致性为关于不同的点。)

注意，这样的空间显然是非空的。$\varphi_1(x_1,x_2)e^{-x_3^2}\in F_{DZ}(x_1,x_2;x_3)$其中$\varphi_1(x_1,x_2)\in\mathscr{D}(x_1,x_2)$。

三、Lewy 方 程 的 解

定义 $(\delta(x_3-i\eta_3),\ \varphi(x_1,x_2,x_3))_{(x_3)}=\varphi(x_1,x_2,i\eta_3)$。

定理一 $E(x_1,y_1,\ x_2,y_2,\ x_3)\in F'_{DZ}(x_1,x_2;\ x_3)$ 且

$$LE=\delta(x_1-y_1,\ x_2-y_2,\ x_3)$$

证明 $$\left|\left(\frac{1}{\partial\pi}\cdot\frac{\delta[x_3+i(x_1^2+x_2^2-y_1^2-y_2^2)]}{x_1-y_1+i(x_2-y_2)},\ \varphi(x_1,x_2,x_3)\right)_{(x_1;x_2,x_3)}\right|$$

$$=\left|\left(\frac{1}{\partial\pi}\cdot\frac{1}{(x_1-y_1)+i(x_2-y_2)},\ \varphi(x_1,x_2,-i(x_1^2+x_2^2-y_1^2-y_2^2)\right)_{(x_1,x_2)}\right|$$

$$\leqslant C(y_1,y_2)\max_{(x_1,x_2)\in K\subset R_2}|\varphi(x_1,x_2,\zeta)|$$

$$|\zeta|\leqslant|x_1^2+x_2^2-y_1^2-y_2^2|\leqslant A$$

K为φ关于x_1,x_2的台。

即知 $E\in F'_{DZ}$。

$$(LE,\varphi)=\left(E,\left[-\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}-2i(x_1+ix_2)\frac{\partial}{\partial x_3}\right]\varphi(x_1,x_2,x_3)\right)$$

$$=\frac{1}{2\pi}\left(\frac{1}{x_1-y_1+i(x_2-y_2)}\left\{\left[-\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}-2i(x_1+ix_2)\frac{\partial}{\partial x_3}\right]\right.\right.$$

注 2) 若对基本函数$\varphi(x_1,x_2,x_3)$关于x_3的富氏变换要求指降性$e^{-A|t_3|}$(A与求解域有关)，则上面的形式运算可以严格化。

注 3) 我们这里限于考虑局部可解性问题，因此关于x_3的降性可以去掉。

$$\cdot\varphi(x_1,x_2,x_3)\Big\}\Big|_{x_3=-i(x_1^2+x_2^2-y_1^2-y_2^2)}\Big)_{(x_1,x_2)}$$

$$=\frac{1}{2\pi}\left(\frac{1}{x_1-y_1+i(x_2-y_2)}\left(-\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}\right)\right.$$
$$\left.\cdot\varphi(x_1,x_2,\ -i(x_1^2+x_2^2-y_1^2-y_2^2))\right)_{(x_1,x_2)}$$

$$=\frac{1}{2\pi}\lim_{\varepsilon\to 0}\iint\limits_{(x_1-y_1)^2+(x_2-y_2)^2\geqslant\varepsilon^2}\frac{1}{(x_1-y_1)+i(x_2-y_2)}\left(-\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}\right)$$
$$\cdot\varphi(x_1,x_2,-i(x_1^2+x_2^2-y_1^2-y_2^2))dx_1dx_2$$

$$=\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int\limits_{(x_1-y_1)^2+(x_2-y_2)^2=\varepsilon^2}\frac{\varphi(x_1,x_2,-i(x_1^2+x_2^2-y_1^2-y_2^2))}{(x_1-y_1)+i(x_2-y_2)}[\cos(n,x)+i\cos(n,y)]ds$$

$$=\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_0^{2\pi}\varphi(x_1,x_2,-i(x_1^2+x_2^2-y_1^2-y_2^2)d\theta$$

$$=\varphi(y_1,y_2,0)=(\delta(x_1-y_1,\ x_2-y_2,\ x_3),\ \varphi(x_1,x_2,x_3)).$$

定理一证毕.

定理二 若 $f\in\mathscr{E}'(x_1,x_2,x_3)$

则方程 $Lu=f$

有解 $u=E\otimes f\ \in F'_{DZ}$

这里 $(E\otimes f,\ \varphi)=(\check{f},\ E_\circ\otimes\check{\varphi})$

$$E_\circ=\frac{\delta[x_3-i(x_1^2+x_2^2-y_1^2-y_2^2)]}{2\pi[(x_1-y_1)+i(x_2-y_2)]}$$

$$E_\circ\otimes\varphi=\left(\frac{\delta[x_3-y_3-i(x_1^2+x_2^2-y_1^2-y_2^2)]}{2\pi[(x_1-y_1)+i(x_1-y_2)]},\ \varphi(y_1,y_2,y_3)\right)_{(y_1,y_2,y_3)}$$

$$\check{\varphi}(x_1,x_2,x_3)=\varphi(-x_1,-x_2,-x_3)$$

证明
$$E_\circ\otimes\check{\varphi}=\left(\frac{\delta[x_3-y_3-i(x_1^2+x_2^2-y_1^2-y_2^2)]}{2\pi[(x_1-y_1)+i(x_2-y_2)]},\ \varphi(-y_1,-y_2,-y_3)\right)_{(y_1,y_2,y_3)}$$

$$=\left(\frac{1}{2\pi[(x_1-y_1)+i(x_2-y_2)]},\varphi(-y_1,-y_2,-x_3+i(x_1^2+x_2^2-y_1^2-y_2^2))\right)_{(y_1,y_2)}$$

$$=\left(\frac{1}{2\pi[X_1+iX_2]},\ \varphi(X_1-x_1,\ X_2-x_2,\ -x_3+i(x_1^2+x_2^2-(X_1-x_1)^2\right.$$
$$\left.-(X_2-x_2)^2)\right)_{(X_1,X_2)}\in C^\infty(R_3)$$

