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On a Method of Constructing Interpolation Formulas via Inverse Series Relations*

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Introduction

In this paper we shall describe a somewhat fruitful method that can be used to obtain various equi-distant interpolation formulas via inverse series relations. The main step of the method is to replace the discrete parameter contained in an inverse series relation by a continuous one, and if necessary, to transform the series summation so as to make it depend continuosly upon the continuous real parameter. We shall give a number of examples illustrating the method. In particular, a kind of simpler interpolation formulas using differences will be derived as a consequence of a generalized Möbius inversion, and it will be expounded that such formulas may be conveniently used to solve interpolation problems defined on $[0,\infty)$ as well as on $[0,\infty)\times[0,\infty)$, etc. Finally a unified explicit formula for piece-wise polynomial interpolation of any degree will be discussed in some detail.

§1. Description of the Method

Suppose that we are given a pair of finite inverse series relations of the form

$$f(n)=\sum_{k=0}^{n}\Phi(n,k)g(k) \qquad (1)$$

$$g(n)=\sum_{k=0}^{n}\Psi(n,k)f(k) \qquad (2)$$

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where $\Phi(n,k)$ and $\Psi(n,k)$ are the kernels of the series transformation satisfying the orthogonality relations

$$\sum_{k=j}^{n}\Phi(n,k)\Psi(k,j)=\sum_{k=j}^{n}\Psi(n,k)\Phi(k,j)=\delta_{n,j}$$

with δ_{nj} denoting the Kronecker delta and $\Phi(n,k)=\Psi(n,k)=0$ for $k>n$. If, in particular, the definition of the kernel function $\Phi(n,k)$ can be extended analytically to be a function $\Phi(x,k)$ of the continuous real variable x in $[0,\infty)$, and if $\langle x\rangle$ denotes the integer nearest to x, (i. e., $(x)=\left[x+\frac{1}{2}\right]$, $[\alpha]$ being the integral part of α), then we may define

$$S_1(f;x)=\sum_{k=0}^{[x]}\Phi(x,k)g(k) \tag{3}$$

$$S_2(f;x)=\sum_{k=0}^{\langle x\rangle}\Phi(x,k)g(k) \tag{4}$$

as a pair of interpolation formulas for a given sequence $\{f(k)\}_0^\infty$ (with $\{g(k)\}$ as a transformed sequence of $\{f(k)\}$), since in fact we have

$$S_1(f;n)=S_2(f;n)=f(n),\quad n=0,\ 1,\ 2,\cdots\cdots \tag{5}$$

Sometimes it may even be possible to introduce a formal series

$$S_3(f:x)\overset{\text{def}}{=}\sum_{k=0}^{\infty}\Phi(x,k)g(k) \tag{6}$$

to represent an interpolation formula for $\{f(k)\}$. But in such case there are involved some convergence problems regarding (6).

Notice that in general neither (3) nor (4) is a continuous function of x. In fact, we have

$$S_1(f;n+)-S_1(f;n-)=\Phi(n,n)g(n),$$

$$S_2(f;(n+\tfrac{1}{2})+)-S_2(f;(n+\tfrac{1}{2})-)=\Phi(n+\tfrac{1}{2},n+1)g(n+1),$$

where $S_i(f;\alpha\pm)$ $(i=1,2)$ denote the limits of $S_i(f;x)$ as $x\to\alpha+$ and $x\to\alpha-$ respectively, and $\Phi(x,k)$ is assumed to be continuous in x for each k. However, a suitable combination of (3) and (4) may yield a continuous interpolation for $\{f(k)\}$. Let us now state a simple proposition as follows.

Proposition Let $S_1(f;x)$ and $S_2(f;x)$ be defined respectively by (3) and (4) in which $\Phi(x;k)$ is a continuous function of x $(0\leqslant x<\infty)$ for each fixed k. Let $\{x\}$ denote the distance of x from $\langle x\rangle$, i. e. $\{x\}=|\langle x\rangle-x|$ with $\langle x\rangle$ denoting the nearest integer to x. Then S(f; x) defined by the following expression

$$S(f;x)=2\{x\}\cdot S_1(f;x)+(1-2\{x\})\cdot S_2(f;x) \tag{7}$$

is a continuous interpolation formula for any given sequence $\{f(k)\}$. In words, $S(f;x)$ is a continuous function of $x(0\leqslant x<\infty)$ satisfying the interpolation condition $S(f;n)=f(n)$, $(n=0, 1, 2, \cdots)$.

