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一类随机代数方程实根的平均个数的界*

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对于这样一类随机代数方程

$$F_n(t)=a_0+a_1t+\cdots\cdots+a_{n-1}t^{n-1}=0,$$

其中 $a_i(i=0,1,2,\cdots\cdots,n-1)$ 是遵从标准正态分布 N(0,1) 的独立的随机变量，其实根的平均个数 EN_{F_n} 历史上有过不少估计，稍近一些有 1943 年 Kac[1]的估计

$$EN_{F_n}\leqslant\frac{2}{\pi}\ln n+\frac{14}{\pi},$$

1965 年 Stevens[2]的估计

$$\frac{2}{\pi}\ln n-0.6<EN_{F_n}<\frac{2}{\pi}\ln n+1.4,$$

1980年骆振华[3]的估计

$$\frac{2}{\pi}\ln n\leqslant EN_{F_n}\leqslant\frac{2}{\pi}\ln n+1.2373,$$

1981年王友菁[4]的估计

$$EN_{F_n}\leqslant\frac{2}{\pi}\ln n+0.9003,$$

及

$$\varliminf_{n\to\infty}\left[EN_{F_n}-\frac{2}{\pi}\ln n\right]\geqslant 0.4671.$$

现在，我们进一步改进王友菁[4]的估计，并求得上下界问题的彻底解决.

引理 1 在 (0,1) 上，$-n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})\uparrow 2x\ln x$.

证 在 (0,1) 上，我们有

$$-n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})=-nx\,(x^{-\frac{1}{n}}-x^{\frac{1}{n}})$$

$$=-nx(e^{-\frac{1}{n}\ln x}-e^{\frac{1}{n}\ln x})$$

$$=-nx\left\{\left[1-\frac{1}{n}\ln x+\frac{1}{2!}\left(\frac{1}{n}\ln x\right)^2-\frac{1}{3!}\left(\frac{1}{n}\ln x\right)^3+\cdots\cdots\right]\right.$$

$$\left.-\left[1+\frac{1}{n}\ln x+\frac{1}{2!}\left(\frac{1}{n}\ln x\right)^2+\frac{1}{3!}\left(\frac{1}{n}\ln x\right)^3+\cdots\cdots\right]\right\}.$$

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$$=2x\ln x+2x\left[\frac{\ln^3x}{3!n^2}+\frac{\ln^5x}{5!n^4}+\frac{\ln^7x}{7!n^6}+\cdots\cdots\right].$$

不难看出，对 (0,1) 上任一固定的 x，数列

$$2x\left[\frac{\mathrm{n}^3x}{3!n^2}+\frac{1}{5!n^4}\ln^5x+\frac{\ln^7x}{7!n^6}+\cdots\cdots\right]$$

是递增趋向于 0 的。可知，在 (0,1) 上函数序列 $-n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})$ 递增收领于 $2x\ln x$。引理得证。

引理 2 在 (0,1) 上，函数序列

$$\sqrt{\frac{1-x^2+n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}{1-x^2-n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}},\qquad n\geqslant 2$$

是单调递降的，且

$$\lim_{n\to\infty}\sqrt{\frac{1-x^2+n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}{1-x^2-n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}}=\sqrt{\frac{1-x^2-2x\ln x}{1-x^2+2x\ln x}}$$

证 由引理 1，在 (0,1) 上，

$$0<1-x^2+n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})\downarrow 1-x^2-2x\ln x,$$

$$0<1-x^2-n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})\uparrow 1-x^2+2x\ln x,$$

$$\therefore\qquad \sqrt{\frac{1-x^2+n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}{1-x^2-n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}}\downarrow\sqrt{\frac{1-x^2-2x\ln x}{1-x^2+2x\ln x}}.$$

引理得证。

现在，我们考虑 EN_{F_n} 的界。在 [4] 中已求得

$$\int_0^1\frac{1-h_n(t)}{1-t^2}dt=\frac{1}{2}\ln n,$$

这里

$$h_n(t)=\frac{nt^{n-1}(1-t^2)}{1-t^{2n}},\quad 0\leqslant t<1.$$

事实上，

$$\int_0^1\frac{1-h_n(t)}{1-t^2}dt=\lim_{S\to 1-0}\int_0^S\left[\frac{1}{1-t^2}-\frac{nt^{n-1}}{1-t^2n}\right]dt$$

$$=\lim_{S\to 1-0}\left[\int_0^S\frac{1}{1-t^2}dt-\int_0^{S^n}\frac{1}{1-x^2}dx\right]$$

