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An Extension of the Parallelogram Characterization of Inner Product Spaces*

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We give in the following a characterization of inner product spaces, of which the well-known parallelogram law and its integral analogue given recently by A. T. Penico and C. V. Stanojevic [1] are special cases.

Theorem: A normed complex linear space N is an inner product space if and only if there exists a measure space (X, B, u) and a pair of orthonormal vectors f and g of $\mathscr{L}^2(X, \mathrm{B}, u)$ with

$$u\{t|\ fg\neq 0\}>0,$$

such that

$$\int\|xf+yg\|^2du\leqslant\|x\|^2+\|y\|^2 \qquad \text{for all } x,\ y\in N.$$

Proof: If N is an inner product space, then for any measure space(X, B, u), for instance the real interval $(0, 1)$ with its Lebesgue measure say, and a pair of orthonormal vectors f and g of $\mathscr{L}^2(X, \mathrm{B}, u)$ with

$$u\{t|\ fg\neq 0\}>0,$$

we have

$$\int\|xy+yg\|^2du=\int(xf+yg,\ xf+yg)du$$
$$=\|x\|^2\int|f|^2du+\|y\|^2\int|g|^2du+2(x,\ y)\mathrm{Re}\int f\bar{g}du$$
$$=\|x\|^2+\|y\|^2$$

for all $x,\ y\in N$. Hence the "only if" part is proved.

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Now suppose that N satisfies the given condition. It is sufficient to restrict to the case where N is a 2-dimensicnal normed real linear space N_2. Let S be the unit sphere of N_2. By Loewner Lemma [2] S can be encompassed by a unique minimal ellipse E which meets S in at least four distinct points x_0, $-x_0$, y_0, $-y_0$, with x_0 and y_0 being linearly independent. This ellipse induces in N_2 an euclidean norm $\|\cdot\|_E$, with E as an unit sphere. Hence $\|z\|_E \leqslant \|z\|$ for all $z \in N_2$. Let a and b be two arbitrary real numbers. Then $\|ax_0\| = \|ax_0\|_E$, $\|by_0\| = by_0\|_E$, and

$$\|ax_0\|^2 + \|by_0\|^2 \geqslant \int \|ax_0 f + by_0 g\|^2 du \geqslant \int \|ax_0 f + by_0 g\|_E^2 du$$

$$= \|ax_0\|_E^2 + \|by_0\|_E^2 = \|ax_0\|^2 + \|by_0\|^2.$$

Hence

$$\int \|ax_0 f + by_0 g\|^2 du = \int \|ax_0 f + by_0 g\|_E^2 du.$$

It follows that

(1) $$\|ax_0 f(t) + by_0 g(t)\| = \|ax_0 f(t) + by_0 g(t)\|_E$$

almost everywhere in u since

$$\|ax_0 f + by_0 g\|^2 - \|ax_0 f + by_0 g\|_E^2 \geqslant 0.$$

That is, if we denote

$$\Lambda_{ab} = \{t \in X \mid \|ax_0 f(t) + by_0 g(t)\| \neq \|ax_0 f(t) + by_0 g(t)\|_E\},$$

then

$$u\{\Lambda_{ab}\} = 0.$$

Let a and b range over all rational numbers and the union of all the corresponding sets Λ_{ab} be denoted by Λ, then

$$u\{\Lambda\} = 0.$$

By the denseness of the rational numbers on the real line and the continuity of norms, we assert that (1) holds for any two real numbers a and b and for all $t \bar{\in} \Lambda$.

Since $f(t) \cdot g(t)$ does not vanish almost everywhere in u, there exists $t_0 \bar{\in} \Lambda$ such that $f(t_0) \neq 0$ and $g(t_0) \neq 0$. Consequently for any real numbers a and b, we have

(2) $$\|ax_0 f(t_0) + by_0 g(t_0)\| = \|ax_0 f(t_0) + by_0 g(t_0)\|_E.$$

With $\{x_0 f(t_0),\ y_0 g(t_0)\}$ as a basis of N_2, (2) implies

$$\|z\| = \|z\|_E \qquad for\ all\ z \in N_2.$$

This proves the theorem.

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It may be of interest to deduce some other characterizations directly from the preceding theorem. Take for example, $X=\{0, 1\}$ and $u=\varepsilon_0+\varepsilon_1$, where

$$\varepsilon_i(E)=\begin{cases}0, & i\bar{\in} E,\\ 1, & i\in E,\quad i=0,\ 1\end{cases}$$

and let

$$f(t)=\frac{1}{\sqrt{2}},\quad t\in X,$$

$$g(t)=\begin{cases}\frac{1}{\sqrt{2}}, & t=0,\\ -\frac{1}{\sqrt{2}}, & t=1.\end{cases}$$

We obtain the well-known parallelogram characterization:

Corollary 1. A normed complex linear space N is an inner product space if and only if

$$\|x+y\|^2+\|x-y\|^2\leqslant 2(\|x\|^2+\|y\|^2)\qquad \text{for all } x,\ y\in N.$$

Again, take $X=[0,\ 2\pi]$ and u being the Lebesgue measure. Let

$$f(t)=\frac{\cos t}{\sqrt{\pi}},\quad t\in[0,\ 2\pi],\qquad g(t)=\frac{\sin t}{\sqrt{\pi}},\quad t\in[0,\ 2\pi].$$

then we obtain the characterization due to Penico-Stanojevic [1]:

Corollary 2. A normed complex linear space N is an inner product space if and only if for all vectors x and y in N

(PS) $$\frac{1}{\pi}\int_0^{2\pi}\|x\cos t+y\sin t\|^2du\leqslant\|x\|^2+\|y\|^2.$$

In fact, the characterization given by Penico and Stanojevic is in a stronger form. They have shown that it is sufficient that (PS) holds for all unit vectors x and y only. This can be observed from the proof given above. For then owing to the continuity of $\cos t$ and $\sin t$ we assert that

$$\|x_0\cos t+y_0\sin t\|=\|x_0\cos t+y_0\sin t\|_E$$

for every $t\in[0,\ 2\pi]$. Since every vector z is of the form $\rho(x_0\cos t+y_0\sin t)$, we obtain $\|z\|=\|z\|_E$ for all $z\in N$ and the conclusion.

References

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2. Day, M. M., Some characterizations of inner-product spaces, *Trans. Amer. Math. Soc.* **62** (1947), 320—337.

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