Observe that $S_1(f;x)$ and $S_2(f;x)$ are continuous everywhere except at those points $x=n$ and $x=n+\frac{1}{2}$ respectively, where $n=0, 1, 2, \cdots$. Thus it suffices to show that $S(f;x)$ given by (7) is continuous at $x=n$ and $x=n+\frac{1}{2}$. As may be verified at once, we have

$$\lim_{x\to n} S(f;x)=S_2(f;n)=S(f;n);$$

$$\lim_{x\to(n+\frac{1}{2})} S(f;x)=S_1(f;n+\frac{1}{2})=S(f;n+\frac{1}{2}).$$

Moreover, it is clear that $S(f;n)=S_2(f;n)=f(n)$ $(n=0, 1, \cdots)$. Hence the proposition.

Certainly there are other forms of combination that may lead continuous $S(f;x)$ similar to (7).

Example 1 The simplest reciprocal pair is the δ-transform corresponding to the case

$$\Phi(n,k)=\Psi(n,k)=(-1)^k\binom{n}{k}.$$

In this case we may write

$$g(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k)=(-1)^n\Delta^n f(0)$$

so that (3), (4) and (6) may be expressed in the for ms

$$S_1(f;x)=\sum_{k=0}^{[x]}\binom{x}{k}\Delta^k f(0), \tag{8.1}$$

$$S_2(f;x)=\sum_{k=0}^{<x>}\binom{x}{k}\Delta^k f(0), \tag{8.2}$$

$$S_3(f;x)\overset{def}{=}\sum_{k=0}^{\infty}\binom{x}{k}\Delta^k f(0), \tag{8.3}$$

respectively. These are the well-known Newton type interpolation formulas.

Example 2 Denote by $B_k (k=0, 1, \cdots)$ the Bernoulli numbers in the even suffix notation so that $B_{2k+1}=0, (k=1, 2, \cdots)$. As may be verified, there are inverse series relations

$$f(n)=\sum_{k=0}^{n}\binom{n}{k}(n-k+1)^{-1}g(k) \tag{9}$$

$$g(n)=\sum_{k=0}^{n}\binom{n}{k}B_kf(n-k) \tag{10}$$

It may be observed that the above equation (9) has been given incorrectly in Riordan's book [7] (cf. Table 3.3; the factor $(k+1)^{-1}$ contained in the first formula should be replaced by $(n-k+1)^{-1}$). Write $B_k\equiv B^k$, $f(k)\equiv f^k$, so that a symbolic abbreviation for $g(n)$ may be written as

$$g(n)=\sum_{k=0}^{n}\binom{n}{k}B^kf^{n-k}\equiv(B+f)^n$$

in accordance with Blissard calculus. Let us now define

$$S_1(f;x)=\sum_{k=0}^{[x]}\binom{x}{k}\frac{(B+f)^k}{x-k+1}, \tag{11}$$

$$S_2(f;x)=\sum_{k=0}^{\langle x\rangle}\binom{x}{k}\frac{(B+f)^k}{x-k+1}, \tag{12}$$

Then the combination of above $S_1(f;x)$ and $S_2(f;x)$ in the form of (7) gives a continuous interpolation for $\{f(k)\}$ in $[0,\infty)$.

Example 3 The so-called Lah numbers are defined by

$$L_{n;k}=(-1)^n\frac{n!}{k!}\binom{n-1}{k-1},\quad k=1(1)n,\quad n=1,\ 2,\ \cdots$$

with $L_{n,0}=\delta_{n,0}$. For any given sequence $\{f(k)\}$ we call the sequence $\{g(n)\}$ given by

$$g(n)=\sum_{k=1}^{n}(-1)^kL_{n\ k}f(k)\equiv g(f;n)$$

the Lah tiansform of $\{f\}$. Then we may define

$$S_1(f;x)=\sum_{k=1}^{[x]}\frac{\Gamma(x+1)}{k!}\binom{x-1}{k-1}g(f;k) \tag{13.1}$$

$$S_2(f;x)=\sum_{k=1}^{\langle x\rangle}\frac{\Gamma(x+1)}{k!}\binom{x-1}{k-1}g(f;k) \tag{13.2}$$

$$S_3(f;x)\overset{def}{=}\sum_{k=1}^{\infty}\frac{\Gamma(x+1)}{k!}\binom{x-1}{k-1}g(f;k) \tag{13.3}$$

These are interpolation formulas for $\{f(k)\}$. In particular the combination of (13.1) and (13.2) in the form of (7) gives a continuous interpolation on $[0,\infty)$.