$$=\lim_{S\to 1-0}\int_{S^n}^S\frac{1}{1-x^2}dx=\lim_{S\to 1-0}\frac{1}{2}\ln\frac{(1+s)(1-s^n)}{(1-s)(1+s^n)}$$

$$=\frac{1}{2}\ln n.$$

因此，根据 Kac[1] 的结果（也可参看 [3]）

$$EN_{F_n}=\frac{4}{\pi}\int_0^1\frac{1}{1-t^2}\sqrt{1-h_n^2(t)}dt,$$

有

$$EN_{F_n}=\frac{4}{\pi}\left[\int_0^1\frac{1-h_n(t)}{1-t^2}dt+\int_0^1\frac{\sqrt{1-h_n^2(t)}-[1-h_n(t)]}{1-t^2}dt\right]$$

$$=\frac{4}{\pi}\cdot\frac{1}{2}\ln n+\frac{4}{\pi}\int_0^1\frac{\sqrt{1-h_n(t)}[\sqrt{1+h_n(t)}-\sqrt{1-h_n(t)}]}{1-t^2}dt$$

$$=\frac{2}{\pi}\ln n+\frac{4}{\pi}\int_0^1\frac{\sqrt{1-h_n(t)}\cdot 2h_n(t)}{(1-t^2)[\sqrt{1+h_n(t)}+\sqrt{1-h_n(t)}]}dt$$

$$=\frac{2}{\pi}\ln n+\frac{8}{\pi}\int_0^1\frac{h_n(t)}{(1-t^2)\left[\sqrt{\frac{1+h_n(t)}{1-h_n(t)}}+1\right]}dt$$

$$=\frac{2}{\pi}\ln n+\frac{8}{\pi}\int_0^1\frac{nt^{n-1}}{(1-t^{2n})\left[\sqrt{\frac{1-t^{2n}+nt^{n-1}(1-t^2)}{1-t^{2n}-nt^{n-1}(1-t^2)}}+1\right]}dt$$

$$=\frac{2}{\pi}\ln n+\frac{8}{\pi}\int_0^1\frac{1}{(1-x^2)\left[\sqrt{\frac{1-x^2+n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}{1-x^2-n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}}+1\right]}dx$$

(令 $t^n=x$)。

根据引理 2，在 (0,1) 上，

$$\sqrt{\frac{1-x^2+n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}{1-x^2-n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}}\downarrow\sqrt{\frac{1-x^2-2x\ln x}{1-x^2+2x\ln x}},$$

∴

$$\frac{1}{\sqrt{\frac{1-x^2+n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}{1-x^2-n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}}+1}\uparrow\frac{1}{\sqrt{\frac{1-x^2-2x\ln x}{1-x^2+2x\ln x}}+1}.$$

于是

$$\sup_{n\geq 2}\left[EN_{F_n}-\frac{2}{\pi}\ln n\right]=\lim_{n\to\infty}\frac{8}{\pi}\int_0^1\frac{dt}{(1-x^2)\left[\sqrt{\frac{1-x^2+n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}{1-x^2-n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}}+1\right]}$$

$$=\frac{8}{\pi}\int_0^1\frac{\sqrt{1-x^2+2x\ln x}}{(1-x^2)[\sqrt{1-x^2-2x\ln x}+\sqrt{1-x^2+2x\ln x}]}dx$$

≈ 0.63.

又由引理 2，

$$\sqrt{\frac{1-x^2+n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}{1-x^2-n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}}\leq\sqrt{\frac{1-x^2+2(x^{\frac{1}{2}}-x^{\frac{3}{2}})}{1-x^2-2(x^{\frac{1}{2}}-x^{\frac{3}{2}})}},\qquad n\geq 2,$$

$$\therefore \quad \frac{1}{\sqrt{\dfrac{1-x^2+n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}{1-x^2-n(x^{1-\frac{1}{n}}-x^{1+\frac{1}{n}})}+1}} \geqslant \frac{1}{\sqrt{\dfrac{1-x^2+2(x^{\frac{1}{2}}-x^{\frac{3}{2}})}{1-x^2-2(x^{\frac{1}{2}}-x^{\frac{3}{2}})}+1}}, \qquad n\geqslant 2.$$