由于 f 紧台，故存在某向量指标 α（分量为非负整数）和紧台连续函数 $F(x_1,x_2,x_3)$ 使 $\check{f}=\frac{\partial^{|\alpha|}F}{\partial x^\alpha}$ 成立（在 f 的台域内）. 所以有

$$|(\check{f},\ E_\circ\otimes\check{\varphi})|\leqslant C\sum_{|\beta|<2\alpha_1+2\alpha_2+\alpha_3}\ \max_{\substack{(x_1,x_2)\in K\subset R_2\\ \zeta\text{在}\zeta\text{平面紧域}}}\left|\frac{\partial^{|\beta|}\varphi(x_1,x_2,\zeta)}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}\partial x_3^{\beta_3}}\right|$$

故有 $u=E\otimes f\in F'_{DZ}$. 再证 $Lu=L(E\otimes f)=f$,

事实上，$(L(E\otimes f),\ \varphi)=(E\otimes f,\ {}'L\varphi)=(\check{f},\ E_\circ\otimes({}'\check{L}\varphi))$

注意 $${}'\check{L}=-\left[\left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right)+2i(x_1+ix_2)\frac{\partial}{\partial x_3}\right],$$

$E_\circ=\dfrac{\delta[x_3-i(x_1^2+x_2^2-y_1^2-y_2^2)]}{2\pi[(x_1-y_1)+i(x_2-y_2)]}$，由和定理一一样的证明，可知

$$\check{L}E_\circ=\delta(x_1-y_1,\ x_2-y_2,\ x_3).$$

故有 $(L(E\otimes f),\varphi)=(\check{f},(\delta(x_1-y_1,\ x_2-y_2,\ x_3-y_3),\check{\varphi}))=(f,\ \varphi)$，

即得 $L(E\otimes f)=f$.

定理二证毕.

这样一来，Lewy 方程的局部可解性(即当$f\in\mathscr{E}'(x_1,x_2,x_3)$就在空间 $F'_{DZ}(x_1,x_2,x_3)$中得到解决. 特别当 $f(x_1,x_2,x_3)$关于 x_3 在 $x_3=0$ 的邻域为解析时，可知在原点邻域有解：

$$u=\frac{1}{2\pi}\iint_G\frac{f(y_1,y_2,x_3+i(x_1^2+x_2^2-y_1^2-y_2^2))}{(x_1-y_1)+i(x_2-y_2)}dy_1dy_2,$$

G为 $x_1=0$，$x_2=0$ 的导邻域.

特别当 $f(x_1,x_2,x_3)=\delta(x_1,x_2)\psi(x_3)$， $\psi(x_3)$ 解析时，可知有解：

$$u=\frac{1}{2\pi}\cdot\frac{\psi(x_3+i(x_1^2+x_2^2))}{x_1+ix_2}.$$

而对方程 $Lu=-i\psi'(x_3)$，其中 $\psi(x_3)$解析，则知有解

$$u=\frac{\psi(x_3)-\psi(x_3+i(x_1^2+x_2^2))}{2(x_1+ix_2)}+g(x_1+ix_2,x_3+i(x_1^2+x_2^2)),$$

g 为关于 x_1,x_2,x_3 解析的任意函数.

特别，当 $\psi=1$，x_3，$\sin x_3$ 时，分别有解 $u=0$，$\dfrac{i(x_1^2+x_2^2)}{2(x_1+ix_2)}$，

$$\frac{\sin x_3-\sin(x_3+i(x_1^2+x_2^2))}{2(x_1+ix_2)}.$$

如果$\psi(x_3)$ 不解析，例如 $\psi(x_3)=|x_3|$，则在解作用于基本函数 $\varphi(x_1,x_2,x_3)$ 时，涉及 $\varphi(x_1,x_2,x_3+i\eta)=\sum_{n=0}^{\infty}\dfrac{\partial^n\varphi(x_1,x_2,x_3)}{\partial x_3^n}\cdot\dfrac{(i\eta)^n}{n!}$. 故解为局部无限阶的，超出普通广义函数的范围.

参考文献

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Local Solvability of Lewy's Equation

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Abstract

In this paper we introduce a space $F_{DZ}(x_1,x_2;x_3)$ which is the direct product of the Space $\mathscr{D}$ in (x_1,x_2), and the space Z of integral functions in x_3. We find out the solutions of Lewy's equation by means of Fourier transformation in the dual space of the F_{DZ}. The fundamental solution of Lewy's equation is

$$E(x_1-y_1,\ x_2-y_2;\ x_3)=\frac{1}{2\pi}\frac{\delta[x_3+i(x_1^2+x_2^2-y_1^2-y_2^2)]}{(x_1-y_1)+i(x_2-y_2)}.$$