Example 4 Define the Legendre sequence transform of $\{f(k)\}$ (cf. Riordan [7], Table 2.5) by the following

$$g(n)=\sum_{k=0}^{n}(-1)^{k+n}\left(\binom{2n+p}{n-k}-\binom{2n+p}{n-k-1}\right)f(k)\equiv g(f;n)$$

where p is any fixed non-negative integer. Then we have the interpolation formulas

$$S_1(f;x)=\sum_{k=0}^{[x]}\binom{x+p+k}{p+2k}g(f;k) \qquad (14.1)$$

$$S_2(f;x)=\sum_{k=0}^{\langle x\rangle}\binom{x+p+k}{p+2k}g(f;k) \qquad (14.2)$$

$$S_3(f;n)\overset{def}{=}\sum_{k=0}^{\infty}\binom{x+p+k}{p+2k}g(f;k) \qquad (14.3)$$

Example 5 Let c and p be positive integers. Define the Legendre-Chebyshev sequence transform of $\{f(k)\}$ (cf. Riordan [7], Table 2.6) by the following

$$g(x)=\sum_{k=0}^{n}\binom{cn+p}{n-k}(-1)^k f(k)\equiv g(f;n)$$

Then we have the interpolation formulas

$$S_1(f;x)=\sum_{k=0}^{[x]}(-1)^k\left(\binom{x+p-1+(c-1)k}{p-1+ck}+c\binom{x+p-1+(c-1)k}{p+ck}\right)g(f;k) \qquad (15.1)$$

$$S_2(f;x)\overset{def}{=}\sum_{k=0}^{\infty}(-1)^k\left(\binom{x+p-1+(c-1)k}{p-1-ck}+c\binom{x+p-1+(c-1)k}{p+ck}\right)g(f;k) \qquad (15.2)$$

Example 6 Let $\{a_k\}$ and $\{b_n\}$ be any given sequences of numbers such that $\psi(x,n)=\prod_{i=1}^{n}(a_i+b_ix)\neq 0$ for $x=0,1,2,\cdots$, with $\psi(x,0)\equiv 1$. Then starting with the following inverse series relation due to Gould and Hsu [2]

$$g(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\psi(k,n)f(k)$$

$$f(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a_{k+1}+kb_{k+1}}{\psi(n,k+1)}g(k)$$

and noticing that $g(n)=(-1)^n\Delta^n(\psi(x,n)f(x))_{x=0}$ we may get the following interpolation series

$$S(f;x)\overset{def}{=}\sum_{k=0}^{\infty}\frac{a_{k+1}+kb_{k+1}}{\psi(x,k+1)}\binom{x}{k}\Delta^k(\psi(x,k)\,f(x))_{x=0} \qquad (16)$$

Evidently for the particular case $a_n=1$, $b_n=0$ $(n=1,2,\cdots)$ the righthand side of (16) will reduce to the Newton interpolation series. Since the interpolation condition $S(f;n)=f(n)$ $(n=0,1,2,\cdots)$ is always fulfiled for whatever given parameters a_n and b_n $(n=1,2,\cdots)$ we see that partial sums of the series (16) may yield a variety of rational interpolation formulas. A detailed investigation of (16) with an eye to making it applicable to numerical analysis has been developed in our previous papers [5] [6]. Actually we have shown that (16) can represent any rational functions (without poles at $x=0,1,2,\cdots$) by suitable choices of the parameters a_n and b_n, and that the differences $F(x,k)=\Delta^k(\psi(x,k)f(x))$ $(k=0,1,2,\cdots)$ satisfy the recurrence relations

$$F(x,k+1)=(a_{k+1}+b_{k+1}x)(F(x+1,k)-F(x,k))+$$
$$+(k+1)b_{k+1}F(x+1,k)$$

with $F(x,0)\equiv f(x)$. Some convergence conditions for (16) have been also discussed previously (loc cit).

§2 A Kind of Interpolation Formulas Using Finite Differences

Any two sequences $\{a_k\}$ and $\{b_k\}$ of real numbers (with $a_0\neq 0$, $b_0\neq 0$) are called a pair of reciprocal sequences if

$$\left(\sum_{k=0}^{\infty}a_k x^k\right)\left(\sum_{k=0}^{\infty}b_k x^k\right)\equiv 1,$$

where the multiplication is performed formally in accordance with Cauchy's rule of product. In order to get a large class of equidistant interpolation formulas we shall make use of the following known result which was actually obtained as a consequence of a generalized Möbius-Rota inversion theorem [3].