因些

$$\inf_{n\geq 2}\left[EN_{F_n}-\frac{2}{\pi}\ln n\right]=\frac{8}{\pi}\int_0^1\frac{\sqrt{1-x^2-2(x^{\frac{1}{2}}-x^{\frac{3}{2}})}\,dx}{(1-x^2)[\sqrt{1-x^2+2(x^{\frac{1}{2}}-x^{\frac{3}{2}})}+\sqrt{1-x^2-2(x^{\frac{1}{2}}-x^{\frac{3}{2}})}]}$$

$$\approx 0.56.$$

对于$n=1$，$F_1(t)=0$不是代表方程了，故可以不考虑。因此，对$n\geqslant 2$，关于$\left[EN_{F_n}-\frac{2}{\pi}\ln n\right]$的上确界和下确界，我们的结果是

定理 对于系数$a_0,a_1,\cdots,a_{n-1}$是独立同分布$N(0,1)$的随机变量的代数方程

$$F_n(t)=a_0+a_1t+\cdots+a_{n-1}t^{n-1}=0,$$

表EN_{F_n}为其实根的平均个数，则

$$EN_{F_n}-\frac{2}{\pi}\ln n,\quad n\geqslant 2$$

是单调递增数列，且

$$\sup_{n\geq 2}\left[EN_{F_n}-\frac{2}{\pi}\ln n\right]=\frac{8}{\pi}\int_0^1\frac{\sqrt{1-x^2+2x\ln x}}{(1-x^2)[\sqrt{1-x^2-2x\ln x}+\sqrt{1-x^2+2x\ln x}]}dx$$

$$\approx 0.63,$$

$$\inf_{n\geq 2}\left[EN_{F_n}-\frac{2}{\pi}\ln n\right]=\frac{8}{\pi}\int_0^1\frac{\sqrt{1-x^2-2(x^{\frac{1}{2}}-x^{\frac{3}{2}})}\,dx}{(1-x^2)[\sqrt{1-x^2+2(x^{\frac{1}{2}}-x^{\frac{3}{2}})}+\sqrt{1-x^2-2(x^{\frac{1}{2}}-x^{\frac{3}{2}})}]}$$

$$\approx 0.56.$$

在本课题的研究过程中，曾多次与王友菁同志讨论，很受启发，作者在此深致谢意！

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The Bounds of Average Number of Real Roots for a Class of Random Algebraic Equations

By *Yu Zhongming* (俞中明)

Abstract

We consider following random algebraic equation

$$F_n(t)=a_0+a_1t+\cdots a_{n-1}t^{n-1}=0,$$

here $a_i(i=0,1,\cdots,n-1)$ are independet random variables with identical distribution $N(0,1)$. The average number EN_{F_n} of real roots of the equation has estimed by many authors, For example. Kac's estimation is

$$EN_{F_n}\leqslant\frac{2}{\pi}\ln n+\frac{14}{\pi},$$

Stevens' estimation is

$$\frac{2}{\pi}\ln n-0.6<EN_{F_n}<\frac{2}{\pi}\ln n+1.4,$$

Lao Zhenhua's estimation is

$$\frac{2}{\pi}\ln n\leqslant EN_{F_n}\leqslant\frac{2}{\pi}\ln n+1.2373,$$

Wang Youjing's estimation is

$$EN_{F_n}\leqslant\frac{2}{\pi}\ln n+0.9003$$

and

$$\lim_{n\to\infty}\left[EN_{F_n}-\frac{2}{\pi}\ln n\right]\geqslant 0.4671.$$

In this paper we attempt to solve completely the problem of estimation of bounds perfectly. Our results are that $\left\{EN_{F_n}-\frac{2}{\pi}\ln n\right\}_{n\geq 2}$ is a increasing sequence of real numbers and

$$\sup_{n\geq 2}\left[EN_{F_n}-\frac{2}{\pi}\ln n\right]=\frac{8}{\pi}\int_0^1\frac{\sqrt{1-x^2+2x\ln x}\,dx}{(1-x^2)[\sqrt{1-x^2-2x\ln x}+\sqrt{1-x^2+2x\ln x}\,]}$$
$$\approx 0.63,$$

$$\inf_{n\geq 2}\left[EN_{F_n}-\frac{2}{\pi}\ln n\right]=\frac{8}{\pi}\int_0^1\frac{\sqrt{1-x^2-2(x^{\frac{1}{2}}-x^{\frac{3}{2}})}\,dx}{(1-x^2)\left[\sqrt{1-x^2+2(x^{\frac{1}{2}}-x^{\frac{3}{2}})}+\sqrt{1-x^2-2(x^{\frac{1}{2}}-x^{\frac{3}{2}})}\right]}$$
$$\approx 0.56.$$