Proposition For any given reciprocal sequences $\{a_k\}$ and $\{b_k\}$ we always have the pair of inverse series relations:

$$f(n)=\sum_{k=1}^{n}\left(\sum_{r=0}^{n-k}\binom{n-k-1}{r-1}a_r\right)g(k) \tag{17}$$

$$g(n)=\sum_{k=1}^{n}\left(\sum_{r=0}^{n-k}\binom{n-k-1}{r-1}b_r\right)f(k) \tag{18}$$

where we have to adopt the convention for binomial coefficients $\binom{0}{0}=\binom{-1}{-1}=1$, $\binom{-1}{n}=\binom{n}{-1}=0$, $(n\neq -1)$.

Let us call $\{g(n)\}$ the transformed sequence of $\{f(k)\}$ and denote it by $g(f;n)=g(n)$, $(n=1,2,\cdots)$. Then we may construct interpolation formulas of the forms

$$S_1(f;x)=\sum_{k=1}^{[x]}\left(\sum_{r=0}^{[x]-k}\binom{x-k-1}{r-1}a_r\right)g(f;k) \tag{19}$$

$$S_2(f;x)=\sum_{k=1}^{\langle x\rangle}\left(\sum_{r=0}^{\langle x\rangle-1}\binom{x-k-1}{r-1}a_r\right)g(f;k) \tag{20}$$

A combination of $S_1(f;x)$ and $S_2(f;x)$ in the form of (7) may yield a continuous interpolation on $[0,\infty)$.

Obviously both (19) and (20) imply infinitely many particular interpolation formulas since $\{a_k\}$ $(a_0\neq 0)$ can be assigned arbitrarily. However, only simply constructed interpolation formulas may be useful for practical computation. In what follows we shall seek for such formulas by taking

$$a_k=\binom{\lambda}{k},\quad b_k=\binom{-\lambda}{k},\qquad (k=0,1,2,\cdots)$$

where λ is a fixed integer not less than 2. Clearly such $\{a_k\}$ and $\{b_k\}$ are reciprocal sequences. Then starting with (17) and (18) we can finally arrive at the following

Theorem 1 Let $\lambda\geqslant 2$ be a fixed integer parameter, and let $\{f(n)\}_{n=0}^{\infty}$ be any given number-sequence with supplemental definition $f(-m)=0$ $(m=1,2,3,\cdots)$. Then the function defined by the following

$$S(f;x)=\sum_{k=1}^{[x]+\lambda}\binom{x+\lambda-k}{\lambda-1}\Delta^{\lambda}f(k-\lambda-1) \tag{21}$$

possesses three properties: (1) it satisfies the interpolation condition $S(f;n)=f(n)$, $n=0,1,2,\cdots$; (2) it is a continuous function of x $(0\leqslant x<\infty)$; (3) for every polynomial $\varphi(x)$ of degree$\leqslant\lambda-1$ defined for $x\geqslant 0$ with supplemental definition $\varphi(-m)=0$ $(m=1,2,\cdots)$ we always have $S(\varphi;x)\equiv\varphi(x)$, $(0\leqslant x<\infty)$.

Proof 1° Making use of the well-known combinatorial identity (cf. Gould [1] formula (3.2), p. 22)

$$\sum_{k=0}^{n}\binom{x+k}{k}\binom{y+n-k}{n-k}=\binom{x+y+n+1}{n} \tag{$*$}$$

we can easily verify that $S(f;x)$ satisfies the interpolation condition. In fact we have, by (21) and ($*$),

$$\begin{aligned}S(f;n)&=\sum_{k=1}^{n+1}\binom{n+\lambda-k}{\lambda-1}\left(\sum_{j=0}^{\lambda}\binom{\lambda}{j}(-1)^{j}f(k-1-j)\right)\\&=\sum_{k=1}^{n+1}\binom{n+\lambda-k}{\lambda-1}\sum_{j=0}^{k-1}\binom{\lambda}{k-1-j}(-1)^{k-1-j}f(j)\\&=\sum_{j=0}^{n}f(j)\sum_{k=j+1}^{n+1}\binom{n+\lambda-k}{\lambda-1}\binom{\lambda}{k-j-1}(-1)^{k-j-1}\\&=\sum_{j=0}^{n}f(j)\sum_{k=j+1}^{n+1}\binom{n+\lambda-k}{\lambda-1}\binom{-\lambda+k-2-j}{k-1-j}\\&=\sum_{j=0}^{n}f(j)\sum_{\nu=0}^{n-j}\binom{n-j-1+\lambda-\nu}{n-j-\nu}\binom{-\lambda-1+\nu}{\nu}\\&=\sum_{j=0}^{n}f(j)\binom{n-j-1}{n-j}=f(n).\end{aligned}$$

2° To prove that $S(f;x)$ is continuous notice first that for nonnegative integer $m<n$ we have $\lim\limits_{x\to m}\binom{x}{n}=0$. Since $S(f;x)$ is continuous everywhere in $[0,\infty)$ except at integer points it suffices to consider the case $x=n$ $(n=1,2,\cdots)$.

Now it is easily found that

$$\lim_{\varepsilon\to 0+} S(f;n+\varepsilon)=\lim_{\varepsilon\to 0+}\sum_{k=1}^{n+1}\binom{n+\varepsilon+\lambda-k}{\lambda-1}\Delta^\lambda f(k-\lambda-1)=S(f;n)=f(n).$$

Similarly we may verify that

$$\lim_{\varepsilon\to 0+} S(f;n-\varepsilon)=f(n).$$

3° Suppose that $f(x)$ is a polynomial of degree $\leqslant\lambda-1$ defined for $x\geqslant 0$ with $f(-m)=0$ $(m=1,2,\cdots\lambda)$. Then

$$\Delta^\lambda f(0)=\Delta^\lambda f(1)=\Delta^\lambda f(2)=\cdots=0.$$

Thus (21) may be rewritten as

$$S(f;x)=\sum_{k=1}^{\lambda}\binom{x+\lambda-k}{\lambda-1}\Delta^\lambda f(k-\lambda-1),\quad (0\leqslant x<\infty).$$

This is of course a polynomial of degree $(\lambda-1)$ and satisfies a set of conditions $S(f;n)=f(n)$, $n=0,1,2,\cdots$. Hence we must have $S(f;x)\equiv f(x)$ $(0\leqslant x<\infty)$. This completes the proof of Theorem 1.

Theorem 2 The continuous interpolating function $S(f;x)$ given by(21) is smooth everywhere in $(0,\infty)$ except at the integer points $x=n$, $(n=1,2,\cdots)$. Moreover

$$S'(f;n+)-S'(f;n-)=\frac{1}{\lambda-1}(-1)^\lambda\Delta^\lambda f(n-1), \tag{22}$$

where $S'(f;n+)$ and $S'(f;n-)$ denote the right and left derivatives of $S(f;x)$ at $x=n$, respectively.

Proof Evidently $S(f;x)$ can be differentiated any number of times with respect to x $(0<x<\infty)$ except at the integer points $x=n$, $(n=1,2,\cdots)$. Moreover.

$$S'(f;n+)=\sum_{k=1}^{n+\lambda}\frac{d}{dx}\binom{x+\lambda-k}{\lambda-1}_{x=n}\cdot\Delta^\lambda f(k-\lambda-1),$$

$$S'(f;n-)=\sum_{k=1}^{n+\lambda-1}\frac{d}{dx}\binom{x+\lambda-k}{\lambda-1}_{x=n}\cdot\Delta^\lambda f(k-\lambda-1),$$

so that we have

$$S'(f;n+)-S'(f;n-)=\frac{d}{dx}\binom{x-n}{\lambda-1}_{x=n}\cdot\Delta^\lambda f(n-1)$$

$$=\frac{d}{dx}\binom{x}{\lambda-1}_{x=0}\Delta^\lambda f(n-1)=\frac{1}{\lambda-1}(-1)^\lambda\Delta^\lambda f(n-1).$$

Hence the theorem.

Though (21) is not a smooth interpolating function it may be effectively used to solve the interpolation problem on $(0,\infty)$ whenever the sequences of data, $\{f(n)\}$, are given empirically. In what follows we state a similar result that can be proved similarly and may be used to treat the interpolation problem on the plane region $[0,\infty)\times[0,\infty)$. For convenience we use $H^{(r,s)}$ to denote the class of bivariate polynomials $\varphi(x,y)$ of the form

$$\varphi(x,y)=\sum_{\substack{0\leqslant\alpha\leqslant r\\0\leqslant\beta\leqslant s}} c_{\alpha\beta}\cdot x^{\alpha}\cdot y^{\beta}$$

In other words, $H^{(r,s)}$ consists of all the bivariate polynomials with highest degrees r in x and s in y, respectively.

Theorem 3 Let $r\geqslant 2$ and $s\geqslant 2$ be positive integer parameters. For any given sequence $\{f(m,n)\}$ $(m,n=0,1,2,\cdots)$ we introduce the supplemental definition $f(-k,\cdot)=f(\cdot,-j)=0$ $(k,j=1,2,3,\cdots)$. Then the interpolating function of the form

$$S(f;x,y)=\sum_{k=1}^{[x+r]}\sum_{j=1}^{[y+s]}\binom{x+r-k}{r-1}\binom{y+s-j}{s-1}\Delta_x^r\Delta_y^s f(k-r-1,j-s-1) \tag{23}$$

with Δ_x Δ_y denoting the difference operators in regard to x and y of f(x, y), respectively, has the following properties (1) $S(f;m,n)=f(m,n)$ $(m,n=0,1,2,\cdots)$; (2) $S(f;x,y)$ is a continuous function of (x,y) in $(0\leqslant x,\ y<\infty)$; (3) for every polynomial $\varphi(x,y)\in H^{(r-1,s-1)}$ defined on $[0,\infty)\times[0,\infty)$ with supplemental definition $\varphi(-k,\cdot)=\varphi(\cdot,-j)=0$ $(k,j=1,2,\cdots)$ we have $S(\varphi;x,y)\equiv\varphi(x,y)$ $(0\leqslant x,y<\infty)$.

As may easily be observed, the continuous surface $z=S(f;x,y)$ defined on $[0,\infty)\times[0,\infty)$ is not smooth on the lines of net: $x=m$ and $y=n$, $(m,n=1,2,3,\cdots)$.

§3 Remarks and Discussion

3.1 Interpolation formulas defined on any set of equidistant knots

The interpolation formula (21) is a formula with step-length unity. Let the knots of interpolation be given by $x_k=x_0+kd$ $(k=0,1,2,\cdots)$, where d is a positive increment. Suppose that we are given the function values $f(x_k)=f(x_0+kd)$ with the supplemental definition

$$f(x_0-md)=0,\quad m=1,2,3,\cdots.$$

Then (21) may be extended to the form (with x in ($[0,\infty)$))

$$S^{(d)}(f;x)=\sum_{k=1}^{[(x-x_0)/d+\lambda]}\binom{(x-x_0)/d+\lambda-k}{\lambda-1}\underset{d}{\Delta^{\lambda}} f(x_0+(k-\lambda-1)d) \tag{24}$$

where $\underset{d}{\Delta}$ denotes the difference operator of increment d. Similarly, we have a bivariate interpolation formula for f(x,y), of the form

$$S^{(d_1,d_2)}(f;x,y)=\sum_{k=1}^{[(x-x_0)/d_1+r]}\sum_{j=1}^{[(y-y_0)/d_2+s]}\binom{(x-x_0)/d_1+r-k}{r-1}\binom{(y-y_0)/d_2+s-j}{s-1}$$
$$\times\underset{d_1}{\Delta_x^r}\underset{d_2}{\Delta_y^s}f(x_0+(k-r-1)d_1,\ y_0+(j-s-1)d_2) \qquad (25)$$

where r and s are positive integers not less than 2, and $\underset{d_1}{\Delta_x}$ and $\underset{d_2}{\Delta_y}$ are partial difference operators with respect to x any y, and using increments d_1 and d_2, respectively.

Formula (25) can be used to approximate bivariate continuous function $z=f(x,y)$ in $[0,\infty)\times[0,\infty)$, when adopting sufficiently small d_1 and d_2. In particular (25) is exact for bivariate polynomials $f(x,y)$ whose highest degrees in x and y do not exceed (r − 1) and (s − 1) respectively.

Example Given $f(x)=\frac{1}{1+25x^2}$. Take $x_0=-1$, $d=0.2$, so that the knots of interpolation may be written $x_k=-1+0.2k(k=0,1,\cdots)$ Supplementing the condition $f(-1-0.2m)=0$ $(m=1,2,\cdots)$, and making use of (24) with $\lambda=4$, we get a table of numerical results in which a comparison has been made with the results given by the Newton interpolation $P_{10}(x)$ consisting of ten terms. (The table is displayed in the close of this paper).

3.2 Comparison with Newton's interpolation formula

1° Formula (24) is an indefinite summation with a variable upper limit so that it is not a type of polynomial interpolation like Newton's.

2° Both Newton's formula and (24) make use of finite differences, but the latter one consists of differences of the definite order λ at distinct points $x_{k-\lambda-1}=x_0+(k-\lambda-1)d$, while Newton's formula employs differences of all orders at a fixed point say x_0.

3° Having fixed the highest order of differences the Newton interpolation formula may satisfy a finite number of interpolation conditions so it possesses only the local property of approximation. Formula (24) (with fixed $\lambda\geqslant 2$) may be used to approximate a function globally on $[0,\infty)$ since it satisfies all the interpolation conditions $S^{(d)}(f;x)=f(x)$, $x=x_0, x_0+d, x_0+2d,\ldots$.

4° As an approximation process for fixed $\lambda\geqslant 2$, the accuracy of (24) may be increased by diminishing the step-length (increment)d. In fact, (24) may converge to a contiuous function f(x) as $d\to 0+$. But this is not the case for Newton's interpolation process. Actually the increasing of difference orders in Newton's formula may even lead to some unforeseen Runge's phenomena. (This may be viewed from the table, loc. cit.)

3.3 A property for displacement

Let us now state and prove the following

Theorem 4 Let $\{f(n)\}_0^\infty$ be any given sequence of numbers with supplemental definition $f(-k)=0$ $(k=1,2,3,\cdots)$. Define the sequence $\{\varphi(n)\}_{-\infty}^{\infty}$ by the following

$$\varphi(n)=\begin{cases} f(n+t) & \text{when } n\geqslant 0 \\ 0 & \text{when } n<0 \end{cases}$$

where t is any fixed positive integer. Then, writing $x=X+t$, we have

$$S(f;x)\equiv S(\varphi;X). \tag{26}$$

That is,

$$\sum_{k=1}^{[x+\lambda]}\binom{x+\lambda-k}{\lambda-1}\Delta^\lambda f(k-\lambda-1)=\sum_{k=1}^{[X+\lambda]}\binom{X+\lambda-k}{\lambda-1}\Delta^\lambda\varphi(k-\lambda-1)$$

Proof The verification of (26) involves somewhat complicated manipulations with some kinds of binomial summations. In particular we shall make use of the combinatorial identity (cf. Gould's Table 3, formula (3.47))

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{x+k}{r}=(-1)^n\binom{x}{r-n}, \qquad (x\text{—real})$$

Clearly we may write

$$S(f\ x)=\sum_{k=1-t}^{[X+\lambda]}\binom{X+\lambda-k}{\lambda-1}\Delta^\lambda f(k-\lambda-1+t)$$
$$=\sum_{k=\lambda+1}^{[X+\lambda]}+\sum_{k=1-t}^{\lambda}=S_1+S_2, \quad \text{say.}$$

As may be observed, S_1 may be rewritten as

$$S_1=\sum_{k=\lambda+1}^{[X+\lambda]}\binom{X+\lambda-k}{\lambda-1}\Delta^\lambda\varphi(k-\lambda-1).$$

For the second sum S_2 we have to reverse the order of the double summation by splitting the resultant sum into three manageble summations, (This requires a little trick). More precisely we have

$$S_2=\sum_{k=1-t}^{\lambda}\binom{X+\lambda-k}{\lambda-1}\sum_{j=k-1-\lambda}^{k-1}(-1)^{k-j-1}\binom{\lambda}{k-j-1}f(j+t)$$
$$=\sum_{j=-t-\lambda}^{-t-1}f(j+t)\sum_{k=1-t}^{j+\lambda+1}(-1)^{k-j-1}\binom{X+\lambda-k}{\lambda-1}\binom{\lambda}{k-j-1}$$
$$+\sum_{j=-t}^{-1}f(j+t)\sum_{k=j+1}^{j+\lambda+1}(-1)^{k-j-1}\binom{X+\lambda-k}{\lambda-1}\binom{\lambda}{k-j-1}$$
$$+\sum_{j=0}^{\lambda-1}f(j+t)\sum_{k=j+1}^{\lambda}(-1)^{k-j-1}\binom{X+\lambda-k}{\lambda-1}\binom{\lambda}{k-j-1}$$
$$=S_{21}+S_{22}+S_{23}, \quad \text{say.}$$

By the supplemental definition we have $S_{21}=0$. Now applying the combinatorial identity cited, we easily find $S_{22}=0$. Indeed,

$$S_{22}=\sum_{j=-t}^{-1} f(j+t)\sum_{k=0}^{\lambda}(-1)^{\lambda-k}\binom{X-j-1+k}{\lambda-1}\binom{\lambda}{\lambda-k}$$

$$=\sum_{j=-t}^{-1} f(j+t)\binom{X-j-1}{-1}=0$$

As regards S_{23} we have

$$S_{23}=\sum_{k=1}^{\lambda}\binom{X+\lambda-k}{\lambda-1}\sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{\lambda}{k-j-1}f(j+t)$$

$$=\sum_{k=1}^{\lambda}\binom{X+\lambda-k}{\lambda-1}\sum_{j=0}^{k-1}(-1)^{j}\binom{\lambda}{j}\varphi(k-j-1)$$

$$=\sum_{k=1}^{\lambda}\binom{X+\lambda-k}{\lambda-1}\Delta^{\lambda}\varphi(k-\lambda-1)$$

Hence we obtain $S(f;x)=S_1+S_2=S_1+S_{23}=S(\varphi;X)$, completing the proof of (26).

Theorem 4 indicates that a longer summation $S(f;x)$ (when x is large) may conveniently be replaced by a shorter one. In practice one may take $t=[x]$ so that $0\leqslant X<1$ and $S(\varphi;X)$ consists of only λ terms.

§4 An Unified Explicit Formula for Piece-Wise Polynomial Interpolation on $[0, \infty)$

It may be of interest to note that a suitable modification of (21) can yield a piece-wise polynomial interpolation formula on $[0, \infty)$ of degree $(\lambda-1)$. Actually what we give below is such a formula

$$\bar{S}(f;x)=\sum_{k=1}^{\lambda+[x/(\lambda-1)]\cdot(\lambda-1)}\binom{x+\lambda-k}{\lambda-1}\Delta^{\lambda}f(k-\lambda-1) \tag{27}$$

Here $\{f(k)\}_0^{\infty}$ is any given sequence with supplemental definition $f(-m)=0$. $m=1, 2,\cdots$.

According to theorem 4 it is readily observed that $\bar{S}(f;x)$, when defined on each interval $[p(\lambda-1), (p+1)(\lambda-1)]$ $(p=0, 1, 2,\ldots)$, satisfies the interpolation condition

$$\bar{S}(f;k)=f(k),\quad k=p(\lambda-1),\ p(\lambda-1)+1,\cdots,p(\lambda-1)+\lambda-1$$

so that $\bar{S}(f;x)$ is the uniquely determined $(\lambda-1)$-th degree polynomial on the interval just mentioned. Thus (27) offers a unified formula for piece-wise polynomial interpolation on $[0, \infty)$.

By employing the argument similar to that used in the proof of theorem 1 we may infer that $\bar{S}(f;x)$ is a continuous function smooth everywhere in $[0, \infty)$ except at the points $x=m\equiv 0$ (mod$(\lambda-1)$), where the derivatives $\bar{S}'(f;x)$ have jumps:

$$\bar{S}'(f;m+)-\bar{S}'(f;m-)=\frac{1}{\lambda-1}(-1)^{\lambda}\Delta^{\lambda}f(m-1).$$

In the numerical table as shown below it is clear that the data given by $\bar{S}^{(d)}(f;x)$ (with $d=0.2$) appear to be much better than those given by Newton's interpolation polynomial $P_{10}(x)$.

Appendix: Numerical Table

x	$f(x)=\frac{1}{1+25x^2}$	$S^{(0.2)}(f;x)$	$\bar{S}^{(0.2)}(f;x)$	$P_{10}(x)$
−1	0.03846	0.03846	0.03846	0.03846
−0.96	0.04160	0.04269	0.04269	1.80438
−0.90	0.04706	0.04841	0.04841	1.57872
−0.86	0.05131	0.05225	0.05225	0.88808
−0.80	0.05882	0.05882	0.05882	0.05882
−0.76	0.06477	0.06913	0.06417	−0.20130
−0.70	0.07547	0.08088	0.07443	−0.22620
−0.66	0.08410	0.08789	0.08320	−0.10832
−0.60	0.1	0.1	0.1	0.1
−0.56	0.11312	0.104	0.11408	0.19873
−0.50	0.13793	0.125	0.14027	0.25376
−0.46	0.15898	0.149	0.16156	0.24145
−0.40	0.2	0.2	0.2	0.2
−0.36	0.23585	0.1864	0.1864	0.18878
−0.30	0.30769	0.25	0.25	0.23535
−0.26	0.37175	0.3344	0.3344	0.31650
−0.20	0.5	0.5	0.5	0.5
−0.16	0.60976	0.7376	0.6224	0.64316
−0.10	0.8	0.95	0.8	0.84340
−0.06	0.91743	1.0096	0.9004	0.94090
0.0	1	1	1	1

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利用反级数关系构造插值公式的一种方法

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摘　要

这篇文章指出，互反级数关系可以成为等距插值公式的一个来源。从原则上说来，只要一对互反级数关系中的“求和核”(级数变换核）容许扩充为连续变量的函数，则相应地便可获得一个插值公式。本文举出一系列例子说明了这个方法。特别，我们从广义Möbius-Rota 反演公式出发，造出了一类借助于差分表出的插值公式。这类公式不同于Newton 插值法，其特点是具有大范围插值性质；并且由于公式中只使用阶数固定的差分，故还能避免出现“Runge 现象”。本文给出了四条定理，论述了这类公式的性质。最后，我们还对具有代数精度的分段（分片）插值法给出了一个统一公式。显然，本文所述方法还值得进一步加以运用并拓广。